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Tight bounds on American option prices

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A B S T R A C T

In contrast to the constant exercise boundary assumed by Broadie and Detemple (1996) [Broadie, M., Detemple, J., 1996. American option valuation: New bounds, approximations, and comparison of existing methods. Review of Financial Studies 9, 1211–1250], we use an exponential function to approximate the early exercise boundary. Then, we obtain lower bounds for American option prices and the optimal exercise boundary which improve the bounds of Broadie and Detemple (1996). With the tight lower bound for the optimal exercise boundary, we further derive a tight upper bound for the American option price using the early exercise premium integral of Kim (1990) [Kim, I.J., 1990. The analytic valuation of American options. Review of Financial Studies 3, 547–572]. The numerical results show that our lower and upper bounds are very tight and can improve the pricing errors of the lower bound and upper bound of Broadie and Detemple (1996) by 83.0% and 87.5%, respectively. The tightness of our upper bounds is comparable to some best accurate/efficient methods in the literature for pricing American options. Moreover, the results also indicate that the hedge ratios (deltas and gammas) of our bounds are close to the accurate values of American options.

1. Introduction

One stream of the American option pricing literature is to derive lower and/or upper bounds for American option values,[1] see e.g. Perrakis (1986), Chen and Yeh (2002), and Chung and Chang (2007).[2] Notably Broadie and Detemple (1996) provide tight lower and upper bounds for American call prices based on the assumption that the early exercise boundary is a constant. In this paper, we provide even tighter bounds for American option prices by making a more realistic and flexible assumption that the early exercise boundary follows an exponential function.[3] Since the optimal exercise boundary is a monotone function in time-to-maturity, the assumed exponential boundary can improve the constant boundary in Broadie and Detemple (1996).

We first derive a tight lower bound for the American call option as the price of a capped (barrier) call option with an exponential exercise policy. Since all admissible exponential exercise policies generate lower price bounds, a tight lower bound for the American call option can be obtained based on maximizing the values of the capped options over the parameters of the exponential exercise barrier. Given the derivative information of the capped call price with respect to parameters of the exponential function, the optimization problem can be easily solved by an iterative procedure.

Next, we obtain a tight lower bound for the optimal exercise boundary based on our tight lower bound for the American call option price. The idea is intuitive and can be described as follows. Note that the optimal exercise boundary (B_t^*) is the intersection point (i.e. the value-matching point) of the early exercise value

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[1] Barone-Adesi (2005) surveys several mainstreams of the development of American option pricing models, including solving the partial differential equation, calculating the early exercise premium integral, using the least-squares approach, and deriving the upper and/or lower bounds for American options. In addition to these models, it is also feasible to price American options by the static hedge portfolio approach in Derman et al. (1995), Carr et al. (1998), and Chung and Shih (forthcoming) or by the linearly implicit scheme with a penalty method approach in Khaliq et al. (2006).

[2] For European option pricing bound literature, please see Chung and Wang (2008) and the references therein.

[3] Although Omberg (1987) and Ingersoll (1998) suggest using an exponential function to approximate the early exercise boundary, they do not provide detailed formulae to solve the lower bound and the optimal exercise boundary. In contrast, we follow the approach of Broadie and Detemple (1996) and derive the essential formulae for solving the lower bound and the optimal exercise boundary.

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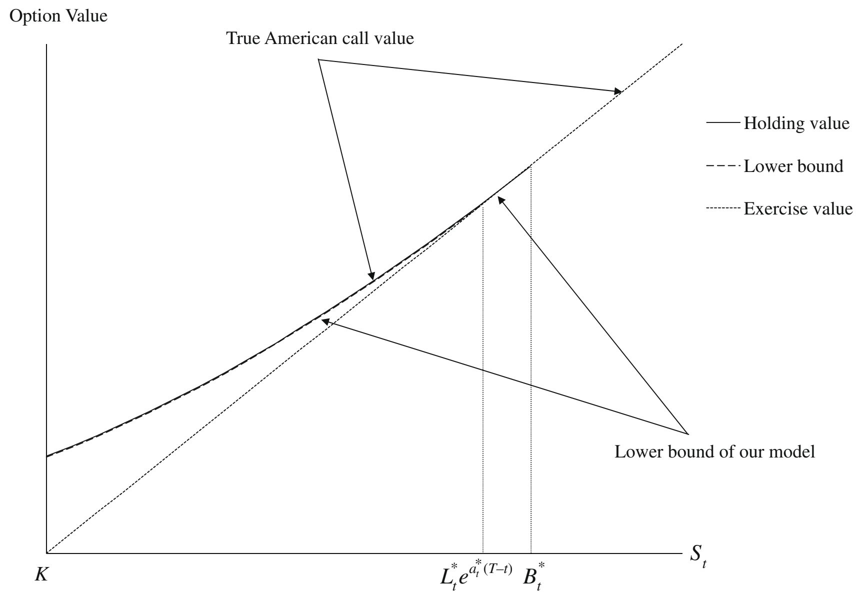

Fig. 1. Illustration of the relationship between the optimal exercise boundary (B_t^*) and our lower bound ($L_t^* e^{a_t^*(T-t)}$). This figure shows the relationship between the optimal exercise boundary (B_t^*) and our lower bound ($L_t^* e^{a_t^*(T-t)}$) for the optimal exercise boundary. Since $C_t^E(S_t, L, a)$ generates the lower price bound of the American call option and (L_t^*, a_t^*) is the solution of $L_t^* e^{a_t^*(T-t)} - K = C_t^E(L_t^* e^{a_t^*(T-t)}, L_t^*, a_t^*)$, it is obvious that $L_t^* e^{a_t^*(T-t)}$ should be smaller than the optimal exercise boundary B_t^*. Moreover, if a tighter lower bound for the American call price is derived, a tighter lower bound for the optimal exercise boundary can be obtained.

and the holding value for American call option as the underlying asset price (S_t) increases from below. Since our lower price bound is always smaller than American option price in the holding region (i.e. the region where $S_t \le B_t^*$), the intersection point of the early exercise value and the holding value for the capped call option under the exponential exercise policy must be less than the optimal exercise boundary B_t^*. Please see Fig. 1 for the ease of understanding our explanation.[4]

Equipped with our tight lower bound for optimal exercise boundary and the integral representation formula for the early exercise premium in Kim (1990), Jacka (1991), and Carr et al. (1992), a tight upper bound for the American call price is then derived in this paper.

Finally, following the method in Broadie and Detemple (1996), we combine both lower and upper bounds, with an optimization and regression exercise, to derive two accurate approximations for American option prices.[5] This optimization regression is based on a set of simulated option contracts, and once equipped with the estimated coefficients, it is possible to approximate American option prices accurately.

Based on the numerical experiments in Tables 1 and 2, we find that our method can improve the errors of the lower bound and upper bound of Broadie and Detemple (1996) by 83.0% and 87.5%, respectively. Moreover, our option pricing bounds are so tight that their accuracy is comparable to the best accurate/efficient methods for pricing American options in the literature. For example, the integral representation formula of Kim (1990), Jacka (1991), and Carr et al. (1992) has been successfully implemented in a series of papers, such as Huang et al. (1996), Ju (1998),[6] and Ibáñez (2003).[7] These papers offer the best speed-accuracy tradeoffs in the literature. Our numerical results indicate that the errors of our upper bounds for pricing long term American put options are only 0.049% which is close to the errors of Ju (1998), i.e. 0.032% (see Table 3 of this study).[8] Similarly, for pricing short term American put options, the errors of our upper bounds and Ibáñez's method are 0.014% and 0.024%, respectively (see Table 4 of this article). Moreover, the two approximations based on our pricing bounds are generally more accurate than Ju (1998) and Ibáñez (2003) for pricing American options.

The rest of this article proceeds as follows. Section 2 provides lower bounds for the American call option price and the optimal exercise boundary based on the assumption that the optimal exercise boundary follows an exponential function. Section 3 applies the lower bound of the optimal exercise boundary to the integral representation formula of Kim (1990), Jacka (1991), and Carr et al. (1992) to obtain an upper bound for the American option price. Section 4 shows the numerical results to analyze the tightness of our lower and upper bounds. Section 5 concludes the paper.

[4] Moreover, it is straightforward to show that a tighter lower bound for the American call price can result in a tighter lower bound for the optimal exercise boundary. Since our lower bound is closer, than the lower bound of Broadie and Detemple (1996), to the American call price, our lower bound for the optimal exercise boundary is also tighter than that of Broadie and Detemple (1996).

[5] Broadie and Detemple (1996) provide two American option price approximations, one based on the lower bound (termed LBA) and one based on both bounds (termed LUBA). Following their approach, we also provide two American option price approximations based on our lower bound and upper bound.

[6] Ju (1998) approximates the early exercise boundary as a multipiece exponential function and substitute it to the early exercise premium integral, derived by Kim (1990), to price American options. Closed-form formulae can be derived and the bases and exponents of the multipiece exponential function can be obtained backward by using *value-matching* and *smooth-pasting* conditions. Thus a two-dimensional Newton–Raphson method must be used to solve the bases and exponents at different times (e.g. see Eqs. (13) and (14) of Ju (1998)).

[7] Ibáñez (2003) introduces a new algorithm to implement the decomposition formula of Kim (1990). He proposes an adjustment of Kim's (1990) discrete-time early exercise premium so that these premiums monotonically converge and therefore it is appropriate to apply them in Richardson extrapolation. Moreover, Ibáñez (2003) also derives the correct order for the error term when applying extrapolation, which is then used to control the error of the extrapolated prices.

[8] Since the pricing error of the proposed method is small, it may be also suitable for pricing long term American-style employee stock options (e.g. see Leon and Vaello-Sebastia (2009)).

Table 1
American call option value bounds and approximations (maturity $T = 0.5$ years).

Option parameter	Asset price	LB1	LB2	UB1	UB2	LBA1	LBA2	LUBA1	LUBA2	True value
$r = 0.03$, $\sigma = 0.2$, $q = 0.07$	80	0.2178	0.2191	0.2196	0.2194	0.2188	0.2194	0.2195	0.2193	0.2194
	90	1.3759	1.3849	1.3885	1.3868	1.3824	1.3863	1.3862	1.3859	1.3864
	100	4.7501	4.7784	4.7919	4.7838	4.7707	4.7827	4.7821	4.7811	4.7826
	110	11.0488	11.0922	11.1253	11.1005	11.0897	11.1008	11.0975	11.0953	11.0977
	120	20.0000	20.0002	20.0575	20.0064	20.0000	20.0119	20.0000	20.0015	20.0005
$r = 0.03$, $\sigma = 0.4$, $q = 0.07$	80	2.6759	2.6871	2.6908	2.6891	2.6889	2.6891	2.6893	2.6882	2.6888
	90	5.6942	5.7186	5.7272	5.7229	5.7207	5.7227	5.7231	5.7210	5.7221
	100	10.1901	10.2329	10.2494	10.2401	10.2350	10.2396	10.2402	10.2367	10.2387
	110	16.1101	16.1731	16.2006	16.1835	16.1758	16.1826	16.1817	16.1782	16.1812
	120	23.2712	23.3504	23.3917	23.3632	23.3559	23.3616	23.3574	23.3558	23.3597
$r = 0.00$, $\sigma = 0.3$, $q = 0.07$	80	1.0287	1.0360	1.0389	1.0375	1.0355	1.0372	1.0373	1.0367	1.0373
	90	3.0981	3.1198	3.1290	3.1241	3.1179	3.1231	3.1232	3.1218	3.1233
	100	6.9845	7.0288	7.0509	7.0373	7.0267	7.0358	7.0355	7.0325	7.0355
	110	12.8818	12.9462	12.9883	12.9585	12.9517	12.9575	12.9531	12.9505	12.9551
	120	20.6501	20.7099	20.7787	20.7233	20.7432	20.7247	20.7208	20.7131	20.7173
$r = 0.07$, $\sigma = 0.3$, $q = 0.03$	80	1.6644	1.6644	1.6644	1.6644	1.6644	1.6644	1.6644	1.6644	1.6644
	90	4.4947	4.4947	4.4947	4.4947	4.4947	4.4947	4.4947	4.4947	4.4947
	100	9.2506	9.2506	9.2506	9.2506	9.2506	9.2506	9.2506	9.2506	9.2506
	110	15.7975	15.7975	15.7975	15.7975	15.7975	15.7975	15.7975	15.7975	15.7975
	120	23.7062	23.7062	23.7062	23.7062	23.7062	23.7062	23.7062	23.7062	23.7062
RMS		0.5033%	0.0674%	0.1633%	0.0205%	0.1229%	0.0179%	0.0163%	0.0262%	

All options have $K = 100$. LB1 and UB1 are lower and upper bounds of Broadie and Detemple (1996). LB2 and UB2 are lower and upper bounds proposed in this paper. LBA1 and LUBA1 are approximations based on Broadie and Detemple's (1996) bounds while LBA2 and LUBA2 are based on our bounds. The "true value" column is calculated from the Binomial Black and Scholes method with Richardson extrapolation (BBSR) and the length of each time step is 0.0001 years.

Table 2
American call option value bounds and approximations (maturity $T = 3$ years).

Option parameter	Asset price	LB1	LB2	UB1	UB2	LBA1	LBA2	LUBA1	LUBA2	True value
$r = 0.03$, $\sigma = 0.2$, $q = 0.07$	80	2.5529	2.5745	2.5891	2.5812	2.5718	2.5787	2.5804	2.5785	2.5800
	90	5.1207	5.1579	5.1865	5.1695	5.1551	5.1655	5.1677	5.1643	5.1670
	100	9.0017	9.0537	9.1023	9.0708	9.0527	9.0653	9.0651	9.0621	9.0660
	110	14.3710	14.4300	14.5037	14.4516	14.4321	14.4448	14.4443	14.4386	14.4434
	120	21.3540	21.4031	21.5060	21.4270	21.4095	21.4182	21.4119	21.4099	21.4139
$r = 0.03$, $\sigma = 0.4$, $q = 0.07$	80	11.2379	11.3101	11.3537	11.3285	11.3155	11.3269	11.3272	11.3225	11.3257
	90	15.6088	15.7023	15.7628	15.7261	15.7065	15.7230	15.7236	15.7175	15.7220
	100	20.6562	20.7698	20.8496	20.7991	20.7698	20.7936	20.7926	20.7874	20.7933
	110	26.3366	26.4678	26.5687	26.5022	26.4592	26.4930	26.4893	26.4870	26.4944
	120	32.6074	32.7522	32.8758	32.7911	32.7301	32.7766	32.7723	32.7720	32.7810
$r = 0.00$, $\sigma = 0.3$, $q = 0.07$	80	5.4631	5.5067	5.5397	5.5202	5.5102	5.5161	5.5199	5.5139	5.5176
	90	8.7658	8.8266	8.8783	8.8459	8.8339	8.8401	8.8435	8.8360	8.8415
	100	13.0477	13.1238	13.1985	13.1490	13.1351	13.1412	13.1415	13.1347	13.1421
	110	18.3473	18.4331	18.5344	18.4634	18.4474	18.4529	18.4530	18.4440	18.4531
	120	24.6849	24.7711	24.9022	24.8053	24.7882	24.7908	24.7974	24.7807	24.7907
$r = 0.07$, $\sigma = 0.3$, $q = 0.03$	80	12.1447	12.1452	12.1453	12.1452	12.1675	12.1452	12.1453	12.1452	12.1452
	90	17.3674	17.3683	17.3684	17.3683	17.3972	17.3683	17.3685	17.3683	17.3683
	100	23.3467	23.3484	23.3486	23.3484	23.3828	23.3484	23.3486	23.3484	23.3484
	110	29.9608	29.9634	29.9639	29.9635	30.0017	29.9634	29.9639	29.9634	29.9635
	120	37.0992	37.1032	37.1040	37.1034	37.1426	37.1032	37.1040	37.1032	37.1033
RMS		0.6188%	0.1174%	0.3213%	0.0401%	0.1403%	0.0162%	0.0159%	0.0380%	

All options have $K = 100$. LB1 and UB1 are lower and upper bounds of Broadie and Detemple (1996). LB2 and UB2 are lower and upper bounds proposed in this paper. LBA1 and LUBA1 are approximations based on Broadie and Detemple's (1996) bounds while LBA2 and LUBA2 are based on our bounds. The "true value" column is calculated from the Binomial Black and Scholes method with Richardson extrapolation (BBSR) and the length of each time step is 0.0001 years.

2. Lower bounds for the American call option value and the optimal exercise boundary

Consider the pricing of an American call option with a strike price K and a fixed maturity date T. Following Black and Scholes (1973) model, we assume that the underlying asset price S_t under the risk-neutral world satisfies

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t, \tag{1}$$

where W_t is a standard Brownian motion process. The volatility σ, the risk-free rate r, and the dividend yield rate q are assumed constant. Let $C_t(S_t)$ denote the American call option price, where the parameters K, T, σ, r, and q are omitted for simplicity.

It is well-known that the valuation of American options is a free boundary problem (see McKean (1965)) and the optimal exercise boundary B_t^* must be solved simultaneously with the valuation problem. Although it is difficult and time consuming to solve B_t^*, the asymptotic behavior of B_t^* has been derived in the literature. For example, Merton (1973) proves that the optimal exercise boundary for the perpetual (i.e. $T \to \infty$) American put option is a constant. Using Merton's technique, one can easily show that the optimal exercise boundary for the perpetual American call option

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Table 3
Price of American put options ($K = 100$, $T = 3$ years, $\sigma = 0.2$, $r = 0.08$).

(S, q)	True value	EXP3	LB2	UB2	LBA2	LUBA2
(80, 0.12)	25.6578	25.6570	25.6572	25.6578	25.6572	25.6577
(90, 0.12)	20.0832	20.0817	20.0829	20.0833	20.0832	20.0832
(100, 0.12)	15.4984	15.4970	15.4982	15.4984	15.5005	15.4984
(110, 0.12)	11.8032	11.8022	11.8031	11.8032	11.8061	11.8032
(120, 0.12)	8.8855	8.8850	8.8854	8.8855	8.8886	8.8855
(80, 0.08)	22.2050	22.2084	22.1963	22.2091	22.2022	22.2032
(90, 0.08)	16.2071	16.2106	16.1973	16.2096	16.2076	16.2062
(100, 0.08)	11.7039	11.7066	11.6953	11.7054	11.7057	11.7038
(110, 0.08)	8.3670	8.3695	8.3512	8.3680	8.3600	8.3664
(120, 0.08)	5.9298	5.9323	5.9247	5.9304	5.9321	5.9300
(80, 0.04)	20.3501	20.3511	20.3448	20.3626	20.3487	20.3512
(90, 0.04)	13.4968	13.5000	13.4853	13.5043	13.4978	13.4959
(100, 0.04)	8.9440	8.9474	8.9320	8.9486	8.9443	8.9437
(110, 0.04)	5.9118	5.9146	5.9016	5.9147	5.9116	5.9120
(120, 0.04)	3.8974	3.8997	3.8896	3.8992	3.8970	3.8976
(80, 0)	20.0000	20.0000	20.0000	20.0155	20.0000	20.0000
(90, 0)	11.6976	11.6991	11.6908	11.7075	11.6970	11.6980
(100, 0)	6.9322	6.9346	6.9235	6.9379	6.9324	6.9319
(110, 0)	4.1550	4.1571	4.1473	4.1583	4.1546	4.1549
(120, 0)	2.5103	2.5119	2.5044	2.5122	2.5095	2.5103
RMS		0.0316%	0.1138%	0.0487%	0.0247%	0.0040%

The parameters are adopted from Table 2 of Ju (1998). The "true value" is calculated from the Binomial Black and Scholes method with Richardson extrapolation (BBSR) and the length of each time step is 0.0001 years. EXP3 represents the America put price estimate using the three-point multipiece exponential boundary method of Ju (1998). LB2 and UB2 are lower and upper bounds proposed in this paper. LBA2 and LUBA2 are approximations based on our bounds.

Table 4
Prices of (short-term) standard American put options.

$S_t = 40$, $r = 0.0488$							
K	T (years)	True value	PEXT	LB2	UB2	LBA2	LUBA2
$\sigma = 0.20$							
35	0.0833	0.0062	0.0062	0.0062	0.0062	0.0062	0.0062
35	0.3333	0.2004	0.2003	0.2002	0.2004	0.2004	0.2004
35	0.5833	0.4328	0.4327	0.4323	0.4329	0.4327	0.4328
40	0.0833	0.8523	0.8523	0.8519	0.8524	0.8516	0.8523
40	0.3333	1.5799	1.5799	1.5786	1.5801	1.5799	1.5799
40	0.5833	1.9905	1.9909	1.9885	1.9910	1.9903	1.9905
45	0.0833	5.0000	5.0000	5.0000	5.0002	5.0000	5.0000
45	0.3333	5.0883	5.0889	5.0871	5.0894	5.0904	5.0883
45	0.5833	5.2670	5.2640	5.2647	5.2684	5.2681	5.2669
$\sigma = 0.30$							
35	0.0833	0.0775	0.0774	0.0774	0.0775	0.0774	0.0775
35	0.3333	0.6976	0.6975	0.6971	0.6977	0.6976	0.6976
35	0.5833	1.2199	1.2197	1.2188	1.2200	1.2199	1.2199
40	0.0833	1.3102	1.3101	1.3098	1.3103	1.3098	1.3103
40	0.3333	2.4827	2.4825	2.4811	2.4830	2.4811	2.4826
40	0.5833	3.1697	3.1697	3.1673	3.1702	3.1697	3.1698
45	0.0833	5.0598	5.0588	5.0588	5.0601	5.0606	5.0595
45	0.3333	5.7057	5.7046	5.7034	5.7063	5.7065	5.7056
45	0.5833	6.2437	6.2423	6.2402	6.2446	6.2440	6.2435
$\sigma = 0.40$							
35	0.0833	0.2467	0.2467	0.2467	0.2467	0.2467	0.2467
35	0.3333	1.3462	1.3461	1.3454	1.3463	1.3460	1.3463
35	0.5833	2.1550	2.1547	2.1535	2.1553	2.1550	2.1552
40	0.0833	1.7685	1.7685	1.7680	1.7686	1.7680	1.7686
40	0.3333	3.3876	3.3874	3.3859	3.3881	3.3857	3.3877
40	0.5833	4.3528	4.3526	4.3500	4.3534	4.3496	4.3528
45	0.0833	5.2870	5.2870	5.2862	5.2873	5.2862	5.2872
45	0.3333	6.5099	6.5106	6.5074	6.5105	6.5097	6.5101
45	0.5833	7.3831	7.3841	7.3791	7.3839	7.3827	7.3832
RMS			0.0236%	0.0618%	0.0136%	0.0336%	0.0045%

The parameters are adopted from Table 1 of Ibáñez (2003). The "true value" is calculated from the Binomial Black and Scholes method with Richardson extrapolation (BBSR) and the length of each time step is 0.0001 years. PEXT represents the extrapolated America put price from three Bermudan prices with 4, 5, and 6 exercise dates with the modified early exercise premium derived by Ibáñez (2003). LB2 and UB2 are lower and upper bounds proposed in this paper. LBA2 and LUBA2 are approximations based on our bounds.

is $\beta K/(\beta - 1)$, where $\beta = \frac{1}{2} - \frac{(r-q)}{\sigma^2} + \sqrt{\left(\frac{(r-q)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$. On the other hand, when $r > q$ and the time to maturity approaches zero (i.e. $t \to T$), Evans et al. (2002, Eq. (1.11)) show that the early exercise boundary near expiration satisfies

$$B_t^* \sim \frac{r}{q} K\left(1 + \sigma\alpha_0\sqrt{2(T-t)}\right),$$

where α_0 is a constant determined by a transcendental equation.

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Inspiring by the above results,[9] we approximate the early exercise boundary of an American call using an exponential function, i.e. $B_t = Le^{a(T-t)}$. The payoff of the approximate American call is either $\max(Le^{a(T-\tau)} - K, 0)$, if the underlying asset price first hits the exponential exercise boundary from below at time τ where $t \le \tau < T$, or $\max(S_T - K, 0)$, otherwise. The American call under the exponential exercise boundary is essentially a capped (barrier) option and its value $(C_t^E(S_t, L, a))$ is given by

$$\begin{aligned} C_t^E(S_t, L, a) = {} & Le^{a(T-t)}\left[\hat{\lambda}_t^{\tilde{b}-\tilde{\beta}} N(\tilde{d}_0) + \hat{\lambda}_t^{\tilde{b}+\tilde{\beta}} N(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t})\right] \\ & - K\left[\hat{\lambda}_t^{\hat{b}-\hat{\beta}} N(\hat{d}_0) + \hat{\lambda}_t^{\hat{b}+\hat{\beta}} N(\hat{d}_0 + 2\hat{\beta}\sigma\sqrt{T-t})\right] \\ & + S_t e^{-q(T-t)}\Big\{N\Big(\hat{d}_1^-(Le^{a(T-t)}) - \sigma\sqrt{T-t}\Big) \\ & - N\Big(\hat{d}^-(K) - \sigma\sqrt{T-t}\Big) \\ & + \hat{\lambda}_t^{\hat{\gamma}-1}\Big[N\Big(\hat{d}^+(K) - \sigma\sqrt{T-t}\Big) \\ & - N\Big(\hat{d}_1^+(Le^{a(T-t)}) - \sigma\sqrt{T-t}\Big)\Big]\Big\} \\ & - Ke^{-r(T-t)}\Big\{N\Big(\hat{d}_1^-(Le^{a(T-t)})\Big) \\ & - N(\hat{d}^-(K)) + \hat{\lambda}_t^{\hat{\gamma}+1}\Big[N(\hat{d}^+(K)) - N\Big(\hat{d}_1^+(Le^{a(T-t)})\Big)\Big]\Big\}, \end{aligned} \tag{2}$$

where $N(.)$ is the cumulative distribution function of a standard normal variable,

$$\hat{\lambda}_t = S_t/(Le^{a(T-t)}), \quad \hat{b} = -\left(r - q + a - \frac{1}{2}\sigma^2\right)\Big/\sigma^2,$$

$$\hat{\beta} = \sqrt{\hat{b}^2 + 2r/\sigma^2},$$

$$\hat{d}_0 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln(\hat{\lambda}_t) - \hat{\beta}\sigma^2(T-t)\right], \quad \tilde{b} = -\left(r + a - q - \frac{1}{2}\sigma^2\right)\Big/\sigma^2,$$

$$\tilde{\beta} = \sqrt{\tilde{b}^2 + 2(r+a)/\sigma^2}, \quad \tilde{d}_0 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln(\hat{\lambda}_t) - \tilde{\beta}\sigma^2(T-t)\right],$$

$$\hat{\gamma} = -2(r - q + a)/\sigma^2,$$

$$\hat{d}^\pm(x) = \frac{1}{\sigma\sqrt{T-t}}\left[\pm\ln(\hat{\lambda}_t) - \ln(L) + \ln(x) - \left(r - q + a - \frac{1}{2}\sigma^2\right)(T-t)\right],$$

$$\hat{d}_1^\pm(x) = \frac{1}{\sigma\sqrt{T-t}}\left[\pm\ln(\hat{\lambda}_t) - \ln(L) + \ln(x) - \left(r - q + 2a - \frac{1}{2}\sigma^2\right)(T-t)\right].$$

Please note the Eq. (2) holds only when both the current stock price and the strike price are below the exponential exercise boundary, i.e. $Le^{a(T-t)} \geqslant \max(S_t, K)$. For completeness, we define $C_t^E(S_t, L, a) = \max(\min(S_t, Le^{a(T-t)}) - K, 0)$ when $Le^{a(T-t)} < \max(S_t, K)$.

Since the policy of exercising when the underlying asset price reaches the exponential exercise boundary is an admissible (but not the optimal) policy for the American option, the above formula for $C_t^E(S_t, L, a)$ gives an immediate lower bound of the American option price $C_t(S_t)$, i.e. $C_t^E(S_t, L, a) \le C_t(S_t)$ for any $\{L, a\}$. Moreover a lower bound is still obtained after maximizing over $\{L, a\}$ subject to the constraint that $Le^{a(T-t)} \ge S_t$,[10] i.e. $\max_{Le^{a(T-t)} \ge S_t} C_t^E(S_t, L, a) \le C_t(S_t)$. Denote the optimal solution $\{\widehat{L}(S_t), \hat{a}(S_t)\}$ as

$$\{\widehat{L}(S_t), \hat{a}(S_t)\} = \arg\max_{Le^{a(T-t)} \ge S_t} C_t^E(S_t, L, a). \tag{3}$$

Hence

$$C_t^l(S_t) = \max_{Le^{a(T-t)} \ge S_t} C_t^E(S_t, L, a) \le C_t(S_t). \tag{4}$$

Note that our lower bound in Eq. (4) improves over the naïve lower bound of the European call value, denoted as $c_t(S_t)$, because $c_t(S_t) = \lim_{L\to\infty} C_t^E(S_t, L, a = 0)$. Moreover, since the constant exercise boundary of Broadie and Detemple (1996) is a special case of our exponential exercise boundary with $a = 0$, our lower bound is also tighter than their lower bound.

Solving $\{\widehat{L}(S_t), \hat{a}(S_t)\}$ is a bi-variate differentiable optimization problem for any given S_t. We apply an iterative procedure to solve the optimization problem. With the formulae of derivatives $\partial C_t^E(S_t, L, a)/\partial L$ and $\partial C_t^E(S_t, L, a)/\partial a$, the optimal solution should satisfy

$$[\partial C_t^E(S_t, L, a)/\partial L]^2 + [\partial C_t^E(S_t, L, a)/\partial a]^2 = 0. \tag{5}$$

We first give an initial guess of the level L_0 and then find the optimal solution of the growth rate a_0 by minimizing the value of the left hand side (LHS) of Eq. (5).[11] Using a_0 we find the optimal solution of L_1 by minimizing the value of the left hand side of Eq. (5). We repeat the above procedure to generate a series of L_i and a_i until Eq. (5) is satisfied.[12] The derivatives $\partial C_t^E(S_t, L, a)/\partial L$ and $\partial C_t^E(S_t, L, a)/\partial a$ are given in Proposition 1 of Appendix A.

Following the same idea of Broadie and Detemple (1996), the lower price bound based on the exponential exercise policy can give a lower bound for the optimal exercise boundary. As the asset price S_t approaches $Le^{a(T-t)}$ from below, we can evaluate the derivatives of the exponential barrier option price with respect to L and a, respectively:

$$\begin{aligned} D_L(L, a, t) &= \lim_{S_t \uparrow Le^{a(T-t)}} \frac{\partial C_t^E(S_t, L, a)}{\partial L}, \\ D_a(L, a, t) &= \lim_{S_t \uparrow Le^{a(T-t)}} \frac{\partial C_t^E(S_t, L, a)}{\partial a}. \end{aligned} \tag{6}$$

The formulae of $D_L(L, a, t)$ and $D_a(L, a, t)$ are also given in Proposition 1 of Appendix A. Let $H(L, a, t) = [D_L(L, a, t)]^2 + [D_a(L, a, t)]^2$ and denote by L_t^* and a_t^* the solutions to the equation

$$H(L, a, t) = 0. \tag{7}$$

It is worth emphasizing that Eq. (7) does not need to be solved recursively, i.e. Eq. (7) can be solved for L_t^* and a_t^* without having first solved for L_s^* and a_s^* for $s \in (t, T]$. We solve Eq. (7) using the same iterative procedure as the one for solving Eq. (5).

It should be noted that $L_t^* e^{a_t^*(T-t)}, t \in [0, T]$, provides a lower boundary for the optimal exercise boundary. Although a rigorous proof of this statement is difficult, the intuition behind it is straightforward.[13] The solution of Eq. (7), (L_t^*, a_t^*), can generate the exponential exercise barrier and thus the lower bound for American call prices through Eq. (2) when the asset price S_t approaches $L_t^* e^{a_t^*(T-t)}$ from below. Please note that Eq. (2) is the formula to calculate the option value of a capped call and its value equals $Le^{a(T-t)} - K$ when $S_t = Le^{a(T-t)}$. Therefore, the solution of Eq. (7) satisfies

$$L_t^* e^{a_t^*(T-t)} - K = C_t^E(L_t^* e^{a_t^*(T-t)}, L_t^*, a_t^*).$$

Since $C_t^E(S_t, L, a)$ generates the lower bounds of the American call prices, $L_t^* e^{a_t^*(T-t)}$ should be the lower-biased early exercise boundary for the American call. Fig. 1 illustrates the above inference and

[9] The results indicate that the first derivative of the early exercise boundary with respective to time is in between 0 (perpetual) and $-\infty$ (short maturity).

[10] If $Le^{a(T-t)} < S_t$, by definition the capped option with the exponential boundary has been exercised and its value equals $\max(Le^{a(T-\tau)} - K, 0)$. Thus the optimization is solved under the constraint that $Le^{a(T-t)} \ge S_t$.

[11] The initial guess is chosen as $L_0 = \max(K, rK/q)$.

[12] The number of iterations is typically within 10 and thus the computation is quick.

[13] The authors thank an anonymous referee for providing the reference of Ibáñez and Paraskevopoulos (forthcoming), which motivates us to derive the following explanation to show that $L_t^* e^{a_t^*(T-t)}$ is indeed a low-biased exercise boundary for the optimal exercise boundary B_t^*.

shows that $L_t^* e^{a_t^*(T-t)}$ is smaller than the optimal exercise boundary B_t^*. In addition, from the figure, we find that if a tighter lower bound for the American call price is derived, a tighter lower bound for the optimal exercise boundary can be obtained.

3. An upper bound for the American call option value

Under Black and Scholes (1973) model, Kim (1990), Jacka (1991), and Carr et al. (1992) derive the following formula for the price of an American call option:

$$C_t(S_t) = c_t(S_t) + \int_t^T \Big[qS_t e^{-q(s-t)} N(d_1(S_t, B_s^*, s-t)) - rKe^{-r(s-t)} N(d_2(S_t, B_s^*, s-t)) \Big] ds, \tag{8}$$

where

$$d_1(x,y,t) = \frac{\ln(x/y) + (r - q + \sigma^2/2)t}{\sigma\sqrt{t}},$$

$$d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}.$$

The second term in the right hand side of Eq. (8) is called the early exercise premium in the literature.[14] The critical exercise boundary solves the following integral equation for B_s^* for all $s \in [t,T]$:

$$B_s^* - K = c_s(B_s^*) + \int_s^T \Big[qB_s^* e^{-q(\tau-s)} N(d_1(B_s^*, B_\tau^*, \tau-s)) - rKe^{-r(\tau-s)} N(d_2(B_s^*, B_\tau^*, \tau-s)) \Big] d\tau. \tag{9}$$

Once B_t^* is obtained, the price of the American option can be calculated easily using Eq. (8). However, solving for B_t^* is usually a time-consuming process because it needs to be solved recursively, i.e. before solving for B_t^* one needs to first solve for B_s^* for $s \in (t,T]$.

Because the early exercise premium in the above formula is decreasing in the boundary, Carr et al. (1992) show that it is possible to bound the American call value analytically. For example, substituting $B_t^* = K$ into Eq. (8) yields an upper bound of the American call. Actually substituting any lower estimates of the critical exercise boundary into Eq. (8) will result in an upper bound of the early exercise premium and thus an upper of the American call price. As a result we can substitute our tight lower bound for optimal exercise boundary into the premium integral of Kim (1990), Jacka (1991), and Carr et al. (1992) to obtain a tight upper bound of the American call price. In other words, the American call option price is bounded above by the following formula:

$$C_t^u(S_t) = c_t(S_t) + \int_t^T \Big[qS_t e^{-q(s-t)} N\big(d_1(S_t, L_s^* e^{a_s^*(T-s)}, s-t)\big) - rKe^{-r(s-t)} N\big(d_2(S_t, L_s^* e^{a_s^*(T-s)}, s-t)\big) \Big] ds, \tag{10}$$

where $L_s^* e^{a_s^*(T-s)}$ is the lower bound on the optimal exercise boundary given by the solution to Eq. (7).

4. Numerical results and discussions

In this section we compare our lower and upper bounds with those of Broadie and Detemple (1996). The comparison is based on the speed of computation and the accuracy of the results. Besides the option pricing bounds, Broadie and Detemple (1996) also propose two approximations for the American option prices. The first approximation is based on the lower bound (LBA) and the second approximation is based on the lower and upper bounds (LUBA). Following Broadie and Detemple (1996), we also develop two approximations based on our lower and upper bounds. Details of our LBA and LUBA are given in Appendix B.

4.1. Comparing the tightness of bounds

We first compare the tightness of our lower and upper bounds with those of Broadie and Detemple (1996). The results reported include (1) the lower bound of Broadie and Detemple (1996) (LB1), (2) our lower bound (LB2), (3) the upper bound of Broadie and Detemple (1996) (UB1), (4) our upper bound (UB2), (5) the approximation based on Broadie and Detemple's lower bound (LBA1), (6) the approximation based on our lower bound (LBA2), (7) the approximation based on Broadie and Detemple's lower and upper bounds (LUBA1), and (8) the approximation based on our lower and upper bounds (LUBA2). The parameters used are adopted from Tables 1 and 2 of Broadie and Detemple (1996). There are 40 options considered in Tables 1 and 2. In this paper, we calculate "true" option values by the Binomial Black and Sholes model with Richardson extrapolation (BBSR) where the length of each time step is 0.0001 years.[15]

As expected, the results in Tables 1 and 2 suggest that our lower and upper bounds are tighter than Broadie and Detemple's (1996) bounds since their bounds are special cases of our bounds with $a = 0$. For short term options, Table 1 shows that the difference between our upper bound and lower bound is generally small. For example, the maximum difference between our upper bound and lower bound is only \$0.0134. The average difference is only \$0.0049 and the difference is smaller than 1 cent for 16 out 20 cases. In contrast, the average difference of Broadie and Detemple's (1996) bounds is about nine times (\$0.0425) of our average difference. Similarly, Table 2 also indicates that our bounds are far tighter than Broadie and Detemple's (1996) bounds for long term options. Even for long term options, the average difference of our bounds is only \$0.0174 which is smaller than the bid-ask spreads observed in the option market.

Based on the numerical results from 40 American call option contracts reported in Tables 1 and 2, we find that the average pricing errors of our lower bounds and upper bounds are 0.0957% and 0.0318%, respectively. In contrast, the average pricing errors of Broadie and Detemple's (1996) lower bound and upper bound are 0.5641% and 0.2549%, respectively. Thus our method can improve the errors of the lower bound and upper bound of Broadie and Detemple (1996) by 83.0% and 87.5%, respectively.

Tables 1 and 2 also show that the pricing error of our LBA2 (an approximation based on our lower bound) is smaller than that of LBA1 which is based on the lower bound of Broadie and Detemple (1996). The average pricing errors of our LBA2 based on 40 options in Tables 1 and 2 are 0.0170%, which improve a lot over LBA1 (0.1319%). However, while our bounds are far more accurate than the bounds of Broadie and Detemple (1996), the approximation based on our bounds (LUBA2) is not necessarily more accurate than their LUBA1. Nevertheless, note that the results in Tables 1 and 2 are illustrative because they are only based on 40 options. This

[14] In fact, according to Ibáñez (2008), American option prices in an incomplete market setting can be decomposed into three components. The first part is priced by arbitrage, the second part depends on a risk orthogonal to the first part, and third part is the early exercise premium.

[15] More specifically, we first employ the BBS method with the length of each time step equaling 0.0001 (0.0002) years to calculate the option values C_1 (C_2). Then apply the Richardson extrapolation $C = 2C_1 - C_2$ to deriving the approximate option value, which is the termed the BBSR estimate with the length of each time step equaling 0.0001 years in this paper.

Table 5
Deltas of American call options (maturity $T = 0.5$ years).

Option parameter	Asset price	LB1	LB2	UB1	UB2	LBA1	LBA2	LUBA1	LUBA2	True value
$r = 0.03$, $\sigma = 0.2$, $q = 0.07$	80	0.04872	0.04904	0.04916	0.04911	0.04896	0.04909	0.04909	0.04908	0.04912
	90	0.20634	0.20768	0.20826	0.20795	0.20727	0.20788	0.20785	0.20783	0.20795
	100	0.48111	0.48341	0.48487	0.48381	0.48299	0.48380	0.48363	0.48356	0.48372
	110	0.77259	0.77235	0.77479	0.77244	0.77444	0.77278	0.77305	0.77225	0.77221
	120	1.00000	0.99701	0.99094	0.99579	1.00000	0.99711	1.00000	0.99666	0.99535
$r = 0.03$, $\sigma = 0.4$, $q = 0.07$	80	0.23114	0.23215	0.23251	0.23233	0.23221	0.23232	0.23233	0.23225	0.23236
	90	0.37506	0.37668	0.37731	0.37694	0.37669	0.37692	0.37695	0.37682	0.37694
	100	0.52308	0.52509	0.52603	0.52540	0.52509	0.52537	0.52533	0.52523	0.52536
	110	0.65773	0.65966	0.66091	0.65995	0.65979	0.65990	0.65964	0.65975	0.65986
	120	0.77060	0.77176	0.77328	0.77195	0.77225	0.77188	0.77155	0.77176	0.77182
$r = 0.00$, $\sigma = 0.3$, $q = 0.07$	80	0.13237	0.13333	0.13372	0.13352	0.13323	0.13347	0.13346	0.13342	0.13354
	90	0.29119	0.29312	0.29405	0.29348	0.29297	0.29341	0.29343	0.29328	0.29347
	100	0.48924	0.49166	0.49330	0.49210	0.49186	0.49208	0.49189	0.49179	0.49200
	110	0.68752	0.68875	0.69110	0.68904	0.69028	0.68917	0.68873	0.68873	0.68884
	120	0.86105	0.85838	0.86135	0.85829	0.86269	0.85862	0.85875	0.85820	0.85798
$r = 0.07$, $\sigma = 0.3$, $q = 0.03$	80	0.19429	0.19429	0.19429	0.19429	0.19429	0.19429	0.19429	0.19429	0.19435
	90	0.37778	0.37778	0.37778	0.37778	0.37778	0.37778	0.37778	0.37778	0.37783
	100	0.57077	0.57077	0.57077	0.57077	0.57077	0.57077	0.57077	0.57077	0.57078
	110	0.73099	0.73099	0.73099	0.73099	0.73099	0.73099	0.73099	0.73099	0.73098
	120	0.84265	0.84265	0.84265	0.84265	0.84265	0.84265	0.84265	0.84265	0.84263
RMS		0.4715%	0.0836%	0.2091%	0.0203%	0.2216%	0.0522%	0.1125%	0.0496%	

All options have $K = 100$. LB1 and UB1 (LB2 and UB2) are deltas calculated from Broadie and Detemple's (our) lower and upper bounds. LBA1 and LUBA1 (LBA2 and LUBA2) are deltas based on the approximations of Broadie and Detemple's (our) bounds. The "true value" column is computed from the extended tree method described in Pelsser and Vorst (1994) using the Binomial Black and Scholes method where the length of each time step is 0.0001 years.

small sample may not fully reflect the superiority of the approximations of LUBA2 over the approximations based on LUBA1.[16]

Besides comparing with Broadie and Detemple (1996), it is also important to contrast our option pricing bounds and two approximations with the best accurate/efficient methods available in the literature for the American options in the Black and Scholes (1973) setting. Since our upper bounds are based on the decomposition approach of Kim (1990), Jacka (1991), and Carr et al. (1992), it may be appropriate to compare our results with other papers which have successfully implemented the decomposition formula. To the best of our knowledge, Ju (1998) and Ibáñez (2003) represent the best speed-accuracy trade-offs using the decomposition approach.

We first compare the errors of our option pricing bounds and two approximation formulae with those of Ju (1998) for pricing long term American put options.[17] The parameters used in Table 3 are adopted from Table 2 of Ju (1998). The "true" American put option values are also computed using the BBSR method with the length of each time step equaling 0.0001 years.[18] The results indicate that our option pricing bounds are generally tight and close to the "true" American put values, especially when $q > r$. The average difference between our upper bound and lower bound is only \$0.01 and the maximum difference is smaller than 2 cents. Moreover, the errors of the proposed method are of similar magnitudes to those of Ju (1998). The RMS relative error of Ju (1998) is 0.032% while the errors of our LB2, UB2, LBA2, and LUBA2 are 0.114%, 0.049%, 0.025%, and 0.004%, respectively. Our LUBA2 is exceptionally accurate for pricing American put options.

We also compare the accuracy of the proposed method with that of Ibáñez (2003) for pricing short term American put options. The parameters applied in Table 4 are adopted from Table 1 of Ibáñez (2003). Again the results suggest that our option pricing bounds and two approximation formulae are accurate for pricing short term American puts. For instance, the RMS relative error of Ibáñez (2003) is 0.024% while the errors of our LB2, UB2, LBA2, and LUBA2 are 0.062%, 0.014%, 0.034%, and 0.005%, respectively.

In summary, the numerical results show that the accuracy of our option pricing bounds, especially the upper bound, is comparable to that of Ju (1998) and Ibáñez (2003), two best accurate/efficient methods for American options in the literature. Moreover, our LBA2 and LUBA2 are generally more accurate than Ju (1998) and Ibáñez (2003).[19]

4.2. Comparing the accuracy of hedge ratios based on bounds

One possible application of our pricing bounds in Eqs. (4) and (10) is to use them to calculate the hedge ratios for American options. For example, deltas of our lower bound and upper bound can be respectively defined as:[20]

[16] In Section 4.3, a similar method as that in Broadie and Detemple (1996) is adopted to compare the accuracy and efficiency of the approximations of LBA1, LBA2, LUBA1, and LUBA2. In the sample of about 2500 options, the LUBA2 based on the exponential exercise barrier on average generates more accurate approximations than the LUBA1 based on the constant exercise barrier.

[17] For brevity, our option pricing bounds and two approximation formulae for American put options are not shown here. The detailed formulae and implementations of our method are available upon request from the authors.

[18] Note that the "true" American option values in Ju (1998) are based on the binomial model with 10,000 time steps. Since the considered American puts are long term ($T = 3$ years) options, a binomial model with 10,000 time steps is not accurate enough as the benchmark values. Thus, we recalculate the "true" American option values using the BBSR method with 10,000 time steps per year.

[19] Note that the computational time of Ibáñez (2003) is close to that of a six-point recursive scheme of Huang et al. (1996) because both methods involve the calculations of Bermudan option prices with 4, 5, and 6 exercise dates. Moreover, the computational time of our LUBA2 is also close to that of Broadie and Detemple's (1996) LUBA1 (see Fig. 2 of this study). According to Table 3 of Ju (1998), the computational time is of the same magnitude for the methods of Huang et al. (1996), Broadie and Detemple (1996), and Ju (1998) (see columns 5, 6, and 12 of Ju's Table 3). Thus we would expect that the computational time of the proposed method is similar to Ju (1998) and Ibáñez (2003).

[20] Since $\widehat{L}$ and $\hat{a}$ are functions of S_t, it is impossible to derive analytical solutions of hedge ratios for the lower bound. However, the numerical derivatives of our lower bound are accurate and easy to compute.

Table 6
Deltas of American call options (maturity $T = 3$ years).

Option parameter	Asset price	LB1	LB2	UB1	UB2	LBA1	LBA2	LUBA1	LUBA2	True value
$r = 0.03$, $\sigma = 0.2$, $q = 0.07$	80	0.19858	0.20000	0.20111	0.20044	0.19991	0.20028	0.20040	0.20024	0.20035
	90	0.31900	0.32062	0.32233	0.32117	0.32070	0.32100	0.32093	0.32086	0.32100
	100	0.46008	0.46132	0.46358	0.46183	0.46158	0.46170	0.46139	0.46144	0.46157
	110	0.61583	0.61584	0.61857	0.61619	0.61618	0.61606	0.61633	0.61576	0.61580
	120	0.78237	0.78022	0.78331	0.78033	0.78087	0.78001	0.77756	0.77993	0.77976
$r = 0.03$, $\sigma = 0.4$, $q = 0.07$	80	0.40187	0.40398	0.40555	0.40452	0.40397	0.40441	0.40451	0.40429	0.40442
	90	0.47163	0.47373	0.47554	0.47428	0.47348	0.47409	0.47403	0.47399	0.47415
	100	0.53712	0.53903	0.54105	0.53956	0.53840	0.53926	0.53901	0.53923	0.53939
	110	0.59825	0.59983	0.60203	0.60032	0.59873	0.59987	0.59964	0.59994	0.60011
	120	0.65524	0.65636	0.65870	0.65677	0.65478	0.65614	0.65636	0.65637	0.65653
$r = 0.00$, $\sigma = 0.3$, $q = 0.07$	80	0.28319	0.28491	0.28656	0.28547	0.28523	0.28531	0.28542	0.28516	0.28534
	90	0.37840	0.38009	0.38218	0.38068	0.38050	0.38050	0.38028	0.38028	0.38048
	100	0.47862	0.47992	0.48242	0.48049	0.48029	0.48026	0.47997	0.48001	0.48021
	110	0.58163	0.58219	0.58502	0.58265	0.58244	0.58233	0.58267	0.58213	0.58229
	120	0.68605	0.68551	0.68861	0.68581	0.68593	0.68531	0.68607	0.68530	0.68536
$r = 0.07$, $\sigma = 0.3$, $q = 0.03$	80	0.48025	0.48028	0.48029	0.48029	0.48096	0.48028	0.48029	0.48028	0.48030
	90	0.56219	0.56225	0.56226	0.56225	0.56287	0.56225	0.56226	0.56225	0.56226
	100	0.63163	0.63171	0.63173	0.63172	0.63220	0.63171	0.63173	0.63171	0.63172
	110	0.68934	0.68946	0.68948	0.68946	0.68972	0.68946	0.68948	0.68946	0.68946
	120	0.73678	0.73693	0.73696	0.73693	0.73688	0.73693	0.73696	0.73693	0.73694
RMS		0.4109%	0.0765%	0.3480%	0.0438%	0.1262%	0.0235%	0.0786%	0.0322%	

All options have $K = 100$. LB1 and UB1 (LB2 and UB2) are deltas calculated from Broadie and Detemple's (our) lower and upper bounds. LBA1 and LUBA1 (LBA2 and LUBA2) are deltas based on the approximations of Broadie and Detemple's (our) bounds. The "true value" column is computed from the extended tree method described in Pelsser and Vorst (1994) using the Binomial Black and Scholes method where the length of each time step is 0.0001 years.

Table 7
Gammas of American call options (maturity $T = 0.5$ years).

Option parameter	Asset price	LB1	LB2	UB1	UB2	LBA1	LBA2	LUBA1	LUBA2	True value
$r = 0.03$, $\sigma = 0.2$, $q = 0.07$	80	0.00894	0.00900	0.00903	0.00902	0.00899	0.00901	0.00901	0.00901	0.00902
	90	0.02273	0.02287	0.02294	0.02289	0.02283	0.02289	0.02289	0.02288	0.02289
	100	0.03022	0.03022	0.03032	0.03021	0.03030	0.03023	0.03018	0.03020	0.03020
	110	0.02676	0.02619	0.02628	0.02614	0.02666	0.02618	0.02643	0.02617	0.02612
	120	0.00000	0.01898	0.00681	0.01269	0.00000	0.01890	0.00000	0.01782	0.01593
$r = 0.03$, $\sigma = 0.4$, $q = 0.07$	80	0.01343	0.01349	0.01352	0.01350	0.01349	0.01350	0.01350	0.01350	0.01350
	90	0.01496	0.01501	0.01504	0.01502	0.01501	0.01502	0.01502	0.01501	0.01502
	100	0.01435	0.01437	0.01440	0.01437	0.01437	0.01437	0.01436	0.01437	0.01437
	110	0.01244	0.01240	0.01243	0.01240	0.01243	0.01240	0.01237	0.01240	0.01239
	120	0.01012	0.01000	0.01003	0.00999	0.01005	0.00999	0.01001	0.00999	0.00999
$r = 0.00$, $\sigma = 0.3$, $q = 0.07$	80	0.01275	0.01283	0.01288	0.01285	0.01282	0.01285	0.01285	0.01284	0.01285
	90	0.01851	0.01860	0.01867	0.01862	0.01861	0.01862	0.01862	0.01860	0.01861
	100	0.02040	0.02039	0.02046	0.02039	0.02046	0.02040	0.02035	0.02038	0.02038
	110	0.01883	0.01859	0.01866	0.01856	0.01879	0.01858	0.01865	0.01857	0.01855
	120	0.01580	0.01525	0.01531	0.01521	0.01561	0.01522	0.01502	0.01524	0.01520
$r = 0.07$, $\sigma = 0.3$, $q = 0.03$	80	0.01612	0.01612	0.01612	0.01612	0.01612	0.01612	0.01612	0.01612	0.01612
	90	0.01970	0.01970	0.01970	0.01970	0.01970	0.01970	0.01970	0.01970	0.01970
	100	0.01816	0.01816	0.01816	0.01816	0.01816	0.01816	0.01816	0.01816	0.01816
	110	0.01364	0.01364	0.01364	0.01364	0.01364	0.01364	0.01364	0.01364	0.01364
	120	0.00880	0.00880	0.00880	0.00880	0.00880	0.00880	0.00880	0.00880	0.00880
RMS		22.3925%	4.2845%	12.7993%	4.5417%	22.3768%	4.1744%	22.3642%	2.6502%	
RMS exclude the 5th option		1.2248%	0.1343%	0.3397%	0.0339%	0.8714%	0.0843%	0.4094%	0.0893%	

All options have $K = 100$. LB1 and UB1 (LB2 and UB2) are gammas calculated from Broadie and Detemple's (our) lower and upper bounds. LBA1 and LUBA1 (LBA2 and LUBA2) are gammas based on the approximations of Broadie and Detemple's (our) bounds. The "true value" column is computed from the extended tree method described in Pelsser and Vorst (1994) using the Binomial Black and Scholes method where the length of each time step is 0.0001 years.

$$
\begin{aligned}
\Delta_t^l(S_t) &= \lim_{\Delta S \to 0}\left(C_t^E(S_t+\Delta S,\widehat{L}(S_t+\Delta S),\hat{a}(S_t+\Delta S)) - C_t^E\left(S_t,\widehat{L}(S_t),\hat{a}(S_t)\right)\right)/\Delta S,\\
\Delta_t^u(S_t) &= e^{-q(T-t)}N(d_1(S_t,K,T-t))\\
&\quad + \int_t^T \Big[qe^{-q(s-t)}N(d_1(S_t,L_s^* e^{a_s^*(T-s)},s-t))\\
&\quad + \Big(qS_t e^{-q(s-t)}n(d_1(S_t,L_s^* e^{a_s^*(T-s)},s-t))\\
&\quad - rKe^{-r(s-t)}n(d_2(S_t,L_s^* e^{a_s^*(T-s)},s-t))\Big)/(S_t\sigma\sqrt{T-s})\Big]ds,
\end{aligned}
\tag{11}
$$

where $n(.)$ is the probability density function of a standard normal variable.

We can also obtain the hedge ratios of Broadie and Detemple's bounds in the same way. For comparison, we also calculate hedge ratios of LBA1, LBA2, LUBA1, and LUBA2. We apply the extended tree method described in Pelsser and Vorst (1994) to compute benchmark values of Δ and Γ using the Binomial Black and Scholes method with the length of each time step equaling 0.0001 years.

Tables 5 and 6 present the deltas of American call options considered in Tables 1 and 2, respectively. The results suggest that the deltas based on bounds and the related approximations are generally accurate. The RMS relative errors range from 0.0203% (UB2) to 0.4715% (LB1) for short term options (maturity $T = 0.5$ years) and from 0.0235% (LBA2) to 0.4109% (LB1) for long term options (maturity $T = 3$ years). Moreover, we also find that the deltas of our bounds can enhance the accuracy of the deltas of Broadie and Detemple's (1996) lower and upper bounds by 81.9% and 88.1%, respectively.

Table 8
Gammas of American call options (maturity $T = 3$ years).

Option parameter	Asset price	LB1	LB2	UB1	UB2	LBA1	LBA2	LUBA1	LUBA2	True value
$r = 0.03$, $\sigma = 0.2$, $q = 0.07$	80	0.01078	0.01082	0.01087	0.01083	0.01083	0.01083	0.01083	0.01082	0.01083
	90	0.01319	0.01318	0.01324	0.01319	0.01321	0.01319	0.01315	0.01318	0.01318
	100	0.01493	0.01485	0.01490	0.01484	0.01486	0.01484	0.01486	0.01483	0.01483
	110	0.01616	0.01599	0.01603	0.01597	0.01599	0.01596	0.01597	0.01597	0.01595
	120	0.01712	0.01686	0.01689	0.01683	0.01695	0.01680	0.01627	0.01684	0.01681
$r = 0.03$, $\sigma = 0.4$, $q = 0.07$	80	0.00718	0.00718	0.00721	0.00719	0.00717	0.00718	0.00717	0.00718	0.00718
	90	0.00677	0.00676	0.00678	0.00676	0.00673	0.00675	0.00673	0.00675	0.00675
	100	0.00633	0.00630	0.00632	0.00630	0.00626	0.00629	0.00627	0.00630	0.00630
	110	0.00590	0.00586	0.00588	0.00586	0.00581	0.00584	0.00586	0.00585	0.00585
	120	0.00550	0.00545	0.00546	0.00544	0.00541	0.00542	0.00549	0.00544	0.00544
$r = 0.00$, $\sigma = 0.3$, $q = 0.07$	80	0.00917	0.00918	0.00923	0.00919	0.00920	0.00919	0.00917	0.00918	0.00919
	90	0.00982	0.00980	0.00984	0.00980	0.00980	0.00979	0.00976	0.00979	0.00979
	100	0.01019	0.01013	0.01017	0.01013	0.01012	0.01012	0.01015	0.01012	0.01012
	110	0.01039	0.01030	0.01033	0.01028	0.01029	0.01027	0.01035	0.01028	0.01027
	120	0.01048	0.01036	0.01038	0.01034	0.01041	0.01031	0.01028	0.01034	0.01033
$r = 0.07$, $\sigma = 0.3$, $q = 0.03$	80	0.00881	0.00881	0.00882	0.00882	0.00881	0.00881	0.00882	0.00881	0.00881
	90	0.00757	0.00757	0.00757	0.00757	0.00756	0.00757	0.00757	0.00757	0.00757
	100	0.00634	0.00634	0.00634	0.00634	0.00632	0.00634	0.00634	0.00634	0.00634
	110	0.00523	0.00523	0.00523	0.00523	0.00521	0.00523	0.00523	0.00523	0.00523
	120	0.00429	0.00429	0.00429	0.00429	0.00425	0.00429	0.00429	0.00429	0.00429
RMS		0.7805%	0.1398%	0.4017%	0.0633%	0.4460%	0.1098%	0.8058%	0.0559%	

All options have $K = 100$. LB1 and UB1 (LB2 and UB2) are gammas calculated from Broadie and Detemple's (our) lower and upper bounds. LBA1 and LUBA1 (LBA2 and LUBA2) are gammas based on the approximations of Broadie and Detemple's (our) bounds. The "true value" column is computed from the extended tree method described in Pelsser and Vorst (1994) using the Binomial Black and Scholes method where the length of each time step is 0.0001 years.

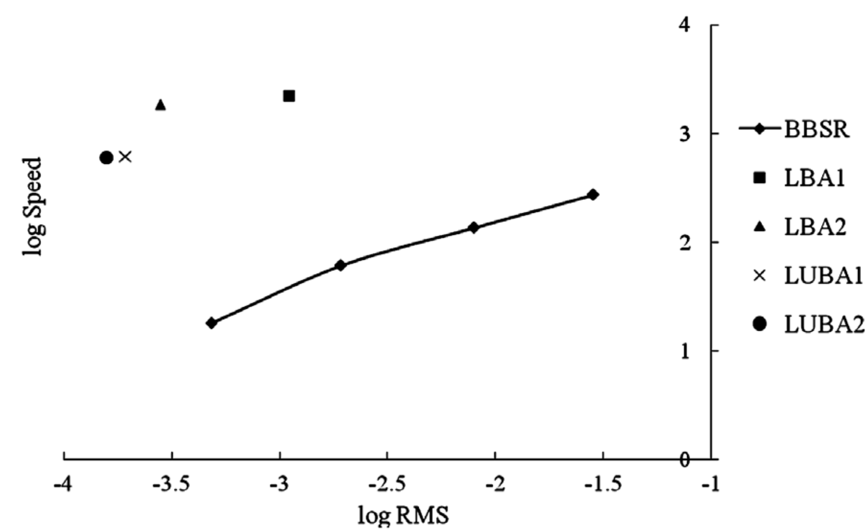

Fig. 2. Speed-accuracy trade-off for all observations with option price $\geqslant 0.5$. RMS relative error is defined as RMS $= \sqrt{\frac{1}{m}\sum_{i=1}^{m}(C_i - \widetilde{C}_i)^2/C_i^2}$, where C_i is the "true" option price (estimated by the BBSR method with the length of each time step to be 0.0001 years), $\widetilde{C}_i$ is the estimated option price. Speed is measured in option prices calculated per second on a Pentium 4 3.4 GHz PC. The BBSR method is based on the length of each time step to be 0.1, 0.05, 0.025, and 0.01 years. Preferred methods are in the upper-left corner.

Tables 7 and 8 illustrate the estimations of gammas based on bounds and their related approximations. The accuracy of gamma estimates is similar to that of price and delta estimates except for the fifth option, a deep in-the-money option, in Table 7. When the fifth option in Table 7 is excluded, the RMS relative errors of gamma estimates based on our LB2, UB2, LBA2, and LUBA2, are 0.1372%, 0.0512%, 0.0982%, and 0.0741%, respectively. Overall, the results in Tables 5–8 demonstrate that the proposed method can generate very accurate delta and gamma estimates.

4.3. Comparing the numerical efficiency of pricing American options

To make a comprehensive analysis of numerical efficiency, we compare our bounds and approximations with those of Broadie and Detemple (1996). Following Broadie and Detemple (1996), the comparison is on the basis of the computational speed and the accuracy of the results over a wide range of option parameters.

We apply the same methodology of Broadie and Detemple (1996) to choose 2500 options to test the results. Volatility (σ) is distributed uniformly between 0.1 and 0.6. Time to maturity is, with probability 0.75, uniform between 0.1 and 1.0 years, and, with probability 0.25, uniform between 1.0 and 5.0 years. The strike price (K) is fixed at 100. The dividend yield (q) is uniform between 0.0 and 0.1. The risk-free rate (r) is, with probability 0.8, uniform between 0.0 and 0.1, and, with probability 0.2, equal to 0.0. Each parameter is drawn independently of the others. Following Broadie and Detemple (1996), 500 sets of σ, q, r, and T are generated, and for each parameter set, five initial stock prices (S_0) are examined from the uniform distribution between 70 and 130.

The accuracy measure used in this paper is the root mean squared (RMS) relative error. RMS relative error is defined as

$$\mathrm{RMS} = \sqrt{\frac{1}{m}\sum_{i=1}^{m} e_i^2},$$

where $e_i = (C_i - \widetilde{C}_i)/C_i$ is the relative error, C_i is the "true" option price (estimated by the BBSR method with the length of each time step equaling 0.0001 years), $\widetilde{C}_i$ is the estimated option price. Same as in Broadie and Detemple (1996), the summation is taken over options in the data set satisfying $C_i \geq 0.5$. Out of the 2500 options, 2285 satisfy this criterion.

Fig. 2 indicates that the approximations based on our bounds and Broadie and Detemple's (1996) bounds are more efficient than the BBSR method. Our LBA2 and LUBA2 can improve the accuracy of Broadie and Detemple's (1996) LBA1 and LUBA1 by 74.7% and 9.2%, respectively.[21] Our LUBA2 is the most accurate one in comparison to the other approximations although it takes more computational time.

5. Conclusion

The optimal exercise boundary is vital for pricing American options. Thus a better approximation of the early exercise boundary yields a better lower bound for the American option price. In contrast to the constant exercise barrier assumed by Broadie and Detemple (1996), this paper uses an exponential function to approximate the early exercise boundary and obtains tight lower bounds for both the American option value and the optimal exercise boundary. Moreover, the tight lower bound of the optimal exercise boundary allows us to derive a tight upper bound of the American option price using the premium integral of Kim (1990), Jacka (1991), and Carr et al. (1992).

The numerical results can be summarized as follows: first of all, the American option prices are bounded tightly between our lower and upper bounds. The average difference between our upper bound and lower bound is only 0.49 cents and the maximum difference is just 1.34 cents for short term options. Secondly, our bounds can improve the pricing errors of the lower bound and upper bound of Broadie and Detemple (1996) by 83.0% and 87.5%, respectively. Moreover, the approximations (LBA2 and LUBA2) based on our bounds are also more accurate than the approximations (LBA1 and LUBA1) based on their bounds. Thirdly, the accuracy of our upper bounds for pricing American is analogous to that of the best accurate/efficient methods in the literature, e.g. Ju (1998) and Ibáñez (2003). Moreover, our LBA2 and LUBA2 are generally more accurate than Ju (1998) and Ibáñez (2003) for pricing American put options. Finally, our pricing bounds and approximations also provide accurate hedge ratios except for deep in-the-money options. The approximation errors of our pricing bounds and approximations for estimating deltas and gammas range from 0.0235% to 0.1398%. The small approximation errors show the superiority of our method to estimate deltas and gammas for American options.

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Appendix A

Proposition 1. *Suppose* $Le^{a(T-t)} \geq \max(S_t, K)$. *Then* $\partial C_t^E(S_t, L, a)/\partial a$ *can be written as*

$$\begin{aligned}
&\partial C_t^E(S_t, L, a)/\partial a \\
&= -Ke^{-r(T-t)}\Bigg\{\left[\frac{-2(T-t)n(\hat{d}^+(K))}{\sqrt{(T-t)}\sigma} + \frac{2(T-t)n(\hat{d}_1^+(Le^{a(T-t)}))}{\sqrt{(T-t)}\sigma}\right]\hat{\lambda}_t^{\hat{\gamma}+1} + \left[N(\hat{d}^+(K)) - N(\hat{d}_1^+(Le^{a(T-t)}))\right]\left[-(\hat{\gamma}+1)(T-t) - \frac{2\ln(\hat{\lambda}_t)}{\sigma^2}\right]\hat{\lambda}_t^{\hat{\gamma}+1}\Bigg\} \\
&\quad - K\Bigg\{\left[\frac{1}{\sqrt{(T-t)}\sigma}n(\hat{d}_0)(T-t)\left(\frac{\hat{b}}{\hat{\beta}} - 1\right)\hat{\lambda}_t^{\hat{b}-\hat{\beta}}\right] + N(\hat{d}_0)\left[-(T-t)(\hat{b}-\hat{\beta}) + \frac{1}{\sigma^2}\left(\frac{\hat{b}}{\hat{\beta}} - 1\right)\ln(\hat{\lambda}_t)\right]\hat{\lambda}_t^{\hat{b}-\hat{\beta}} \\
&\quad + \left[n\left(\hat{d}_0 + 2\hat{\beta}\sigma\sqrt{T-t}\right)\frac{-\sqrt{(T-t)}}{\sigma}\left(\frac{\hat{b}}{\hat{\beta}} + 1\right)\hat{\lambda}_t^{\hat{b}+\hat{\beta}}\right] + N\left(\hat{d}_0 + 2\hat{\beta}\sigma\sqrt{T-t}\right)\left[-(T-t)(\hat{b}+\hat{\beta}) - \frac{1}{\sigma^2}\left(\frac{\hat{b}}{\hat{\beta}} + 1\right)\ln\left(\hat{\lambda}_t\right)\right]\hat{\lambda}_t^{\hat{b}+\hat{\beta}}\Bigg\} \\
&\quad + Le^{a(T-t)}(T-t)\left[N(\tilde{d}_0)\hat{\lambda}_t^{\tilde{b}-\tilde{\beta}} + N(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t})\hat{\lambda}_t^{\tilde{b}+\tilde{\beta}}\right] \\
&\quad + Le^{a(T-t)}\Bigg\{\left[\frac{-1}{\sqrt{(T-t)}\sigma}n(\tilde{d}_0)(T-t)\left(1 + \frac{1-\tilde{b}}{\tilde{\beta}}\right)\hat{\lambda}_t^{\tilde{b}-\tilde{\beta}}\right] + N(\tilde{d}_0)\left[-(T-t)(\tilde{b}-\tilde{\beta}) - \frac{1}{\sigma^2}\left(1 + \frac{1-\tilde{b}}{\tilde{\beta}}\right)\ln(\hat{\lambda}_t)\right]\hat{\lambda}_t^{\tilde{b}-\tilde{\beta}} \\
&\quad + \left[n(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t})\frac{-\sqrt{(T-t)}}{\sigma}\left(1 - \frac{1-\tilde{b}}{\tilde{\beta}}\right)\hat{\lambda}_t^{\tilde{b}+\tilde{\beta}}\right] + N(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t})\left[-(T-t)(\tilde{b}+\tilde{\beta}) - \frac{1}{\sigma^2}\left(1 - \frac{1-\tilde{b}}{\tilde{\beta}}\right)\ln(\hat{\lambda}_t)\right]\hat{\lambda}_t^{\tilde{b}+\tilde{\beta}}\Bigg\} \\
&\quad + S_t e^{-q(T-t)}\Bigg\{\left[\frac{-2(T-t)n(\hat{d}^+(K) - \sigma\sqrt{T-t})}{\sqrt{(T-t)}\sigma} + \frac{2(T-t)n\left(\hat{d}_1^+(Le^{a(T-t)}) - \sigma\sqrt{T-t}\right)}{\sqrt{(T-t)}\sigma}\right]\hat{\lambda}_t^{\hat{\gamma}-1}. \\
&\quad + \left[N(\hat{d}^+(K) - \sigma\sqrt{T-t}) - N\left(\hat{d}_1^+(Le^{a(T-t)}) - \sigma\sqrt{T-t}\right)\right]\left[-(\hat{\gamma}-1)(T-t) - \frac{2\ln(\hat{\lambda}_t)}{\sigma^2}\right]\hat{\lambda}_t^{\hat{\gamma}-1}\Bigg\},
\end{aligned}$$

[21] Moreover, our results (not reported here) show that LB2 and UB2 can improve the accuracy of Broadie and Detemple's 1996) LB1 and UB1 by 82.4% and 78.3%, respectively.

$\partial C_t^E(S_t, L, a)/\partial L$ can be written as

$$\begin{aligned}
&\partial C_t^E(S_t, L, a)/\partial L \\
&= e^{a(T-t)}\left[N(\tilde{d}_0)\hat{\lambda}_t^{\tilde{b}-\tilde{\beta}} + N\left(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t}\right)\hat{\lambda}_t^{\tilde{b}+\tilde{\beta}}\right] \\
&\quad - Ke^{-r(T-t)}\left\{\frac{n(\hat{d}_1^-(Le^{a(T-t)}))}{L\sqrt{(T-t)}\sigma} + \left[-\frac{2n(\hat{d}^+(K))}{L\sqrt{(T-t)}\sigma} + \frac{n(\hat{d}_1^+(Le^{a(T-t)}))}{L\sqrt{(T-t)}\sigma}\right]\hat{\lambda}_t^{\hat{\gamma}+1} + \left(-\frac{\hat{\gamma}+1}{L}\right)\left[N(\hat{d}^+(K)) - N(\hat{d}_1^+(Le^{a(T-t)}))\right]\hat{\lambda}_t^{\hat{\gamma}+1}\right\} \\
&\quad - K\left\{-\frac{1}{L\sqrt{(T-t)}\sigma}\left[n(\hat{d}_0)\hat{\lambda}_t^{\hat{b}-\hat{\beta}} + n(\hat{d}_0 + 2\hat{\beta}\sigma\sqrt{T-t})\hat{\lambda}_t^{\hat{b}+\hat{\beta}}\right] + \left(-\frac{\hat{b}-\hat{\beta}}{L}\right)N(\hat{d}_0)\hat{\lambda}_t^{\hat{b}-\hat{\beta}} + \left(-\frac{\hat{b}+\hat{\beta}}{L}\right)N\left(\hat{d}_0 + 2\hat{\beta}\sigma\sqrt{T-t}\right)\hat{\lambda}_t^{\hat{b}+\hat{\beta}}\right\} \\
&\quad + Le^{a(T-t)}\left\{\frac{-1}{L\sqrt{(T-t)}\sigma}\left[n(\tilde{d}_0)\hat{\lambda}_t^{\tilde{b}-\tilde{\beta}} + n\left(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t}\right)\hat{\lambda}_t^{\tilde{b}+\tilde{\beta}}\right] + \left(-\frac{\tilde{b}-\tilde{\beta}}{L}\right)N(\tilde{d}_0)\hat{\lambda}_t^{\tilde{b}-\tilde{\beta}} + \left(-\frac{\tilde{b}+\tilde{\beta}}{L}\right)N\left(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t}\right)\hat{\lambda}_t^{\tilde{b}+\tilde{\beta}}\right\} \\
&\quad + S_t e^{-q(T-t)}\left\{\frac{n(\hat{d}_1^-(Le^{a(T-t)}) - \sigma\sqrt{T-t})}{L\sqrt{(T-t)}\sigma} + \left[-\frac{2n(\hat{d}^+(K) - \sigma\sqrt{T-t})}{L\sqrt{(T-t)}\sigma} + \frac{n\left(\hat{d}_1^+(Le^{a(T-t)}) - \sigma\sqrt{T-t}\right)}{L\sqrt{(T-t)}\sigma}\right]\hat{\lambda}_t^{\hat{\gamma}-1} + \left(-\frac{\hat{\gamma}-1}{L}\right)\left[N\left(\hat{d}^+(K) - \sigma\sqrt{T-t}\right).\right.\right. \\
&\quad \left.\left. - N\left(\hat{d}_1^+(Le^{a(T-t)}) - \sigma\sqrt{T-t}\right)\right]\hat{\lambda}_t^{\hat{\gamma}-1}\right\},
\end{aligned}$$

$D_a(L, a, t)$ can be written as

$$\begin{aligned}
&D_a(L, a, t) \\
&= \lim_{S_t \uparrow Le^{a(T-t)}} \frac{\partial C_t^E(S_t, L, a)}{\partial a} = -Ke^{-r(T-t)}\left\{\left[\frac{-2(T-t)n(\hat{d}^+(K))}{\sqrt{(T-t)}\sigma} + \frac{2(T-t)n(\hat{d}_1^+(Le^{a(T-t)}))}{\sqrt{(T-t)}\sigma}\right] + \left[N(\hat{d}^+(K)) - N(\hat{d}_1^+(Le^{a(T-t)}))\right][-(\hat{\gamma}+1)(T-t)]\right\} \\
&\quad - K\left\{\left[\frac{1}{\sqrt{(T-t)}\sigma}n(\hat{d}_0)(T-t)\left(\frac{\hat{b}}{\hat{\beta}} - 1\right)\right]\right. \\
&\quad \left. + N(\hat{d}_0)\left[-(T-t)(\hat{b}-\hat{\beta})\right] + \left[n\left(\hat{d}_0 + 2\hat{\beta}\sigma\sqrt{T-t}\right)\frac{-\sqrt{(T-t)}}{\sigma}\left(\frac{\hat{b}}{\hat{\beta}} + 1\right)\right] + N\left(\hat{d}_0 + 2\hat{\beta}\sigma\sqrt{T-t}\right)\left[-(T-t)(\hat{b}+\hat{\beta})\right]\right\} \\
&\quad + Le^{a(T-t)}(T-t)\left[N(\tilde{d}_0) + N\left(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t}\right)\right] \\
&\quad + Le^{a(T-t)}\left\{\left[\frac{-1}{\sqrt{(T-t)}\sigma}n(\tilde{d}_0)(T-t)\left(1 + \frac{1-\tilde{b}}{\tilde{\beta}}\right)\right]\right. \\
&\quad \left. + N(\tilde{d}_0)\left[-(T-t)(\tilde{b}-\tilde{\beta})\right] + \left[n\left(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t}\right)\frac{-\sqrt{(T-t)}}{\sigma}\left(1 - \frac{1-\tilde{b}}{\tilde{\beta}}\right)\right] + N\left(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t}\right)\left[-(T-t)(\tilde{b}+\tilde{\beta})\right]\right\} \\
&\quad + Le^{a(T-t)}e^{-q(T-t)}\left\{\left[\frac{-2(T-t)n(\hat{d}^+(K) - \sigma\sqrt{T-t})}{\sqrt{(T-t)}\sigma} + \frac{2(T-t)n\left(\hat{d}_1^+(Le^{a(T-t)}) - \sigma\sqrt{T-t}\right)}{\sqrt{(T-t)}\sigma}\right].\right. \\
&\quad \left. + \left[N\left(\hat{d}^+(K) - \sigma\sqrt{T-t}\right) - N\left(\hat{d}_1^+(Le^{a(T-t)}) - \sigma\sqrt{T-t}\right)\right][-(\hat{\gamma}-1)(T-t)]\right\},
\end{aligned}$$

and $D_L(L, a, t)$ can be written as

$$\begin{aligned}
&D_L(L, a, t) \\
&= \lim_{S_t \uparrow Le^{a(T-t)}} \frac{\partial C_t^E(S_t, L, a)}{\partial L} = e^{a(T-t)}\left[N(\tilde{d}_0) + N\left(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t}\right)\right] \\
&\quad - Ke^{-r(T-t)}\left\{\frac{n(\hat{d}_1^-(Le^{a(T-t)}))}{L\sqrt{(T-t)}\sigma} + \left[-\frac{2n(\hat{d}^+(K))}{L\sqrt{(T-t)}\sigma} + \frac{n(\hat{d}_1^+(Le^{a(T-t)}))}{L\sqrt{(T-t)}\sigma}\right] + \left(-\frac{\hat{\gamma}+1}{L}\right)\left[N(\hat{d}^+(K)) - N(\hat{d}_1^+(Le^{a(T-t)}))\right]\right\} \\
&\quad - K\left\{-\frac{1}{L\sqrt{(T-t)}\sigma}\left[n(\hat{d}_0) + n(\hat{d}_0 + 2\hat{\beta}\sigma\sqrt{T-t})\right] + \left(-\frac{\hat{b}-\hat{\beta}}{L}\right)N(\hat{d}_0) + \left(-\frac{\hat{b}+\hat{\beta}}{L}\right)N(\hat{d}_0 + 2\hat{\beta}\sigma\sqrt{T-t})\right\} \\
&\quad + Le^{a(T-t)}\left\{\frac{-1}{L\sqrt{(T-t)}\sigma}\left[n(\tilde{d}_0) + n\left(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t}\right)\right] + \left(-\frac{\tilde{b}-\tilde{\beta}}{L}\right)N(\tilde{d}_0) + \left(-\frac{\tilde{b}+\tilde{\beta}}{L}\right)N\left(\tilde{d}_0 + 2\tilde{\beta}\sigma\sqrt{T-t}\right)\right\} \\
&\quad + Le^{a(T-t)}e^{-q(T-t)}\left\{\frac{n(\hat{d}_1^-(Le^{a(T-t)}) - \sigma\sqrt{T-t})}{L\sqrt{(T-t)}\sigma} + \left[-\frac{2n\left(\hat{d}^+(K) - \sigma\sqrt{T-t}\right)}{L\sqrt{(T-t)}\sigma} + \frac{n\left(\hat{d}_1^+(Le^{a(T-t)}) - \sigma\sqrt{T-t}\right)}{L\sqrt{(T-t)}\sigma}\right].\right. \\
&\quad \left. + \left(-\frac{\hat{\gamma}-1}{L}\right)\left[N\left(\hat{d}^+(K) - \sigma\sqrt{T-t}\right) - N\left(\hat{d}_1^+(Le^{a(T-t)}) - \sigma\sqrt{T-t}\right)\right]\right\}.
\end{aligned}$$

Appendix B

B.1. LBA2: Approximation based on our lower bound

In this paper, a process similar to that in Broadie and Detemple (1996) is adopted to convert our lower bound to the approximate American option value LBA2. The LBA2 is assumed to follow

$$\mathrm{LBA2} = \hat{\lambda}_1 C_t^l(S_t),$$

where the adjusting parameter $\hat{\lambda}_1 = \hat{\lambda}_1(S_t, K, T, r, q)$ is a function of S_t, K, T, r, and q.

The details of deciding $\hat{\lambda}_1(S_t, K, T, r, q)$ are as follows. First we define $x_1 = 1$, $x_2 = T$, $x_3 = \sqrt{T}$, $x_4 = S_t/K$, $x_5 = r$, $x_6 = q$, $x_7 = \min(r/(\max(q, 10^{-5}), 5)$, $x_8 = x_7^2, x_9 = (C_t^l(S_t) - c_t(S_t))/K, x_{10} = x_9^2, x_{11} = C_t^l(S_t)/c_t(S_t), x_{12} = S_t/\hat{L}(S_t)$, and $x_{13} = \hat{a}(S_t)$. Then derive an intermediate variable y_l via the following equation.

$$\begin{aligned} y_l = {} & 1.002\mathrm{E}{+}00 x_1 + 1.647\mathrm{E}{-}04 x_2 + 8.245\mathrm{E}{-}05 x_3 - 1.336\mathrm{E}{-}03 x_4 \\ & - 3.679\mathrm{E}{-}03 x_5 + 1.035\mathrm{E}{-}02 x_6 + 1.220\mathrm{E}{-}04 x_7 - 6.357\mathrm{E}{-}04 x_8 \\ & - 1.035\mathrm{E}{-}02 x_9 + 1.292\mathrm{E}{-}02 x_{10} - 2.726\mathrm{E}{-}04 x_{11} \\ & + 3.976\mathrm{E}{-}04 x_{12} - 4.452\mathrm{E}{-}04 x_{13}. \end{aligned}$$

Finally, $\hat{\lambda}_1(S_t, K, T, r, q)$ can be derived by

$$\hat{\lambda}_1(S_t, K, T, r, q) = \begin{cases} 1 & \text{if} C^l(S_t) = c(S_t) \text{ or } C^l(S_t) \le S_t - K \\ \max(\min(y_l, 1.008), 1) & \text{otherwise} \end{cases},$$

where the maximum value of $\hat{\lambda}_1(S_t, K, T, r, q)$ is assumed to be 1.008 because our lower bounds are always within 0.79% of the true option values. The coefficients for y_l are derived from a regression based on the randomly generated 2500 option contracts in Section 4.3.[22]

B.2. LUBA2: Approximation based on our lower and upper bounds

The process to derive the approximate option value LUBA2 based on the information of $C_t^l(S_t)$ and $C_t^u(S_t)$ is elaborated as follows. First, consider a linearly weighted average relation between the LUBA2 and the lower and upper bounds $C_t^l(S_t)$ and $C_t^u(S_t)$.

$$\mathrm{LUBA2} = \hat{\lambda}_2 C_t^l(S_t) + (1 - \hat{\lambda}_2) C_t^u(S_t),$$

where the weighted average parameter $\hat{\lambda}_2 = \hat{\lambda}_2(S_t, K, T, r, q)$ is a function of S_t, K, T, r, and q.

To determine the function $\hat{\lambda}_2(S_t, K, T, r, q)$, we first define $x_1 = 1$, $x_2 = T$, $x_3 = \sqrt{T}$, $x_4 = r$, $x_5 = q$, $x_6 = \min(r/(\max(q, 10^{-5}), 5)$, $x_7 = x_6^2, x_8 = dC_t^l(S_t)/dS_t$,[23] $x_9 = x_8^2, x_{10} = (C_t^l(S_t) - c_t(S_t))/K, x_{11} = x_{10}^2$, $x_{12} = C_t^l(S_t)/c_t(S_t)$, $x_{13} = (C_t^u(S_t) - C_t^l(S_t))/K$, $x_{14} = C_t^u(S_t)/C_t^l(S_t)$, $x_{15} = S_t/(L_t^* e^{a_t^*(T-t)})$, $x_{16} = x_{15}^2$, $x_{17} = S_t/L_t^*$, $x_{18} = a_t^*$, $x_{19} = S_t/\hat{L}(S_t)$, and $x_{20} = \hat{a}(S_t)$. Then we calculate an intermediate variable y_u via the following equation.

$$\begin{aligned} y_u = {} & 2.329\mathrm{E}{-}01 x_1 - 2.384\mathrm{E}{-}02 x_2 + 1.457\mathrm{E}{-}01 x_3 + 3.718\mathrm{E}{-}02 x_4 \\ & + 1.849\mathrm{E}{-}01 x_5 - 3.111\mathrm{E}{-}01 x_6 + 2.447\mathrm{E}{-}01 x_7 - 1.887\mathrm{E}{-}01 x_8 \\ & + 3.801\mathrm{E}{-}01 x_9 + 3.556\mathrm{E}{-}01 x_{10} - 6.465\mathrm{E}{-}01 x_{11} + 4.622\mathrm{E}{-}02 x_{12} \\ & + 6.454\mathrm{E}{-}02 x_{13} - 2.170\mathrm{E}{-}01 x_{14} + 8.079\mathrm{E}{-}02 x_{15} \\ & + 2.202\mathrm{E}{-}01 x_{16} + 6.245\mathrm{E}{-}01 x_{17} - 2.970\mathrm{E}{-}01 x_{18} \\ & - 4.320\mathrm{E}{-}01 x_{19} + 2.964\mathrm{E}{-}01 x_{20}. \end{aligned}$$

[22] In this paper, two separately random sets of option contracts are generated following the same rule described in Section 4.3. One is used to estimate $\hat{\lambda}_1$ and $\hat{\lambda}_2$, and the other is used to compute the RMS-speed results in Fig. 2. For the set to determine $\hat{\lambda}_1$ and $\hat{\lambda}_2$, the number of qualified option contracts (i.e. option price $\geqslant 0.5$) is 2277.

[23] In this paper, the term $x_8 = dC_t^l(S_t)/dS_t$ is approximated by using a numerical differentiation with respect to S_t given the same exponential exercise barrier $\hat{L}(S_t)e^{\hat{a}(S_t)(T-t)}$.

Finally, $\hat{\lambda}_2(S_t, K, T, r, q)$ can be derived according to the following rule:

$$\hat{\lambda}_2(S_t, K, T, r, q) = \begin{cases} 1 & \text{if } C^l(S_t) = c(S_t) \text{ or } C^l(S_t) \leqslant S_t - K \\ \max(\min(y_u, 1), 0) & \text{otherwise} \end{cases}.$$

Based on the above framework for determining $\hat{\lambda}_2(S_t, K, T, r, q)$, a weighted regression is employed to find the coefficients in the formula of y_u. More specifically, the target function is

$$\min \sum_i \left(\frac{C_i^u - C_i^l}{C_i} \right) \left(\frac{\hat{\lambda}_2 C_i^l + (1 - \hat{\lambda}_2) C_i^u - C_i}{C_i} \right)^2,$$

where C_i^l, C_i^u, and C_i are the lower and upper bounds and the true value for the ith option contract. The intuition behind the weighted regression is that if the lower and upper bounds are very tight, the value of $\hat{\lambda}_2$ become less important in predicting C_i. Via performing this weighted regression on the randomly generated 2500 option contracts, the coefficients in the equation of y_u can be determined.

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