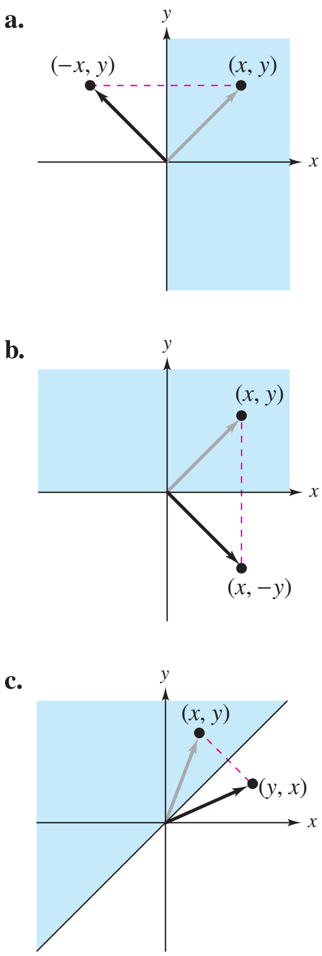
6.5 Applications of Linear Transformations

- Identify linear transformations defined by reflections, expansions, contractions, or shears in R^2.
- Use a linear transformation to rotate a figure in R^3.



Reflections in R^2
Figure 6.11

THE GEOMETRY OF LINEAR TRANSFORMATIONS IN R^2

This section gives geometric interpretations of linear transformations represented by 2×2 elementary matrices. Following a summary of the various types of 2×2 elementary matrices are examples that examine each type of matrix in more detail.

Elementary Matrices for Linear Transformations in R^2

Reflection in y-Axis

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection in x-Axis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection in Line $y = x$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Horizontal Expansion $(k > 1)$ or Contraction $(0 < k < 1)$

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Vertical Expansion $(k > 1)$ or Contraction $(0 < k < 1)$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Horizontal Shear

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical Shear

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

EXAMPLE 1 Reflections in R^2

The transformations defined by the following matrices are called **reflections.** These have the effect of mapping a point in the xy-plane to its "mirror image" with respect to one of the coordinate axes or the line $y = x$, as shown in Figure 6.11.

a. Reflection in the y-axis:

$$T(x, y) = (-x, y)$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

b. Reflection in the x-axis:

$$T(x, y) = (x, -y)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

c. Reflection in the line $y = x$:

$$T(x, y) = (y, x)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

EXAMPLE 2 Expansions and Contractions in R^2

The transformations defined by the following matrices are called **expansions** or **contractions,** depending on the value of the positive scalar k.

a. Horizontal contractions and expansions: $T(x, y) = (kx, y)$

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$$

b. Vertical contractions and expansions: $T(x, y) = (x, ky)$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$$

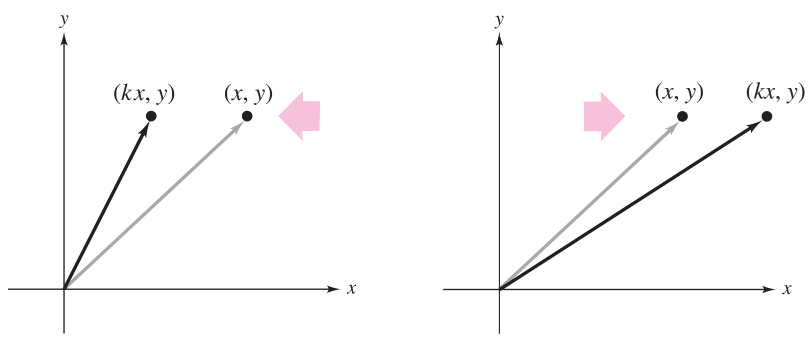
Note in Figures 6.12 and 6.13 that the distance the point (x, y) moves by a contraction or an expansion is proportional to its x- or y-coordinate. For instance, under the transformation represented by

$$T(x, y) = (2x, y)$$

the point $(1, 3)$ would move one unit to the right, but the point $(4, 3)$ would move four units to the right. Under the transformation represented by

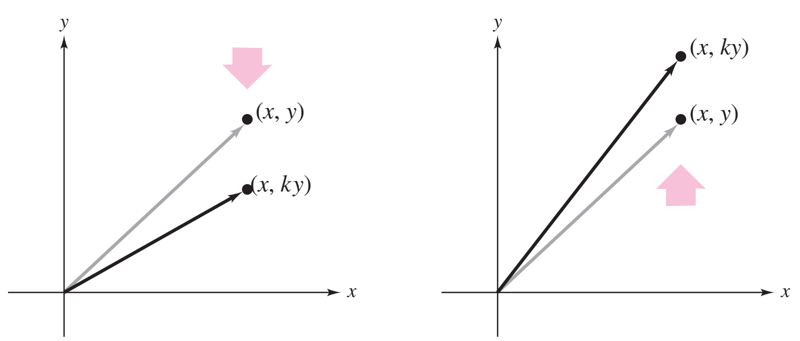
$$T(x, y) = \left(x, \tfrac{1}{2}y\right)$$

the point $(1, 4)$ would move two units down, but the point $(1, 2)$ would move one unit down.



Contraction $(0 < k < 1)$ Expansion $(k > 1)$

Figure 6.12



Contraction $(0 < k < 1)$ Expansion $(k > 1)$

Figure 6.13

Another type of linear transformation in R^2 corresponding to an elementary matrix is called a **shear,** as described in Example 3.

EXAMPLE 3 Shears in R^2

The transformations defined by the following matrices are shears.

$$T(x, y) = (x + ky, y) \qquad\qquad T(x, y) = (x, y + kx)$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$

a. A horizontal shear represented by

$$T(x, y) = (x + 2y, y)$$

is shown in Figure 6.14. Under this transformation, points in the upper half-plane "shear" to the right by amounts proportional to their y-coordinates. Points in the lower half-plane "shear" to the left by amounts proportional to the absolute values of their y-coordinates. Points on the x-axis do not move by this transformation.

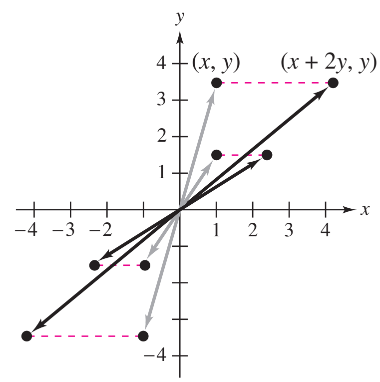


Figure 6.14

b. A vertical shear represented by

$$T(x, y) = (x, y + 2x)$$

is shown in Figure 6.15. Here, points in the right half-plane "shear" upward by amounts proportional to their x-coordinates. Points in the left half-plane "shear" downward by amounts proportional to the absolute values of their x-coordinates. Points on the y-axis do not move.

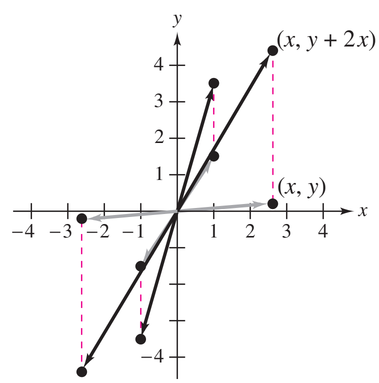
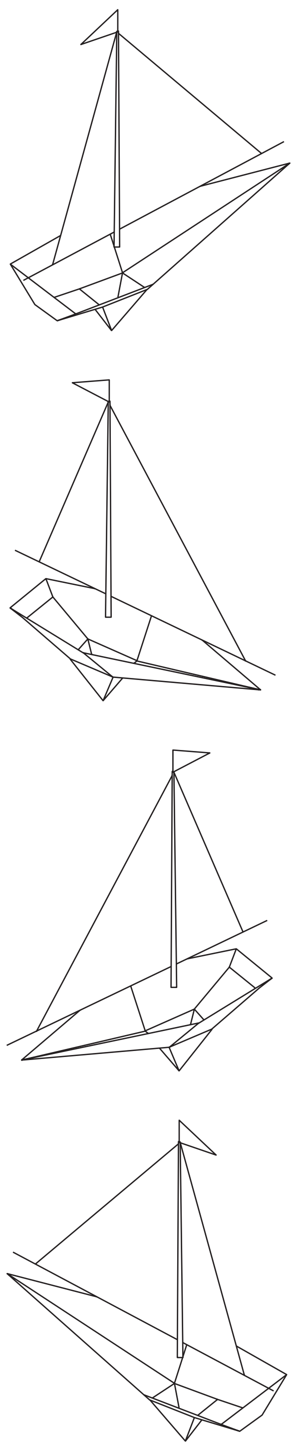


Figure 6.15



LINEAR ALGEBRA APPLIED

The use of computer graphics is common in many fields. By using graphics software, a designer can "see" an object before it is physically created. Linear transformations can be useful in computer graphics. To illustrate with a simplified example, only 23 points in R^3 make up the images of the toy boat shown in the figure at the left. Most graphics software can use such minimal information to generate views of an image from any perspective, as well as color, shade, and render as appropriate. Linear transformations, specifically those that produce rotations in R^3, can represent the different views. The remainder of this section discusses rotation in R^3.

ROTATION IN R^3

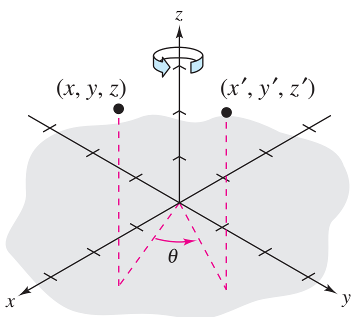


Figure 6.16

In Example 7 in Section 6.1, you saw how a linear transformation can be used to rotate figures in R^2. Here you will see how linear transformations can be used to rotate figures in R^3.

Suppose you want to rotate the point (x, y, z) counterclockwise about the z-axis through an angle θ, as shown in Figure 6.16. Letting the coordinates of the rotated point be (x', y', z'), you have

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}.$$

Example 4 shows how to use this matrix to rotate a figure in three-dimensional space.

EXAMPLE 4 Rotation About the z-Axis

The eight vertices of the rectangular prism shown in Figure 6.17 are as follows.

$V_1(0, 0, 0)$ $V_2(1, 0, 0)$

$V_3(1, 2, 0)$ $V_4(0, 2, 0)$

$V_5(0, 0, 3)$ $V_6(1, 0, 3)$

$V_7(1, 2, 3)$ $V_8(0, 2, 3)$

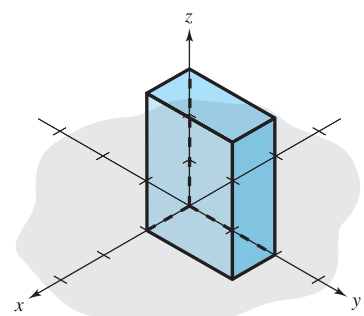


Figure 6.17

Find the coordinates of the vertices after the prism is rotated counterclockwise about the z-axis through (a) $\theta = 60°$, (b) $\theta = 90°$, and (c) $\theta = 120°$.

SOLUTION

a. The matrix that yields a rotation of 60° is

$$A = \begin{bmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying this matrix by the column vectors corresponding to each vertex produces the following rotated vertices.

$V_1'(0, 0, 0)$ $V_2'(0.5, 0.87, 0)$ $V_3'(-1.23, 1.87, 0)$ $V_4'(-1.73, 1, 0)$

$V_5'(0, 0, 3)$ $V_6'(0.5, 0.87, 3)$ $V_7'(-1.23, 1.87, 3)$ $V_8'(-1.73, 1, 3)$

Figure 6.18(a) shows a graph of the rotated prism.

b. The matrix that yields a rotation of 90° is

$$A = \begin{bmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and Figure 6.18(b) shows a graph of the rotated prism.

c. The matrix that yields a rotation of 120° is

$$A = \begin{bmatrix} \cos 120° & -\sin 120° & 0 \\ \sin 120° & \cos 120° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and Figure 6.18(c) shows a graph of the rotated prism.

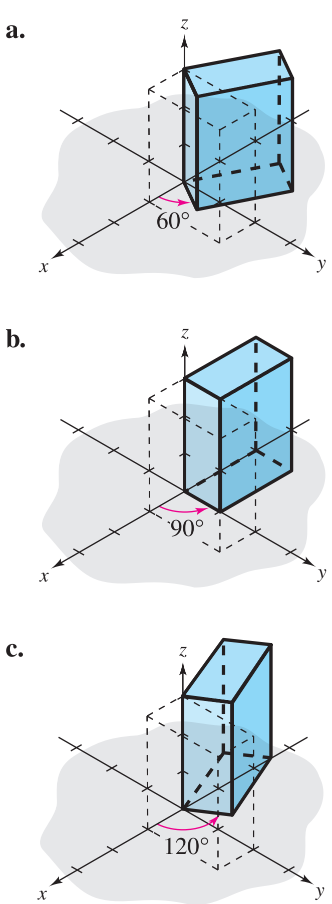
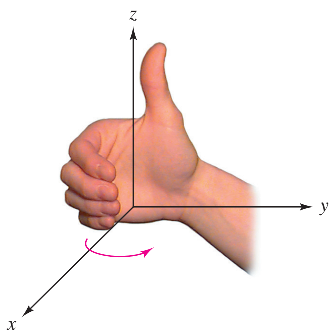


Figure 6.18

REMARK

To illustrate the right-hand rule, imagine the thumb of your right hand pointing in the positive direction of an axis. The cupped fingers will point in the direction of counterclockwise rotation. The figure below shows counterclockwise rotation about the z-axis.



Example 4 uses matrices to perform rotations about the z-axis. Similarly, you can use matrices to rotate figures about the x- or y-axis. The following summarizes all three types of rotations.

Rotation About the x-Axis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

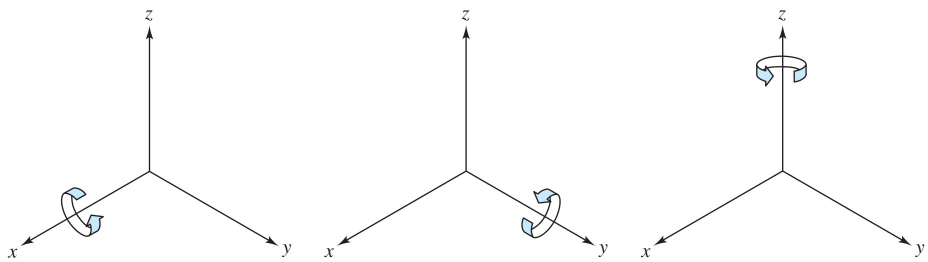
Rotation About the y-Axis

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Rotation About the z-Axis

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In each case, the rotation is oriented counterclockwise (using the "right-hand rule") relative to the indicated axis, as shown in Figure 6.19.



Rotation about x-axis   Rotation about y-axis   Rotation about z-axis

Figure 6.19

EXAMPLE 5 Rotation About the x-Axis and y-Axis

a. The matrix that yields a rotation of 90° about the x-axis is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ \\ 0 & \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Figure 6.20(a) shows the prism from Example 4 rotated 90° about the x-axis.

b. The matrix that yields a rotation of 90° about the y-axis is

$$A = \begin{bmatrix} \cos 90^\circ & 0 & \sin 90^\circ \\ 0 & 1 & 0 \\ -\sin 90^\circ & 0 & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Figure 6.20(b) shows the prism from Example 4 rotated 90° about the y-axis.

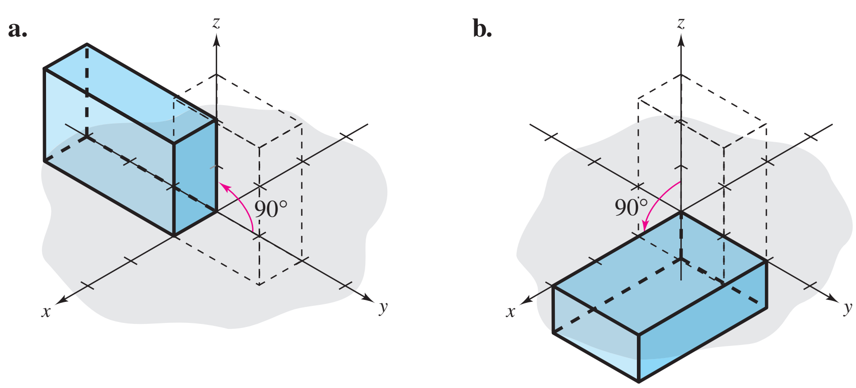


Figure 6.20

Simulation

Explore this concept further with an electronic simulation available at *www.cengagebrain.com.*

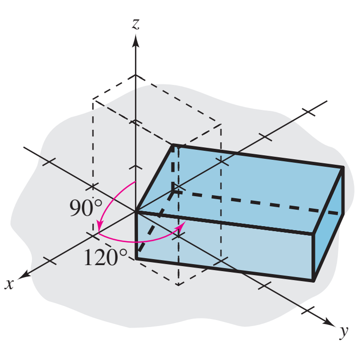


Figure 6.21

Rotations about the coordinate axes can be combined to produce any desired view of a figure. For instance, Figure 6.21 shows the prism from Example 4 rotated 90° about the y-axis and then 120° about the z-axis.