3.4 Applications of Determinants

- Find the adjoint of a matrix and use it to find the inverse of the matrix.
- Use Cramer's Rule to solve a system of n linear equations in n variables.
- Use determinants to find area, volume, and the equations of lines and planes.

THE ADJOINT OF A MATRIX

So far in this chapter, you have studied procedures for evaluating, and properties of, determinants. In this section, you will study an explicit formula for the inverse of a nonsingular matrix and use this formula to derive a theorem known as Cramer's Rule. You will then solve several applications of determinants.

Recall from Section 3.1 that the cofactor C_{ij} of a square matrix A is defined as $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the ith row and the jth column of A. The **matrix of cofactors** of A has the form

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

The transpose of this matrix is called the **adjoint** of A and is denoted by adj(A). That is,

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

EXAMPLE 1 Finding the Adjoint of a Square Matrix

Find the adjoint of $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$.

SOLUTION

The cofactor C_{11} is given by

$$\begin{bmatrix} -1 & \cancel{3} & \cancel{2} \\ \cancel{0} & -2 & 1 \\ \cancel{1} & 0 & -2 \end{bmatrix} \rightarrow C_{11} = (-1)^2\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} = 4.$$

Continuing this process produces the following matrix of cofactors of A.

$$\begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

The transpose of this matrix is the adjoint of A. That is, $\text{adj}(A) = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$.

The adjoint of a matrix A is useful for finding the inverse of A, as indicated in the next theorem.

REMARK

Theorem 3.10 is not particularly efficient for calculating inverses. The Gauss-Jordan elimination method discussed in Section 2.3 is much better. Theorem 3.10 is theoretically useful, however, because it provides a concise formula for the inverse of a matrix.

THEOREM 3.10 The Inverse of a Matrix Given by Its Adjoint

If A is an $n \times n$ invertible matrix, then $A^{-1} = \dfrac{1}{\det(A)}\operatorname{adj}(A)$.

PROOF

Begin by proving that the product of A and its adjoint is equal to the product of the determinant of A and I_n. Consider the product

$$A[\operatorname{adj}(A)] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}.$$

The entry in the ith row and jth column of this product is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}.$$

If $i = j$, then this sum is simply the cofactor expansion of A in its ith row, which means that the sum is the determinant of A. On the other hand, if $i \neq j$, then the sum is zero. (Try verifying this.)

$$A[\operatorname{adj}(A)] = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A)I$$

Because A is invertible, $\det(A) \neq 0$ and you can write

$$\frac{1}{\det(A)}A[\operatorname{adj}(A)] = I \quad \text{or} \quad A\left[\frac{1}{\det(A)}\operatorname{adj}(A)\right] = I.$$

By Theorem 2.7 and the definition of the inverse of a matrix, it follows that

$$\frac{1}{\det(A)}\operatorname{adj}(A) = A^{-1}.$$

REMARK

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the adjoint of A is simply

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Moreover, if A is invertible, then from Theorem 3.10 you have

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

which agrees with the result in Section 2.3.

EXAMPLE 2 Using the Adjoint of a Matrix to Find Its Inverse

Use the adjoint of A to find A^{-1}, where $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$.

SOLUTION

The determinant of this matrix is 3. Using the adjoint of A (found in Example 1), the inverse of A is

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}.$$

Check that this matrix is the inverse of A by showing that $AA^{-1} = I = A^{-1}A$.

CRAMER'S RULE

Cramer's Rule, named after Gabriel Cramer (1704–1752), uses determinants to solve a system of n linear equations in n variables. This rule applies only to systems with unique solutions. To see how Cramer's Rule works, take another look at the solution described at the beginning of Section 3.1. There, it was pointed out that the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

has the solution

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

when $a_{11}a_{22} - a_{21}a_{12} \neq 0$. Each numerator and denominator in this solution can be represented as a determinant, as follows.

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad a_{11}a_{22} - a_{21}a_{12} \neq 0$$

The denominator for x_1 and x_2 is simply the determinant of the coefficient matrix A of the original system. The numerators for x_1 and x_2 are formed by using the column of constants as replacements for the coefficients of x_1 and x_2 in $|A|$. These two determinants are denoted by $|A_1|$ and $|A_2|$, as follows.

$$|A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \quad \text{and} \quad |A_2| = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

You have $x_1 = \dfrac{|A_1|}{|A|}$ and $x_2 = \dfrac{|A_2|}{|A|}$. This determinant form of the solution is called **Cramer's Rule.**

EXAMPLE 3 Using Cramer's Rule

Use Cramer's Rule to solve the system of linear equations.

$$\begin{aligned} 4x_1 - 2x_2 &= 10 \\ 3x_1 - 5x_2 &= 11 \end{aligned}$$

SOLUTION

First find the determinant of the coefficient matrix.

$$|A| = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = -14$$

Because $|A| \neq 0$, you know the system has a unique solution, and applying Cramer's Rule produces

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} = \frac{-28}{-14} = 2$$

and

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} = \frac{14}{-14} = -1.$$

The solution is $x_1 = 2$ and $x_2 = -1$.

Cramer's Rule generalizes easily to systems of n linear equations in n variables. The value of each variable is given as the quotient of two determinants. The denominator is the determinant of the coefficient matrix, and the numerator is the determinant of the matrix formed by replacing the column corresponding to the variable being solved for with the column representing the constants. For instance, the solution for x_3 in the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \text{is} \quad x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

THEOREM 3.11 Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the solution of the system is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where the ith column of A_i is the column of constants in the system of equations.

PROOF

Let the system be represented by $AX = B$. Because $|A|$ is nonzero, you can write

$$X = A^{-1}B = \frac{1}{|A|}\operatorname{adj}(A)B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

If the entries of B are $b_1, b_2, \ldots, b_n$, then $x_i = \frac{1}{|A|}(b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni})$, but the sum (in parentheses) is precisely the cofactor expansion of A_i, which means that $x_i = |A_i|/|A|$, and the proof is complete.

EXAMPLE 4 Using Cramer's Rule

Use Cramer's Rule to solve the system of linear equations for x.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x \qquad + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

SOLUTION

The determinant of the coefficient matrix is $|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10.$

Because $|A| \neq 0$, you know that the solution is unique, so apply Cramer's Rule to solve for x, as follows.

$$x = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{(1)(-1)^5\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix}}{10} = \frac{(1)(-1)(-8)}{10} = \frac{4}{5}$$

REMARK

Try applying Cramer's Rule to solve for y and z. You will see that the solution is $y = -\frac{3}{2}$ and $z = -\frac{8}{5}$.

AREA, VOLUME, AND EQUATIONS OF LINES AND PLANES

Determinants have many applications in analytic geometry. One application is in finding the area of a triangle in the xy-plane.

Area of a Triangle in the xy-Plane

The area of a triangle with vertices

$$(x_1, y_1), (x_2, y_2), \text{ and } (x_3, y_3)$$

is

$$\text{Area} = \pm\tfrac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

where the sign $(\pm)$ is chosen to give a positive area.

PROOF

Prove the case for $y_i > 0$. Assume that $x_1 \le x_3 \le x_2$ and that (x_3, y_3) lies above the line segment connecting (x_1, y_1) and (x_2, y_2), as shown in Figure 3.1. Consider the three trapezoids whose vertices are

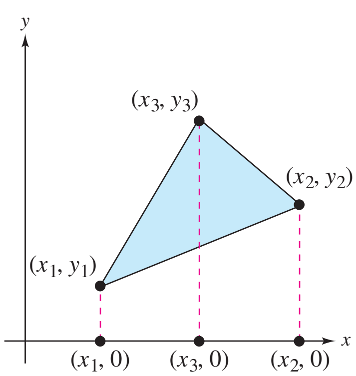


Figure 3.1

Trapezoid 1: $(x_1, 0), (x_1, y_1), (x_3, y_3), (x_3, 0)$
Trapezoid 2: $(x_3, 0), (x_3, y_3), (x_2, y_2), (x_2, 0)$
Trapezoid 3: $(x_1, 0), (x_1, y_1), (x_2, y_2), (x_2, 0)$.

The area of the triangle is equal to the sum of the areas of the first two trapezoids minus the area of the third trapezoid. So,

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}(y_1 + y_3)(x_3 - x_1) + \tfrac{1}{2}(y_3 + y_2)(x_2 - x_3) - \tfrac{1}{2}(y_1 + y_2)(x_2 - x_1) \\ &= \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2) \\ &= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \end{aligned}$$

If the vertices do not occur in the order $x_1 \le x_3 \le x_2$ or if the vertex (x_3, y_3) is not above the line segment connecting the other two vertices, then the formula above may yield the negative of the area. So, use $\pm$ and choose the correct sign to give a positive area.

EXAMPLE 5 Finding the Area of a Triangle

Find the area of the triangle whose vertices are

$$(1, 0), \quad (2, 2), \quad \text{and} \quad (4, 3).$$

SOLUTION

It is not necessary to know the relative positions of the three vertices. Simply evaluate the determinant

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -\frac{3}{2}$$

and conclude that the area of the triangle is $\frac{3}{2}$ square units.

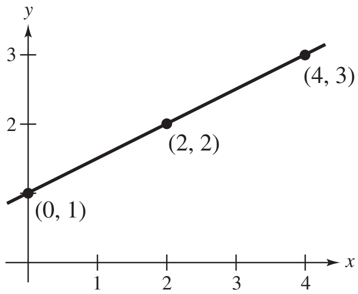


Figure 3.2

Suppose the three points in Example 5 had been on the same line. What would have happened had you applied the area formula to three such points? The answer is that the determinant would have been zero. Consider, for instance, the three collinear points (0, 1), (2, 2), and (4, 3), as shown in Figure 3.2. The determinant that yields the area of the "triangle" that has these three points as vertices is

$$\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = 0.$$

If three points in the xy-plane lie on the same line, then the determinant in the formula for the area of a triangle is zero. The following generalizes this result.

Test for Collinear Points in the *xy*-Plane

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

The test for collinear points can be adapted to another use. That is, when you are given two points in the xy-plane, you can find an equation of the line passing through the two points, as follows.

Two-Point Form of the Equation of a Line

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\det\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0.$$

EXAMPLE 6 Finding an Equation of the Line Passing Through Two Points

Find an equation of the line passing through the two points

$(2, 4)$ and $(-1, 3)$.

SOLUTION

Let $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (-1, 3)$. Applying the determinant formula for the equation of a line produces

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, expand by cofactors in the first row to obtain the following.

$$\begin{aligned} x\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} &= 0 \\ x(1) - y(3) + 1(10) &= 0 \\ x - 3y + 10 &= 0 \end{aligned}$$

So, an equation of the line is $x - 3y = -10$.

The formula for the area of a triangle in the plane has a straightforward generalization to three-dimensional space, which is presented without proof as follows.

Volume of a Tetrahedron

The volume of a tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is

$$\text{Volume} = \pm\frac{1}{6}\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}$$

where the sign $(\pm)$ is chosen to give a positive volume.

EXAMPLE 7 Finding the Volume of a Tetrahedron

Find the volume of the tetrahedron whose vertices are $(0, 4, 1)$, $(4, 0, 0)$, $(3, 5, 2)$, and $(2, 2, 5)$, as shown in Figure 3.3.

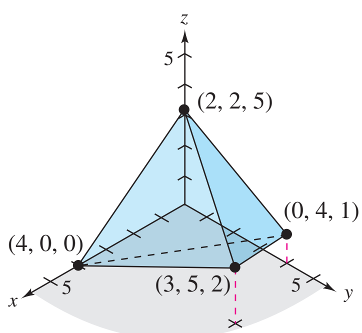


Figure 3.3

SOLUTION

Using the determinant formula for the volume of a tetrahedron produces

$$\frac{1}{6}\begin{vmatrix} 0 & 4 & 1 & 1 \\ 4 & 0 & 0 & 1 \\ 3 & 5 & 2 & 1 \\ 2 & 2 & 5 & 1 \end{vmatrix} = \frac{1}{6}(-72) = -12.$$

So, the volume of the tetrahedron is 12 cubic units.

If four points in three-dimensional space lie in the same plane, then the determinant in the formula for the volume of a tetrahedron is zero. So, you have the following test.

Test for Coplanar Points in Space

Four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) are coplanar if and only if

$$\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0.$$

An adaptation of this test is the determinant form of an equation of a plane passing through three points in space, as follows.

Three-Point Form of the Equation of a Plane

An equation of the plane passing through the distinct points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is given by

$$\det\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0.$$

EXAMPLE 8 **Finding an Equation of the Plane Passing Through Three Points**

Find an equation of the plane passing through the three points $(0, 1, 0)$, $(-1, 3, 2)$, and $(-2, 0, 1)$.

SOLUTION

Using the determinant form of the equation of a plane produces

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 3 & 2 & 1 \\ -2 & 0 & 1 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, subtract the fourth column from the second column to obtain

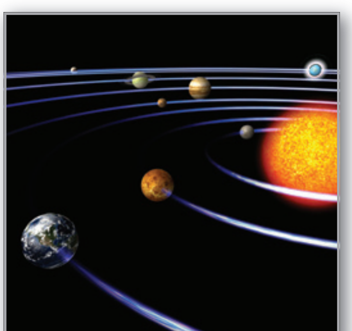
$$\begin{vmatrix} x & y-1 & z & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 2 & 1 \\ -2 & -1 & 1 & 1 \end{vmatrix} = 0.$$

Now, expanding by cofactors in the second row yields

$$x\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} - (y-1)\begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} + z\begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix} = 0$$

$$x(4) - (y-1)(3) + z(5) = 0.$$

This produces the equation $4x - 3y + 5z = -3$.



LINEAR ALGEBRA APPLIED

According to Kepler's First Law of Planetary Motion, the orbits of the planets are ellipses, with the sun at one focus of the ellipse. The general equation of a conic section (such as an ellipse) is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

To determine the equation of the orbit of a planet, an astronomer can find the coordinates of the planet along its orbit at five different points (x_i, y_i), where $i = 1, 2, 3, 4,$ and 5, and then use the determinant

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5x_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix}.$$

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