1.3 Applications of Systems of Linear Equations

- Set up and solve a system of equations to fit a polynomial function to a set of data points.
- Set up and solve a system of equations to represent a network.

Systems of linear equations arise in a wide variety of applications. In this section you will look at two applications, and you will see more in subsequent chapters. The first application shows how to fit a polynomial function to a set of data points in the plane. The second application focuses on networks and Kirchhoff's Laws for electricity.

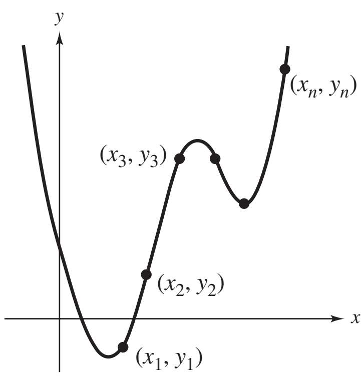

Figure 1.4

POLYNOMIAL CURVE FITTING

Suppose n points in the xy-plane

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

represent a collection of data and you are asked to find a polynomial function of degree $n - 1$

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$$

whose graph passes through the specified points. This procedure is called **polynomial curve fitting.** If all x-coordinates of the points are distinct, then there is precisely one polynomial function of degree $n - 1$ (or less) that fits the n points, as shown in Figure 1.4.

To solve for the n coefficients of $p(x)$, substitute each of the n points into the polynomial function and obtain n linear equations in n variables $a_0, a_1, a_2, \ldots, a_{n-1}$.

$$\begin{aligned} a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_{n-1}x_1^{n-1} &= y_1 \\ a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_{n-1}x_2^{n-1} &= y_2 \\ &\vdots \\ a_0 + a_1x_n + a_2x_n^2 + \cdots + a_{n-1}x_n^{n-1} &= y_n \end{aligned}$$

Example 1 demonstrates this procedure with a second-degree polynomial.

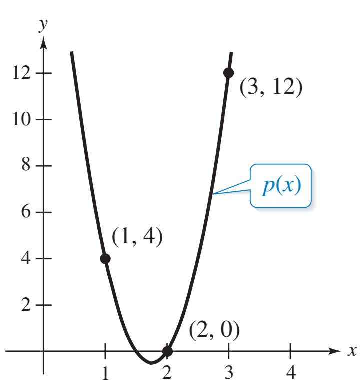

Figure 1.5

Simulation
Explore this concept further with an electronic simulation available at *www.cengagebrain.com*.

EXAMPLE 1 Polynomial Curve Fitting

Determine the polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points (1, 4), (2, 0), and (3, 12).

SOLUTION

Substituting $x = 1$, 2, and 3 into $p(x)$ and equating the results to the respective y-values produces the system of linear equations in the variables a_0, a_1, and a_2 shown below.

$$\begin{aligned} p(1) &= a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2 = 4 \\ p(2) &= a_0 + a_1(2) + a_2(2)^2 = a_0 + 2a_1 + 4a_2 = 0 \\ p(3) &= a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2 = 12 \end{aligned}$$

The solution of this system is

$$a_0 = 24, \; a_1 = -28, \text{ and } a_2 = 8$$

so the polynomial function is

$$p(x) = 24 - 28x + 8x^2.$$

Figure 1.5 shows the graph of p.

EXAMPLE 2 Polynomial Curve Fitting

Find a polynomial that fits the points

$(-2, 3), (-1, 5), (0, 1), (1, 4)$, and $(2, 10)$.

SOLUTION

Because you are given five points, choose a fourth-degree polynomial function

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4.$$

Substituting the given points into $p(x)$ produces the following system of linear equations.

$$\begin{aligned} a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 &= 3 \\ a_0 - a_1 + a_2 - a_3 + a_4 &= 5 \\ a_0 &= 1 \\ a_0 + a_1 + a_2 + a_3 + a_4 &= 4 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 &= 10 \end{aligned}$$

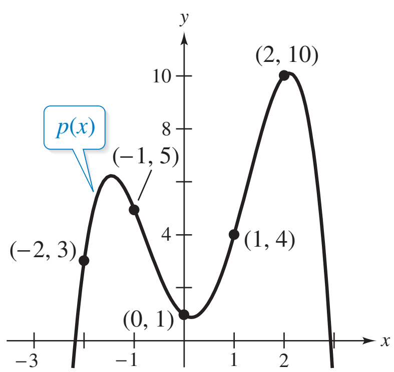

Figure 1.6

The solution of these equations is

$$a_0 = 1, \quad a_1 = -\tfrac{30}{24}, \quad a_2 = \tfrac{101}{24}, \quad a_3 = \tfrac{18}{24}, \quad a_4 = -\tfrac{17}{24}$$

which means the polynomial function is

$$\begin{aligned} p(x) &= 1 - \tfrac{30}{24}x + \tfrac{101}{24}x^2 + \tfrac{18}{24}x^3 - \tfrac{17}{24}x^4 \\ &= \tfrac{1}{24}(24 - 30x + 101x^2 + 18x^3 - 17x^4). \end{aligned}$$

Figure 1.6 shows the graph of p.

The system of linear equations in Example 2 is relatively easy to solve because the x-values are small. For a set of points with large x-values, it is usually best to *translate* the values before attempting the curve-fitting procedure. The next example demonstrates this approach.

EXAMPLE 3 Translating Large x-Values Before Curve Fitting

Find a polynomial that fits the points

$$\underbrace{(2006, 3)}_{(x_1, y_1)}, \quad \underbrace{(2007, 5)}_{(x_2, y_2)}, \quad \underbrace{(2008, 1)}_{(x_3, y_3)}, \quad \underbrace{(2009, 4)}_{(x_4, y_4)}, \quad \underbrace{(2010, 10)}_{(x_5, y_5)}.$$

SOLUTION

Because the given x-values are large, use the translation $z = x - 2008$ to obtain

$$\underbrace{(-2, 3)}_{(z_1, y_1)}, \quad \underbrace{(-1, 5)}_{(z_2, y_2)}, \quad \underbrace{(0, 1)}_{(z_3, y_3)}, \quad \underbrace{(1, 4)}_{(z_4, y_4)}, \quad \underbrace{(2, 10)}_{(z_5, y_5)}.$$

This is the same set of points as in Example 2. So, the polynomial that fits these points is

$$\begin{aligned} p(z) &= \tfrac{1}{24}(24 - 30z + 101z^2 + 18z^3 - 17z^4) \\ &= 1 - \tfrac{5}{4}z + \tfrac{101}{24}z^2 + \tfrac{3}{4}z^3 - \tfrac{17}{24}z^4. \end{aligned}$$

Letting $z = x - 2008$, you have

$$p(x) = 1 - \tfrac{5}{4}(x - 2008) + \tfrac{101}{24}(x - 2008)^2 + \tfrac{3}{4}(x - 2008)^3 - \tfrac{17}{24}(x - 2008)^4.$$

EXAMPLE 4 An Application of Curve Fitting

Find a polynomial that relates the periods of the three planets that are closest to the Sun to their mean distances from the Sun, as shown in the table. Then test the accuracy of the fit by using the polynomial to calculate the period of Mars. (In the table, the mean distance is given in astronomical units, and the period is given in years.)

Planet	*Mercury*	*Venus*	*Earth*	*Mars*
Mean Distance	0.387	0.723	1.000	1.524
Period	0.241	0.615	1.000	1.881

SOLUTION

Begin by fitting a quadratic polynomial function

$$p(x) = a_0 + a_1x + a_2x^2$$

to the points

$$(0.387, 0.241), (0.723, 0.615), \text{ and } (1, 1).$$

The system of linear equations obtained by substituting these points into $p(x)$ is

$$\begin{aligned} a_0 + 0.387a_1 + (0.387)^2a_2 &= 0.241 \\ a_0 + 0.723a_1 + (0.723)^2a_2 &= 0.615 \\ a_0 + a_1 + a_2 &= 1. \end{aligned}$$

The approximate solution of the system is

$$a_0 \approx -0.0634, \quad a_1 \approx 0.6119, \quad a_2 \approx 0.4515$$

which means that an approximation of the polynomial function is

$$p(x) = -0.0634 + 0.6119x + 0.4515x^2.$$

Using $p(x)$ to evaluate the period of Mars produces

$$p(1.524) \approx 1.918 \text{ years.}$$

Note that the actual period of Mars is 1.881 years. Figure 1.7 compares the estimate with the actual period graphically.

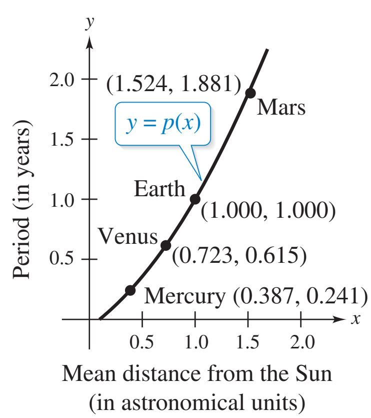

Figure 1.7

As illustrated in Example 4, a polynomial that fits some of the points in a data set is not necessarily an accurate model for other points in the data set. Generally, the farther the other points are from those used to fit the polynomial, the worse the fit. For instance, the mean distance of Jupiter from the Sun is 5.203 astronomical units. Using $p(x)$ in Example 4 to approximate the period gives 15.343 years—a poor estimate of Jupiter's actual period of 11.860 years.

The problem of curve fitting can be difficult. Types of functions other than polynomial functions may provide better fits. For instance, look again at the curve-fitting problem in Example 4. Taking the natural logarithms of the given distances and periods produces the following results.

Planet	*Mercury*	*Venus*	*Earth*	*Mars*
Mean Distance (x)	0.387	0.723	1.000	1.524
ln x	-0.949	-0.324	0.0	0.421
Period (y)	0.241	0.615	1.000	1.881
ln y	-1.423	-0.486	0.0	0.632

Now, fitting a polynomial to the logarithms of the distances and periods produces the *linear relationship*

$$\ln y = \frac{3}{2} \ln x$$

shown in Figure 1.8.

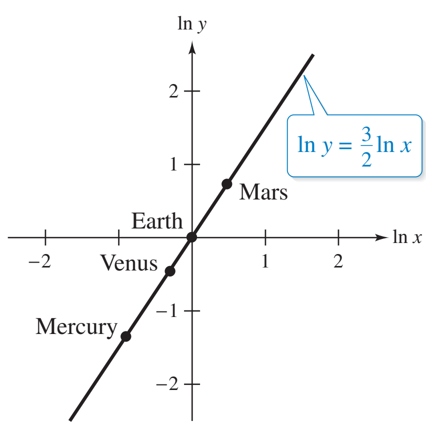

Figure 1.8

From $\ln y = \frac{3}{2} \ln x$, it follows that $y = x^{3/2}$, or $y^2 = x^3$. In other words, the square of the period (in years) of each planet is equal to the cube of its mean distance (in astronomical units) from the Sun. Johannes Kepler first discovered this relationship in 1619.

LINEAR ALGEBRA APPLIED

Researchers in Italy studying the acoustical noise levels from vehicular traffic at a busy three-way intersection on a college campus used a system of linear equations to model the traffic flow at the intersection. To help formulate the system of equations, "operators" stationed themselves at various locations along the intersection and counted the numbers of vehicles going by. *(Source: Acoustical Noise Analysis in Road Intersections: A Case Study, Guarnaccia, Claudio, Recent Advances in Acoustics & Music, Proceedings of the 11th WSEAS International Conference on Acoustics & Music: Theory & Applications, June, 2010)*

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NETWORK ANALYSIS

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering. In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction. For instance, the junction shown in Figure 1.9 has 25 units flowing into it, so there must be 25 units flowing out of it. You can represent this with the linear equation

$$x_1 + x_2 = 25.$$

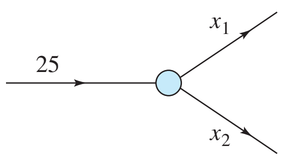


Figure 1.9

Because each junction in a network gives rise to a linear equation, you can analyze the flow through a network composed of several junctions by solving a system of linear equations. Example 5 illustrates this procedure.

EXAMPLE 5 Analysis of a Network

Set up a system of linear equations to represent the network shown in Figure 1.10. Then solve the system.

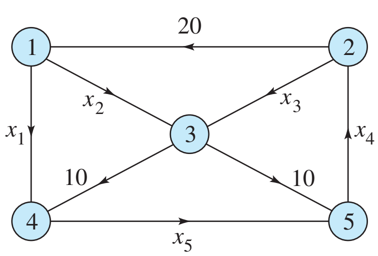


Figure 1.10

SOLUTION

Each of the network's five junctions gives rise to a linear equation, as follows.

$$\begin{aligned}
x_1 + x_2 \qquad\qquad &= 20 \quad &\text{Junction 1}\\
x_3 - x_4 \qquad &= -20 \quad &\text{Junction 2}\\
x_2 + x_3 \qquad\qquad &= 20 \quad &\text{Junction 3}\\
x_1 \qquad\qquad - x_5 &= -10 \quad &\text{Junction 4}\\
-x_4 + x_5 &= -10 \quad &\text{Junction 5}
\end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{bmatrix}.$$

Gauss-Jordan elimination produces the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the matrix above, you can see that

$$x_1 - x_5 = -10, \quad x_2 + x_5 = 30, \quad x_3 - x_5 = -10, \quad \text{and} \quad x_4 - x_5 = 10.$$

Letting $t = x_5$, you have

$$x_1 = t - 10, \quad x_2 = -t + 30, \quad x_3 = t - 10, \quad x_4 = t + 10, \quad x_5 = t$$

where t is any real number, so this system has infinitely many solutions.

In Example 5, suppose you could control the amount of flow along the branch labeled x_5. Using the solution of Example 5, you could then control the flow represented by each of the other variables. For instance, letting $t = 10$ would reduce the flow of x_1 and x_3 to zero, as shown in Figure 1.11.

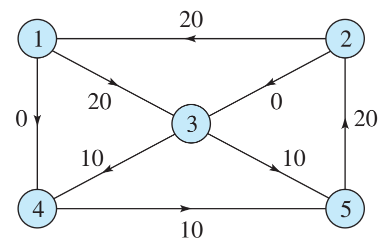


Figure 1.11

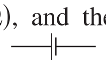
You may be able to see how the type of network analysis demonstrated in Example 5 could be used in problems dealing with the flow of traffic through the streets of a city or the flow of water through an irrigation system.

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as **Kirchhoff's Laws.**

REMARK

A closed path is a sequence of branches such that the beginning point of the first branch coincides with the end point of the last branch.

1. All the current flowing into a junction must flow out of it.
2. The sum of the products IR (I is current and R is resistance) around a closed path is equal to the total voltage in the path.

In an electrical network, current is measured in amperes, or amps (A), resistance is measured in ohms (Ω), and the product of current and resistance is measured in volts (V). The symbol  represents a battery. The larger vertical bar denotes where the current flows out of the terminal. The symbol  denotes resistance. An arrow in the branch indicates the direction of the current.

EXAMPLE 6 Analysis of an Electrical Network

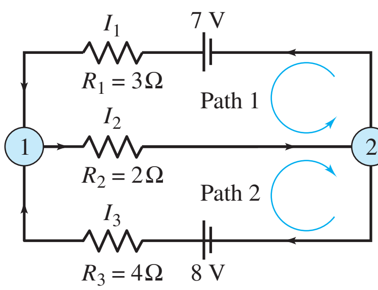


Figure 1.12

Determine the currents I_1, I_2, and I_3 for the electrical network shown in Figure 1.12.

SOLUTION

Applying Kirchhoff's first law to either junction produces

$$I_1 + I_3 = I_2 \qquad \text{Junction 1 or Junction 2}$$

and applying Kirchhoff's second law to the two paths produces

$$\begin{aligned} R_1I_1 + R_2I_2 &= 3I_1 + 2I_2 = 7 && \text{Path 1} \\ R_2I_2 + R_3I_3 &= 2I_2 + 4I_3 = 8. && \text{Path 2} \end{aligned}$$

So, you have the following system of three linear equations in the variables I_1, I_2, and I_3.

$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ 3I_1 + 2I_2 &= 7 \\ 2I_2 + 4I_3 &= 8 \end{aligned}$$

Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 2 & 4 & 8 \end{bmatrix}$$

produces the reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which means $I_1 = 1$ amp, $I_2 = 2$ amps, and $I_3 = 1$ amp.

EXAMPLE 7 Analysis of an Electrical Network

Determine the currents I_1, I_2, I_3, I_4, I_5, and I_6 for the electrical network shown in Figure 1.13.

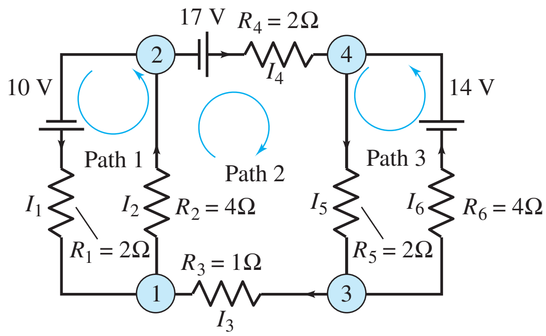


Figure 1.13

SOLUTION

Applying Kirchhoff's first law to the four junctions produces

$$\begin{aligned} I_1 + I_3 &= I_2 && \text{Junction 1}\\ I_1 + I_4 &= I_2 && \text{Junction 2}\\ I_3 + I_6 &= I_5 && \text{Junction 3}\\ I_4 + I_6 &= I_5 && \text{Junction 4} \end{aligned}$$

and applying Kirchhoff's second law to the three paths produces

$$\begin{aligned} 2I_1 + 4I_2 \qquad\qquad\qquad\qquad &= 10 && \text{Path 1}\\ 4I_2 + I_3 + 2I_4 + 2I_5 \qquad &= 17 && \text{Path 2}\\ 2I_5 + 4I_6 &= 14. && \text{Path 3} \end{aligned}$$

You now have the following system of seven linear equations in the variables I_1, I_2, I_3, I_4, I_5, and I_6.

$$\begin{aligned} I_1 - I_2 + I_3 \qquad\qquad\qquad &= 0\\ I_1 - I_2 \qquad + I_4 \qquad\qquad &= 0\\ I_3 \qquad - I_5 + I_6 &= 0\\ I_4 - I_5 + I_6 &= 0\\ 2I_1 + 4I_2 \qquad\qquad\qquad\qquad &= 10\\ 4I_2 + I_3 + 2I_4 + 2I_5 \qquad &= 17\\ 2I_5 + 4I_6 &= 14 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0\\ 1 & -1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 1 & -1 & 1 & 0\\ 2 & 4 & 0 & 0 & 0 & 0 & 10\\ 0 & 4 & 1 & 2 & 2 & 0 & 17\\ 0 & 0 & 0 & 0 & 2 & 4 & 14 \end{bmatrix}.$$

Using Gauss-Jordan elimination, a graphing utility, or a software program, solve this system to obtain

$$I_1 = 1, \quad I_2 = 2, \quad I_3 = 1, \quad I_4 = 1, \quad I_5 = 3, \quad \text{and} \quad I_6 = 2$$

meaning $I_1 = 1$ amp, $I_2 = 2$ amps, $I_3 = 1$ amp, $I_4 = 1$ amp, $I_5 = 3$ amps, and $I_6 = 2$ amps.