數値分析

Chapter 12
Numerical Methods for Partial-Differential Equations

12.1 Introduction

- Two-variable partial differential equations.

- elliptic equation (橢圓)
  $\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y)$

- parabolic equation (拋物線)
  $\frac{\partial u}{\partial t}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0$

- hyperbolic equation (雙曲線)
  $\alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)$

12.2 Finite-Difference Methods for Elliptic Problems

- $\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y)$ on $R = \{(x,y) | a < x < b, c < y < d\}$
  with $u(x,y) = g(x,y)$ for $(x,y) \in S$

- 目標是要求 $u(x,y)$ 函數. 首先將 $X - Y$ 平面離散化, 若在 given 任意 (x,y) 下, 都能找出 u 之值, 就約當找出 $u(x,y)$ 之函數 (在 x,y 切夠細的情況下)

- $x_i = a + ih$, $i = 0, 1, \cdots, n$
  $y_i = c + jk$, $j = 0, 1, \cdots, m$
  $\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} - \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(\xi_i, y_j)$
  $\frac{\partial^2 u}{\partial y^2}(x_i, y_j) = \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} - \frac{k^2}{12} \frac{\partial^4 u}{\partial y^4}(x_i, \eta_j)$

- 代入 partial differential equation, 且令 $w_{i,j} = u(x_i, y_j)$
  $\Rightarrow$
  $2[(\frac{h}{k})^2 + 1]w_{ij} - (w_{i+1,j} + w_{i-1,j}) - (\frac{h}{k})^2(w_{i,j+1} + w_{i,j-1}) = -h^2 f(x_i, y_j)$
  P.472 Figure 12.5

- boundary conditions
  $w_{0j} = g(x_0, y_j), w_{nj} = g(x_n, y_j), w_{i0} = g(x_i, y_0), w_{im} = g(x_i, y_m)$
  P.472 Figure 12.6
  一次解中間 $P_1, \cdots, P_{12}$ 的 w_{ij}

- P.473 Example 1
  9個未知數 9個方程式 $\Rightarrow$ 有 exact solution

- 當 matrix 小 (order $<$ 100), 用 Gaussian elimination 求 exact solution
  當 matrix 大, 用 iterative method (Gauss-Seidal) 求 approximated solution.

12.3 Finite-Difference Methods for Parabolic Problems

- $\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ for $0 < x < l$ and $t > 0$
  with $u(0,t) = g(0)$, $u(l,t) = g(l)$ for $t > 0$
  $u(x,0) = f(x)$ for $0 \le x \le l$

- Forward difference method (Explicit)
  $\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j)$
  $\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)$

- 代入 partial differential equation, 且令 $w_{ij} = u(x_i, t_j)$
  $\Rightarrow \frac{w_{i,j+1} - w_{ij}}{k} - \alpha^2 \frac{w_{i+1,j} - 2w_{ij} + w_{i-1,j}}{h^2} = 0$
  $\Rightarrow w_{i,j+1} = (1 - \frac{2\alpha^2 k}{h^2}) w_{ij} + \alpha^2 \frac{k}{h^2}(w_{i+1,j} + w_{i-1,j})$
  (第 j 期可推第 $j+1$ 期)

- 令 $\lambda = \alpha^2(k/h^2)$, $w^{(0)} = (f(x_1), f(x_2), \cdots, f(x_{n-1}))^t$
  $w^{(j)} = (w_{1j}, w_{2j}, \cdots, w_{n-1,j})^t$
  $\Rightarrow w^{(j)} = A \cdot w^{(j-1)}$,

$$\text{where } A = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & \cdots & 0 & \lambda & (1-2\lambda) \end{bmatrix}_{(m-1)\times(m-1)} .$$

- P.484 Figure 12.9

- P.480 Example 1

- Error Propagation
  $w_1 = A(w^{(0)} + e^{(0)}) = Aw^{(0)} + Ae^{(0)}$
  $\Rightarrow w^{(n)}$ is stable if $\|A^n e^{(0)}\| \leq \|e^{(0)}\|$
  當 $\rho(A^n) = (\rho(A))^n \leq 1 \Rightarrow \|A^n e^{(0)}\| \leq \|e^{(0)}\|$
  $\Rightarrow$ The forward-difference is stable only if $\rho(A) \leq 1$

- The stable condition is $0 \leq \lambda \leq \frac{1}{2}$
  $\Rightarrow \alpha^2 \frac{k}{h^2} \leq \frac{1}{2}$ (當 h 小時, k 要更小, 才能造成 stable)

- Backward difference method (Implicit)
  $\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i,t_j)-u(x_i,t_{j-1})}{k} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j)$
  $\Rightarrow \frac{w_{ij}-w_{i,j-1}}{k} - \alpha^2 \frac{w_{i+1,j}-2w_{ij}+w_{i-1,j}}{h^2} = 0$
  P.483 Figure 12.8

- 整理後得
  $(1+2\lambda)w_{ij} - \lambda w_{i+1,j} - \lambda w_{i-1,j} = w_{i,j-1}$
  $\Rightarrow Aw^{(j)} = w^{(j-1)}$

$$\text{where } A = \begin{bmatrix} (1+2\lambda) & -\lambda & 0 & \cdots & 0 \\ -\lambda & (1+2\lambda) & -\lambda & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\lambda \\ 0 & \cdots & 0 & -\lambda & (1+2\lambda) \end{bmatrix}_{(m-1)\times(m-1)}.$$

- P.484 Example 2

- P.486 Table 12.4 (用 Explicit 不收斂, 但用 Implicit 收斂)

- Explicit: $w^{(j)} = Aw^{(j-1)}$, 收斂條件 $\rho(A) \le 1$
  Implicit: $Aw^{(j)} = w^{(j-1)} \Rightarrow w^{(j)} = A^{-1}w^{(j-1)}$,
  收斂條件 $\rho(A^{-1}) \le 1$
  因 $\min \rho(A) > 1 \Rightarrow \max \rho(A^{-1}) < 1$
  $\Rightarrow$ Implicit 收斂之條件與 λ 是 independent

- Implicit 之弱點, error 爲 $O(k+h^2)$
  可否有機會, 使得 error 降爲 $O(k^2+h^2)$?
  用 Implicit + Explicit

- Forward: $\frac{w_{i,j+1}-w_{i,j}}{k} - \alpha^2 \frac{w_{i+1,j}-2w_{ij}+w_{i-1,j}}{h^2} = 0$
  with error $\frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j) + O(h^2)$
  Backward: $\frac{w_{i,j+1}-w_{i,j}}{k} - \alpha^2 \frac{w_{i+1,j+1}-2w_{i,j+1}+w_{i-1,j+1}}{h^2} = 0$
  with error $-\frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \hat{\mu}_j) + O(h^2)$
  assume $\frac{\partial^2 u}{\partial t^2}(x_i, \hat{\mu}_j) \approx \frac{\partial^2 u}{\partial t^2}(x_i, \mu_j)$
  $\Rightarrow$
  $\frac{w_{i,j+1}-w_{i,j}}{k} - \frac{\alpha^2}{2}[\frac{w_{i+1,j}-2w_{ij}+w_{i-1,j}}{h^2} + \frac{w_{i+1,j+1}-2w_{i,j+1}+w_{i-1,j+1}}{h^2}] = 0$

- P.488 Example 3

- P.489 Table 12.5 比只用 Implicit 效果好

12.4 Finite-Difference Methods for Hyperbolic Problems

- $\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0$
  s.t. $u(0,t) = u(l,t) = 0$ for $t > 0$
  $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$

- 差分代入後
  $\frac{w_{i,j+1}-w_{i,j}+w_{i,j-1}}{k^2} - \alpha^2 \frac{w_{i+1,j}-2w_{ij}+w_{i-1,j}}{h^2} = 0$
  $\Rightarrow w_{i,j+1} = 2(1-\lambda^2)w_{ij} + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}$
  where $\lambda = \alpha\frac{k}{h}$

- $$\begin{bmatrix} w_{1,j+1} \\ w_{2,j+1} \\ \vdots \\ \vdots \\ w_{n-1,j+1} \end{bmatrix}$$

  $$= \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & (1-\lambda^2) \end{bmatrix} \begin{bmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ \vdots \\ w_{n-1,j} \end{bmatrix} - \begin{bmatrix} w_{1,j-1} \\ w_{2,j-1} \\ \vdots \\ \vdots \\ w_{n-1,j-1} \end{bmatrix}$$

  P.494 Figure 12.10

- 用 $j=0$, 加上一階條件, 先估 $j=1$, $w_{i1} = w_{i0} + kg(x_i)$
  再用 $j=0$ 與 $j=1$, 估 $j=2$

- 另一種估 w_{i1} 之方法
  $w_{i1} = w_{i0} + k\frac{\partial u}{\partial t}(x,0) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \hat{\mu}_j)$
  $= w_{i0} + kg(x_i) + \frac{k^2}{2}(\alpha^2\frac{\partial^2 u}{\partial x^2}(x_i, \hat{\mu}_j))$
  $= w_{i0} + kg(x_i) + \frac{\alpha^2 k^2}{2}f''(x_i)$
  $\Rightarrow w_{i1} = (1-\lambda^2)f(x_i) + \frac{\lambda^2}{2}f(x_{i+1}) + \frac{\lambda^2}{2}f(x_{i-1}) + kg(x_i)$
  P.496 Example 1
  P.497 Table 12.6