數值分析

Chapter 8
Approximation Theory

8.2 Discrete Least Squares Approximation

"Best Fit": it needs not agree precisely with the data at any point.

1. $\min(\max|y_i - (a_1x_i + a_0)|)$
2. $\min \sum|y_i - (a_1x_i + a_0)|$
3. $\min \sum(y_i - (a_1x_i + a_0))^2$

- Linear Least Squares

The general problem of fitting the best least squares line to a collection of data $\{(x_i, y_i)\}_{i=1}^m$ involves minimizing the total error $E_2(a_0, a_1) = \sum\limits_{i=1}^{m}(y_i - (a_1x_i + a_0))^2$ with respect to the parameters a_0 and a_1. For a minimum to occur, we need

$$0 = \frac{\partial}{\partial a_0}\sum_{i=1}^{m}(y_i - (a_1x_i + a_0))^2 = 2\sum_{i=1}^{m}(y_i - (a_1x_i + a_0))(-1)$$

and

$$0 = \frac{\partial}{\partial a_1}\sum_{i=1}^{m}(y_i - (a_1x_i + a_0))^2 = 2\sum_{i=1}^{m}(y_i - (a_1x_i + a_0))(-x_i)$$

These equations simplify to the normal equations

$$a_0\sum_{i=1}^{m} x_i + a_1\sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_iy_i \text{ and } a_0m + a_1\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i$$

The linear least squares solution for a given collection of data $\{(x_i, y_i)\}_{i=1}^m$ has the form $y = a_1x + a_0$, where

$$a_0 = \frac{(\sum\limits_{i=1}^{m} x_i^2)(\sum\limits_{i=1}^{m} y_i) - (\sum\limits_{i=1}^{m} x_iy_i)(\sum\limits_{i=1}^{m} x_i)}{m(\sum\limits_{i=1}^{m} x_i^2) - (\sum\limits_{i=1}^{m} x_i)^2}$$

and

$$a_1 = \frac{m(\sum\limits_{i=1}^{m} x_iy_i) - (\sum\limits_{i=1}^{m} x_i)(\sum\limits_{i=1}^{m} y_i)}{m(\sum\limits_{i=1}^{m} x_i^2) - (\sum\limits_{i=1}^{m} x_i)^2}$$

P.326 Example 1

- Polynomial Least Squares

  The problem of approximating a set of data, $\{(x_i, y_i)|i = 1, 2, ..., m\}$ with

  $P_n(x) = a_n x^n + a_{n-1}x_{n-1} + ... + a_1 x + a_0$

  of degree $n < m - 1$ using least squares is handled in a similar manner. It requires choosing the constants a_0, a_1, ... , a_n to minimize the total least squares error:

  $E_2 = \sum_{i+1}^{m} (y_i - P_n(x_i))^2$

  For E_2 to be minimized, it is necessary that $\frac{\partial E_2}{\partial a_j} = 0$ for each $j = 0, 1, ..., n$. This gives n+1 normal equations in the n+1 unknowns, a_j,

  $$a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 + ... + a_n \sum_{i=1}^{m} x_i^n = \sum_{i=1}^{m} y_i x_i^0$$
  $$a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 + a_2 \sum_{i=1}^{m} x_i^3 + ... + a_n \sum_{i=1}^{m} x_i^{n+1} = \sum_{i=1}^{m} y_i x_i^1$$
  $$\vdots$$
  $$a_0 \sum_{i=1}^{m} x_i^n + a_1 \sum_{i=1}^{m} x_i^{n+1} + a_2 \sum_{i=1}^{m} x_i^{n+2} + ... + a_n \sum_{i=1}^{m} x_i^{2n} = \sum_{i=1}^{m} y_i x_i^n$$

  P.328 Example 2.

- Least Square of Arbitrary Function $\phi_j(x)$

  $\min_{a_j} \sum_{i=1}^{m} [y_i - a_n\phi_n(x_i) - a_{n-1}\phi_{n-1}(x_i) - ... - a_0\phi_0(x_i)]^2$

  $\frac{\partial}{\partial a_j} = 2 \sum_{i=1}^{m} [y_i - a_n\phi_n(x_i) - a_{n-1}\phi_{n-1}(x_i) - ... - a_0\phi_0(x_i)](-\phi_j(x_i)) = 0$

$$\Rightarrow \sum_{k=0}^{n} a_k[\sum_{i=1}^{m} \phi_k(x_i)\phi_j(x_i)] = \sum_{i=1}^{m} y_i\phi_j(x_i)$$

$$\begin{bmatrix} \sum_{i=1}^{m} \phi_0(x_i)\phi_j(x_i) & \sum_{i=1}^{m} \phi_1(x_i)\phi_j(x_i) & \cdots & \sum_{i=1}^{m} \phi_n(x_i)\phi_j(x_i) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_j \\ \vdots \\ \vdots \\ a_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{m} y_i\phi_0(x_i) \\ \sum_{i=1}^{m} y_i\phi_1(x_i) \\ \vdots \\ \sum_{i=1}^{m} y_i\phi_j(x_i) \\ \vdots \\ \vdots \\ \sum_{i=1}^{m} y_i\phi_n(x_i) \end{bmatrix}$$

8.3 Continuous Least Squares Approximation

- 不只是 m 個點逼近, 而是整個函數逼近, 所以並非已知 (x_i, y_i), 而是已知 $f(x)$ 與區間 $[a, b]$

- To minimize the error
  $E(a_0, a_1..., a_n) = \int_a^b (f(x) - P_n(x))^2 dx = \int_a^b (f(x) - \sum_{k=0}^{n} a_k x^k)^2 dx$
  A necessary condition for the numbers a_0, a_1,..., a_n to minimize the total error E is that
  $\frac{\partial E}{\partial a_j} = 0$ for each $j = 0, 1..., n$

We can expand the integrand in this expression to

$E = \int_a^b [f(x)]^2 dx - 2 \sum_{k=0}^{n} a_k \int_a^b x^k f(x) dx + \int_a^b (\sum_{k=0}^{n} a_k x^k)^2 dx$

so

$\frac{\partial E}{\partial a_j} = -2 \int_a^b x^j f(x) dx + 2 \sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx$

$\Rightarrow \sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = \int_a^b x^j f(x) dx$, for each $j = 0, 1, ..., n$

P.333 Example 1

缺點: 第一, x^{j+k} 這麼高的次方, 有 round off 之問題.

第二, n=2 之計算結果, n=3 不能再用.

解救法: 原本 n+1 equation 中都含有 n+1 變數,

現在希望每個 equation 中只有一個變數.

- The set of function $\{\phi_0, \phi_1 ... \phi_n\}$ is said to be linearly independent on $[a, b]$ if,

  $c_0\phi_0(x) + c_1\phi_1(x) + ... + c_n\phi_n(x) = 0$ for all $x \in [a, b]$

  only when $c_0 = c_1 = ... = c_n = 0$

- Weight function (assign varying degrees of importance to approximations on certain portions of the intervals)

  $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1, 1]$

  Suppose $\{\phi_0, \phi_1, ..., \phi_n\}$ is a set of linearly indep functions on $[a, b]$, w is a weight function for $[a, b]$, and, for $f \in C[a, b]$, a linear combination

  $P(x) = \sum_{k=0}^{n} a_k \phi_k(x)$

  is sought to minimize the error

  $E(a_0, a_1, ..., a_n) = \int_a^b w(x)[f(x) - \sum_{k=0}^{n} a_k \phi_k(x)]^2 dx$

  This problem reduces to the situation considered at the beginning of this section in the special case when $w(x) \equiv 1$ and $\phi_k(x) = x^k$.

FOC $\Rightarrow 0 = \frac{\partial E}{\partial a_j} = 2\int_a^b w(x)[f(x) - \sum_{k=0}^{n} a_k\phi_k(x)]\phi_j(x)dx$

$\Rightarrow \int_a^b w(x)f(x)\phi_j(x)dx = \sum_{k=0}^{n} a_k \int_a^b w(x)\phi_k(x)\phi_j(x)dx$ for each $j = 0, 1, ..., n$

> The set of functions $\{\phi_0, \phi_1, ..., \phi_n\}$ is said to be orthogonal for the interval $[a, b]$ with respect to the weight function w if $\int_a^b w(x)\phi_k(x)\phi_j(x)dx = \begin{cases} 0, & \text{when } j \neq k \\ \alpha_k > 0, & \text{when } j = k \end{cases}$

$\Rightarrow \int_a^b w(x)f(x)\phi_j(x)dx = a_j \int_a^b w(x)[\phi_j(x)]^2dx = a_j\alpha_j$

$\Rightarrow a_j = \frac{1}{\alpha_j}\int_a^b w(x)f(x)\phi_j(x)dx$

$(= \frac{\int_a^b w(x)f(x)\phi_j(x)dx}{\int_a^b w(x)[\phi_j(x)]^2dx})$

- <u>Recursive Generation of Orthogonal Polynomials</u>

  The set of polynomials $\{\phi_0, \phi_1, ..., \phi_n\}$ defined in the following way is linearly indep and orthogonal on $[a, b]$ with respect to the weight function w

  $\phi_0(x) \equiv 1$, $\phi_1(x) = x - B_1$

  where

  $B_1 = \frac{\int_a^b xw(x)[\phi_0(x)]^2dx}{\int_a^b w(x)[\phi_0(x)]^2dx}$

  and when $k \geq 2$

  $\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x)$ (recursive equation)

  where

  $B_k = \frac{\int_a^b xw(x)[\phi_{k-1}(x)]^2dx}{\int_a^b w(x)[\phi_{k-1}(x)]^2dx}$ and $C_k = \frac{\int_a^b xw(x)\phi_{k-1}(x)\phi_{k-2}(x)dx}{\int_a^b w(x)[\phi_{k-2}(x)]^2dx}$

  Moreover, for any polynomial $\phi_k(x)$ of degree $k < n$

  $\int_a^b w(x)\phi_n(x)\phi_k(x)dx = 0$

EX 3. $\{P_n(x)\}$ is orthogonal on [-1,1] with respect to the weight function $w(x) \equiv 1$. $P_n(1) = 1$ for each n. Using the recursive procedure, $P_0(x) \equiv 1$, so

$B_1 = \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} dx} = 0$ and $P_1(x) = (x - B_1)P_0(x) = x$

Also

$B_2 = \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx} = 0$ and $C_2 = \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} 1 dx} = \frac{1}{3}$

so

$P_2(x) = (x - B_2)P_1(x) - C_2 P_0(x) = (x-0)x - \frac{1}{3} * 1 = x^2 - \frac{1}{3}$

$B_3 = 0$, $C_3 = \frac{4}{15} \Rightarrow P_3(x) = xP_2(x) - \frac{4}{15}P_1(x) = x^3 - \frac{3}{5}x$

$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$

$P_5(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$

8.4 Chebyshev Polynomials

- $\{T_n(x)\}$ are orthogonal on $(-1, 1)$ with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$

  For x $\in$ [-1,1], define

  $T_n(x) = \cos(n \cos^{-1} x)$ for each $n \geq 0$

  $T_0(x) = \cos 0 = 1$ and $T_1(x) = \cos(\cos^{-1} x) = x$

  令$\theta = \cos^{-1} x$, $T_n(\theta(x)) \equiv T_n(\theta) = \cos(n\theta)$, where $\theta \in [0, \pi]$ (因 $x \in (-1, 1)$)

  because $T_{n+1}(\theta) = \cos(n\theta + \theta) = \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta$ and

  $T_{n-1}(\theta) = \cos(n\theta - \theta) = \cos(n\theta)\cos\theta + \sin(n\theta)\sin\theta$

  therefore $T_{n+1}(\theta) = 2\cos(n\theta)\cos\theta - T_{n-1}(\theta)$

  $\Rightarrow T_{n+1} = 2\cos(n \cos^{-1} x)x - T_{n-1}(x) = 2T_n(x)x - T_{n-1}(x)$

  Since $T_0(x)$ and $T_1(x)$ are both polynomials in x, $T_{n+1}(x)$ will be a polynomial in x for each n

  P.341 Figure 8.8

- To show the orthogonal of the Chebyshev polynomials, consider
  $\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \int_{-1}^{1} \frac{\cos(n\cos^{-1}x)\cos(m\cos^{-1}x)}{\sqrt{1-x^2}}dx$
  令 $\theta = \cos^{-1}x \Rightarrow d\theta = -\frac{1}{\sqrt{1-x^2}}dx$ (三角形圖示)
  and
  $\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = -\int_{\pi}^{0} \cos(n\theta)\cos(m\theta)d\theta = \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)d\theta$

  $\| \; cos(n\theta)cos(m\theta)d\theta = \frac{1}{2}[cos(n+m)\theta + cos(n-m)\theta]$

  $\Rightarrow \int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \frac{1}{2}\int_{0}^{\pi} \cos((n+m)\theta)d\theta + \frac{1}{2}\int_{0}^{\pi} \cos((n-m)\theta)d\theta$
  $= [\frac{1}{2(n+m)}\sin(n+m)\theta + \frac{1}{2(n-m)}\sin(n-m)\theta]_{0}^{\pi} = 0$
  Similarly, $\int_{-1}^{1} \frac{[T_n(x)]^2}{\sqrt{1-x^2}}dx = \frac{\pi}{2}$ for each $n \geq 1$

- Zeros and Extrema of Chebyshev Polynomials
  The Chebyshev polynomial $T_n(x)$, of degree $n \geq 1$, has n simple zeros in [-1,1] at
  $\overline{x}_k = \cos(\frac{2k-1}{2n}\pi)$ for each $k = 1, 2, ..., n$
  Moreover, T_n assumes its absolute extrema at
  $\overline{x}'_k = \cos(\frac{k\pi}{n})$ with $T_n(\overline{x}'_k) = (-1)^{-k}$ for each $k = 0, 1, ..., n$
  P.341 Figure 8.8

  $$\begin{array}{lll} \text{if } n=1 \Rightarrow & k=1 & \Rightarrow \overline{x}_1 = \cos\frac{\pi}{2} = 0 \\ & k=0,1 & \Rightarrow \overline{x}'_0 = \cos 0, \overline{x}'_1 = \cos\pi = -1 \\ \text{if } n=2 \Rightarrow & k=1,2 & \Rightarrow \overline{x}_1 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \overline{x}_2 = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} \\ & k=0,1,2 & \Rightarrow \overline{x}'_0 = \cos 0 = 1, \overline{x}'_1 = \cos\frac{\pi}{2} = 0, \\ & & \quad \overline{x}'_2 = \cos\pi = -1 \end{array}$$

- monic Chebyshev polynomial (使 leading coefficient = 1)

  $\tilde{T}_0 = 1$, $\tilde{T}_n = \frac{1}{2^{n-1}} T_n(x)$

  此時 zeros 與 extrem 之 $\overline{x}_k$ 與 $\overline{x}'_k$ 與原本之 Chebyshev 都一樣. 但極值會隨 $n \uparrow$ 而 $\downarrow$, $\tilde{T}_n(\overline{x}'_k) = \frac{(-1)^k}{2^{n-1}}$

  (最大最小值之變動會越來越小)

- The error form for the Lagrange polynomial applied to the interval $[-1, 1]$ states that if x_0, x_1,..., x_n are distinct numbers in the interval $[-1, 1]$ and if $f \in C^{n+1}[-1, 1]$, then, each $x \in [-1, 1]$, a number $\xi(x)$ exists in $(-1, 1)$ with

  $f(x) - P(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!}(x - x_0)...(x - x_n)$

  where $P(x)$ is the Lagrange interpolating polynomial. There is no control over $\xi(x)$, so to minimize the error by shrewd placement of the nodes x_0, x_1,..., x_n is equivalent to choosing x_0, x_1,..., x_n to minimize the quantity

  $|(x - x_0)...(x - x_n)|$

  throughout the interval $[-1, 1]$

  此時若取 x_0, x_1,..., x_n 爲 zeros of $\tilde{T}_{n+1}(x)$

  $\Rightarrow (x - \overline{x}_0)(x - \overline{x}_1) \ldots (x - \overline{x}_n) = \tilde{T}_{n+1}(x)$ (因 $\tilde{T}_{n+1}(x)$ 爲 n 次 polynomial , 且在 $\overline{x}_0, \overline{x}_1 \ldots \overline{x}_n$ 都爲0)

  且因 $\max\limits_{x \in [-1,1]} \mid \tilde{T}_{n+1} \mid = \frac{1}{2^n}$

  $\Rightarrow \max\limits_{x \in [-1,1]} \mid f(x) - P(x) \mid \leq \frac{1}{2^n (n+1)!} \max\limits_{x \in [-1,1]} \mid f^{n+1}(x) \mid$

- The technique for choosing points to minimize the interpolating error can be easily extended to a general closed interval [a,b] by using the change if variable

  $\tilde{x}_k = \frac{1}{2}[(b - a)\overline{x}_k + a + b]$

  to transform the numbers $\overline{x}_k$ in the interval $[-1, 1]$ into the corresponding numbers in the interval $[a, b]$

  P.344 Example 1, P.345 Table 8.4

8.5 Rational Function Approximation

- 一般之 polynomial 有 oscillation 之傾向, 亦即 error 會正負交錯, 雖然 error term 有 upper bound, 但通常會大於平均.

- 這節提出的方法, 是想要將 error 平均散佈在 approximation interval 中.

- A rational function r of degree N has the form

  $r(x) = \frac{p(x)}{q(x)}$

  where $p(x)$and $q(x)$are polynomials whose degree sum to N

  Consider the Pade approximation technique

  $f(x)-r(x) = f(x)-\frac{p(x)}{q(x)} = \frac{f(x)q(x)-p(x)}{q(x)} = \frac{f(x)\sum_{i=0}^{m} q_i x^i - \sum_{i=0}^{n} p_i x_i}{q(x)}$

  此時 f(x) 用 Maclurin Expansion , 且希望找 q_i, P_i, s.t. $f^{(k)}(0) - r^{(k)}(0) = 0$, for k=0,1,...,N

- Ex1. The Maclaurin series expansion for e^{-x} is $\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^i$

  To find the Pade approximation to e^{-x} of degree 5 with $n = 3$ and $m = 2$ requires choosing p_0, p_1, p_2, p_3, q_1, and q_2 so that the coefficients of x^k for $k = 0, 1, ..., 5$ are zero in the expression

  $(1-x+\frac{x^2}{2}-\frac{x^3}{6}+...)(1+q_1x+q_2x^2)-(p_0+p_1x+p_2x^2+p_3x^3)$

  解6個聯立方程式

  $\Rightarrow p_0 = 1,\ p_1 = \frac{-3}{5},\ p_2 = \frac{3}{20},\ p_3 = \frac{-1}{60},\ q_1 = \frac{2}{5}$ and $q_2 = \frac{1}{20}$

  so the Pade approximation is

  $r(x) = \frac{1-\frac{3}{5}x+\frac{3}{20}x^2-\frac{1}{60}x^3}{1+\frac{2}{5}x+\frac{1}{20}x^2}$

  P.349 Table 8.5 ($r(x)$ vs. $P_5(x)$)

雖然 Pade approximation 是根據 Taylor Series 來, 但依然增進了 accuracy.

Using nested multiplication:

$P_5(x) = ((((\frac{-1}{120}x + \frac{1}{24})x - \frac{1}{6})x + \frac{1}{2})x - 1)x + 1$

5個"×", 5個"+", "−"

$r(x) = \frac{((-\frac{1}{60}x+\frac{3}{20})x-\frac{3}{5})x+1}{(\frac{1}{20}x+\frac{2}{5})x+1}$ 5個"×", 5個"+","−", 1個"÷"

$= \frac{-1}{3}x + \frac{17}{3} + \frac{\frac{-152}{3}}{x+\frac{117}{19}+\frac{\frac{3125}{361}}{x+\frac{35}{19}}}$ 1個"×",5個"+","−", 2個"÷"

(continuous-fraction 可增進計算效率)

用 T_k 取代 Pade approximation 中的 x_k, 形成 general Chebyshev rational function.

8.6 Trigonometric Polynomial Approximation

- 適合用在周期性函數之估計.
- For each positive n, the set τ_n of trigonometric polynomials of degree less than or equal to n is the set of all linear combinations of $\{\phi_0, \phi_1, ..., \phi_{2n-1}\}$, where

  $\phi_0(x) = \frac{1}{\sqrt{2\pi}}$ (共1個)

  $\phi_k(x) = \frac{1}{\sqrt{\pi}} \cos kx$ for each $k = 1, 2, ..., n$ (共 n 個)

  and

  $\phi_{n+k}(x) = \frac{1}{\sqrt{\pi}} \sin kx$ for each $k = 1, 2, ..., n-1$ (共 n-1個)

> $\{\phi_0, \phi_1, ..., \phi_{2n-1}\}$ is orthonormal on $[-\pi, \pi]$
> with respect to the weight function $w(x) \equiv 1$
> If $k \neq j$ and $j \neq 0$
> $\int_{-\pi}^{\pi} \phi_{n+k}(x)\phi_j(x)dx$
> $= \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin kx \frac{1}{\sqrt{\pi}} \cos jxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin kx \cos jxdx$
> $=$ (by積化合差) $\frac{1}{2\pi}\int_{-\pi}^{\pi}[\sin(k+j)x + \sin(k-j)x]dx$
> $= \frac{1}{2\pi}[\frac{-\cos(k+j)x}{k+j} - \frac{\cos(k-j)x}{k-j}]_{-\pi}^{\pi} = 0$
> 其他之證明類似

Given $f \in C[-\pi, \pi]$, the continuous least squares approximation by function in τ_n is defined by

$S_n(x) = \sum_{k=0}^{2n-1} a_k \phi_k(x)$

(if $n \to \infty$, $S_n(x)$ is called the Fourier series of f)

where

$a_k = \int_{-\pi}^{\pi} f(x)\phi_k(x)dx$ for each $k = 0, 1, ..., 2n-1$

(P.337 $a_k = \frac{\int_{-pi}^{\pi} w(x)\phi_k(x)f(x)dx}{\int_{-\pi}^{\pi} w(x)[\phi_k(x)]^2dx} =$ (因爲 $w(x) = 1$) $\frac{\int_{-\pi}^{\pi} \phi_k(x)f(x)dx}{\int_{-\pi}^{\pi} [\phi_k(x)]^2dx} = \int_{-\pi}^{\pi} f(x)\phi_k(x)dx$)

$$\begin{aligned} \alpha_k &= \int_{-\pi}^{\pi} w(x)[\phi_j(x)]^2dx \\ &= \int_{-\pi}^{\pi} 1(\tfrac{1}{\sqrt{\pi}}\cos kx)^2dx \\ &= \int_{-\pi}^{\pi} \tfrac{1}{\pi}\tfrac{1}{2}(\cos 2kx + \cos 0)dx \\ &= \tfrac{1}{\pi}\tfrac{1}{2}(\tfrac{\sin 2kx}{2k} + x) \Big|_{-\pi}^{\pi} \\ &= 1 \end{aligned}$$

- Ex 1. $f(x) = |x|$ for $-\pi < x < \pi$

  $a_0 = \int_{-\pi}^{\pi} |x| \frac{1}{\sqrt{2\pi}} dx = \frac{2}{\sqrt{2\pi}} \int_0^{\pi} x dx = \frac{\sqrt{2}\pi^2}{2\sqrt{\pi}}$

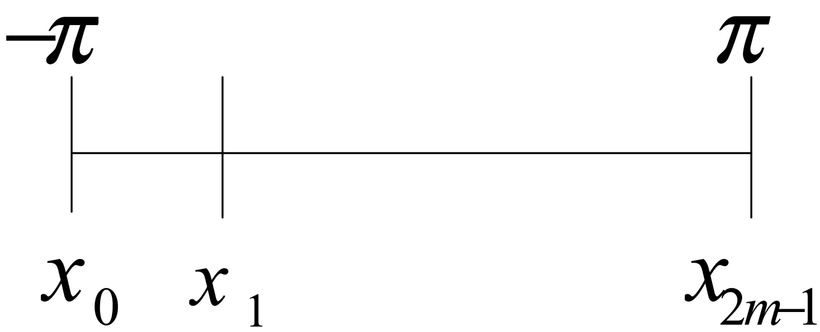
  $a_k = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} |x| \cos kx dx = \frac{2}{\sqrt{\pi}} \int_0^{\pi} x \cos kx dx$

  $= \frac{2}{\sqrt{\pi}k^2}[(-1)^k - 1]$,for each $k = 1, 2, ..., n$

  $b_k = a_{n+k} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} |x| \sin kx dx = 0$

  $\Rightarrow S_n(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=0}^{n} \frac{(-1)^k - 1}{k^2} \cos kx$

- 之前的 $S_n(x)$ 是 match 整個 $f(x)$, 但 Trigonometric Polynomial 也可用在 discrete varsion least square (意即 match 點, 而非 match $f(x)$)



$\Rightarrow x_j = -\pi + \frac{j}{m}\pi$, for $j = 0, 1, \ldots, 2m-1$

- 取 $S_n(x) = \sum\limits_{k=0}^{2n-1} a_k \phi_k(x)$

  其中 $\hat{\phi}_0(x) = \frac{1}{2}$

  $\hat{\phi}_k(x) = \cos kx,\ k = 1, 2, \ldots, n$

  $\hat{\phi}_{n+k}(x) = \sin kx,\ k = 1, 2, \ldots, n-1$

- $\min\limits_{a_k, b_k} \sum\limits_{j=0}^{2m-1} \{y_j - [\frac{a_0}{2} + a_n \cos nx_j + \sum\limits_{k=1}^{n-1} (a_k \cos kx_j + b_k \sin kx_j)]\}^2$

  因 equally space in $[-\pi, \pi]$

  $\Rightarrow \sum\limits_{j=0}^{2m-1} \hat{\phi}_k(x_j) \hat{\phi}_j(x_j) = 0$

  $\Rightarrow$ orthogonal set of functions

  $\Rightarrow a_k = \dfrac{\sum\limits_{j=0}^{2m-1} \phi_k(x_j) f(x_j)}{\sum\limits_{j=0}^{2m-1} \phi_k(x_j)^2}$ (其中 $\sum\limits_{j=0}^{m} \phi_k(x_j)^2 = m$)

$\Rightarrow a_k = \frac{1}{m} \sum\limits_{j=0}^{2m-1} y_i \cos kx_j$

$b_k = \frac{1}{m} \sum\limits_{j=0}^{2m-1} y_i \sin kx_j$

P.356 Example 2.

8.7 Fast Fourier Transforms

- 加速上小節用 Trigonometric Polynomial 來做 discrete version least square approximation.

- $S_n(x) = \frac{a_0}{2} + a_n \cos nx + \sum_{k=1}^{n-1}(a_k \cos kx + b_k \sin kx)$

  where

  $a_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j \cos kx_j$ for $k = 0, 1, ..., n$

  and

  $b_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j \sin kx_j$ for $k = 1, ..., n-1$

  Use the form with $n = m$ for interpolation if we make a minor modification. Replacing the term a_m with $\frac{a_m}{2}$

  $\Rightarrow S_m(x) = \frac{a_0 + a_m \cos mx}{2} + \sum_{k=1}^{m-1}(a_k \cos kx + b_k \sin kx)$

  The nodes are given, for each $j = 0, 1, ..., 2m-1$, by

  $x_j = -\pi + (\frac{j}{m})\pi$

  and the coefficients, for each $k = 0, 1, ..., m$, as

  $a_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j \cos kx_j$ and $b_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j \sin kx_j$

  用 Trigonometric Polynomial, 需 $2m$ 個點 $\Rightarrow$ 計算量太多, 且 round off error 也太大.

  Instead of directly evaluating the constants a_k and b_k, the Fast Fourier Transform (FFT) procedure computes the complex coefficients c_k in the formula

  $F(x) = \frac{1}{m}\sum_{k=0}^{2m-1} c_k e^{ikx}$

  where

$c_k = \sum_{j=0}^{2m-1} y_j e^{\frac{\pi ijk}{m}}$ for each $k = 0...2m-1$

$\|$ 因爲 $e^{iz} = \cos z + i \sin z$

$$\Rightarrow \frac{1}{m} c_k e^{-i\pi k} = \frac{1}{m} \sum_{j=0}^{2m-1} y_j e^{\pi ijk/m} e^{-i\pi k}$$
$$= \frac{1}{m} \sum_{j=0}^{2m-1} y_j e^{ik(-\pi + (\pi j/m))}$$
$$= \frac{1}{m} \sum_{j=0}^{2m-1} y_j (\cos kx_j + i \sin kx_j)$$
$$\Rightarrow \frac{1}{m} c_k e^{-i\pi k} = a_k + ib_k$$

- Ex 1. $m = 2$ and $x_j = -\pi + (\frac{j}{2})\pi$ for $j = 0, 1, 2, 3$
  The polynomial is given by
  $S_2(x) = \frac{a_0 + a_2 \cos 2x}{2} + a_1 \cos x + b_1 \sin x$
  where the coefficient are
  $a_0 = \frac{1}{2}[y_0 \cos 0 + y_1 \cos 0 + y_2 \cos 0 + y_3 \cos 0]$
  $a_1 = \frac{1}{2}[y_0 \cos x_0 + y_1 \cos x_1 + y_2 \cos x_2 + y_3 \cos x_3]$
  $a_2 = \frac{1}{2}[y_0 \cos 2x_0 + y_1 \cos 2x_1 + y_2 \cos 2x_2 + y_3 \cos 2x_3]$
  $b_1 = \frac{1}{2}[y_0 \sin x_0 + y_1 \sin x_1 + y_2 \sin x_2 + y_3 \sin x_3]$

- 若用 FFT 來看:
  $c_0 = y_0 e^0 + y_1 e^0 + y_2 e^0 + y_3 e^0$
  $c_1 = y_0 e^0 + y_1 e^{\frac{\pi i}{2}} + y_2 e^{\pi i} + y_3 e^{\frac{3\pi i}{2}}$
  $c_2 = y_0 e^0 + y_1 e^{\pi i} + y_2 e^{2\pi i} + y_3 e^{3\pi i}$
  $c_3 = y_0 e^0 + y_1 e^{\frac{3\pi i}{2}} + y_2 e^{3\pi i} + y_3 e^{\frac{9\pi i}{2}}$
  and $a_k + ib_k = \frac{1}{2} c_k e^{-i\pi k}$
  $a_0 = \frac{1}{2}\text{Re}(c_0)$

$a_1 = \frac{1}{2}\text{Re}(c_1 e^{-\pi i})$

$a_2 = \frac{1}{2}\text{Re}(c_2 e^{-2\pi i})$

$b_1 = \frac{1}{2}\text{Im}(c_1 e^{-\pi i})$

- Discrete Least Square Approximation

  Given (x_i, y_i), $i = 1, \ldots, m$

  (i) 用 $Y = a_1 X + a_0$

  (ii) 用 $Y = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$

  (iii) 用 $Y = a_n \phi_n(X) + a_{n-1}\phi_{n-1}(X) + \cdots + a_1\phi_1(X) + a_0\phi_0(X)$

  用 $n+1$ 個函數來逼近 $j = 0, \ldots, n$

  $\min\limits_{a_j, j=0,\ldots,n} \sum\limits_{i=1}^{m} [y_i - a_n\phi_n(x_i) - a_{n-1}\phi_{n-1}(x_i) - \cdots - a_j\phi_j(x_i) - \ldots - a_0\phi_0(x_i)]^2$

  $\frac{\partial}{\partial a_j} = 2\sum\limits_{i=1}^{m} [y_i - a_n\phi_n(x_i) - a_{n-1}\phi_{n-1}(x_i) - \cdots - a_j\phi_j(x_i) - \ldots - a_0\phi_0(x_i)][-\phi_j(x_i)] = 0$

  $\Rightarrow \sum\limits_{k=0}^{n} a_k [\sum\limits_{i=1}^{m} \phi_k(x_i)\phi_j(x_i) = \sum\limits_{i=1}^{m} y_i\phi_j(x_i)$

$$\begin{bmatrix} \sum\limits_{i=1}^{m} \phi_0(x_i)\phi_j(x_i) & \cdots & \cdots & \sum\limits_{i=1}^{m} \phi_n(x_i)\phi_j(x_i) \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_j \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum\limits_{i=1}^{n} y_i\phi_0(x_i) \\ \vdots \\ \sum\limits_{i=1}^{n} y_i\phi_j(x_i) \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

- Continuous Least Square Approximation

  Given $f(x)$ on $[a, b]$

(i) 用 n 次 polynomial 作 Approximation.

$$\left(\begin{array}{ll} \text{可用, 但缺點} & \text{1. n=2 之計算, n=3 不行用} \\ & \text{2. n 增加, 矩陣大, 不好解} \end{array} \right)$$

解救法: 原本 n+1 equation 中都含有 n+1 變數,
現在希望每個 equation 中只有一個變數.

(ii) 用 orthogonal functions 即可達成
(Linearly independent sets of polynomials)

$$\text{orthogonal functions} \left\{ \begin{array}{l} \text{Chebyshev Polynomials} \\ \text{Trignometric functions } \tau_n \\ \text{若 } n \to \infty \longrightarrow \text{Fourier Series} \end{array} \right.$$

- Rational Function Approximation
  (可改善 polynomail 之 oscillation error)
  可改善 Taylor series 之 error term (使其更小).