數値分析

Chapter 7

Iterative Methods for Solving Linear System

- Iterative method (相對於 direct method): 解不精準, 但有效率 (not exact, but efficient), 可減少 computation 與 round off error.

- 因爲要判定兩個 vector 是否夠接近, 所以先介紹一些 linear algebra, 來衡量 distance between vectors and matrices.

- <u>vector norm on R^n</u>

  A vector norm on R^n is a function, $\|\cdot\|$, from R^n into R with the following properties:

  (i) $\|\mathrm{x}\| \geq 0$ for all $\mathrm{x} \in R^n$

  (ii) $\|\mathrm{x}\| = 0 \Leftrightarrow \mathrm{x} = (0, \ldots, 0)^t \equiv 0$

  (iii) $\|\alpha\mathrm{x}\| = |\alpha|\|\mathrm{x}\|$ for all $\alpha \in R$ and $\mathrm{x} \in R^n$

  (iv) $\|\mathrm{x}+\mathrm{y}\| \leq \|\mathrm{x}\| + \|\mathrm{y}\|$ for all x, x $\in R^n$

- Two types of norms on R^n:

  $\|\mathrm{x}\|_2 = \{\sum_{i=1}^{n} x_i^2\}^{1/2}$

  $\|\mathrm{x}\|_\infty = \max_{1\leq i\leq n} |x_i|$

- Distance between two vectors $\|\mathrm{x}-\mathrm{y}\|_2$ and $\|\mathrm{x}-\mathrm{y}\|_\infty$

- 無論用 $\|\mathrm{x}\|_\infty$ 或是 $\|\mathrm{x}\|_2$ 來看收斂是一樣的:

  $\|\mathrm{x}\|_\infty = \max_{i=1,\ldots,n} |x_i| = |x_j|$

  因爲 $\|\mathrm{x}\|_\infty^2 = |x_j|^2 = x_j^2 \leq \sum_{i=1}^{n} x_i^2 = \|\mathrm{x}\|_2^2$

  $\Rightarrow l_2$ 收斂 imply l_∞ 收斂.

  因爲 $\|\mathrm{x}\|_2^2 = \sum_{i=1}^{n} x_i^2 \leq \sum_{i=1}^{n} x_j^2 = nx_j^2 = n\|\mathrm{x}\|_\infty^2$

  $\Rightarrow l_\infty$ 收斂 imply l_2 收斂.

- Matrix norm

  $\|A\|$: $R^{2n} \to R$

  要滿足

  (i) $\|\mathrm{A}\| \geq 0$

  (ii) $\|\mathrm{A}\| = 0$, if and only if A is O, the matrix with all zero entries

  (iii) $\|\alpha A\| = |\alpha|\|A\|$

  (iv) $\|A + B\| \leq \|A\| + \|B\|$

  (v) $\|AB\| \leq \|A\|\|B\|$ (比起 vector norm 多了這項條件)

- Natural Matrix Norm

  $\|A\| = \max\limits_{\|\mathrm{x}\|=1} \|A\mathrm{x}\|$

- l_∞ Norm

  $\|A\|_\infty = \max\limits_{\|\mathrm{x}\|_\infty=1} \|A\mathrm{x}\|_\infty$

  (找出所有 $\|\mathrm{x}\|_\infty = 1$ 之 x 來試找最大的 $\|A\mathrm{x}\|_\infty$)

- l_2 Norm

  $\|A\|_2 = \max\limits_{\|\mathrm{x}\|_2=1} \|A\mathrm{x}\|_2$

  P.293 Figure 7.3 and 7.4

- In fact, $\|A\|_\infty = \max\limits_{\|\mathrm{x}\|_\infty=1} \|A\mathrm{x}\|_\infty = \max\limits_{1\leq j\leq n} \sum\limits_{j=1}^{n} |a_{ij}|$

  (把 row 作絕對值加總後選最大的值 $= \|A\|_\infty$)

  (因爲要找所有 $\|X\|_\infty = 1$ 之 X 來試, 所以找

  $X = (\pm 1, \pm 1, \cdots \pm 1)^t$ 來試即可, 共 2^n 種可能)

  (此時之 $\|AX\|_\infty$ 會比 X 用 $(\pm 1, 0.8, -0.3, \cdots, 0.2)$

  來的大)

7.3 Eigenvalues and Eigenvectors

- $\|A - \lambda I\| = 0$

  $A\mathrm{x} = \lambda \mathrm{x} \Rightarrow (A - \lambda I)\mathrm{x} = 0$

  $\Rightarrow$ 滿足上式之 x is called an eigenvector of A corresponding to the eigenvalue λ

  See Figure 7.5

  P.297 Example 1

- if $A\mathrm{x} = \lambda \mathrm{x}$

  $\Rightarrow \mathrm{k}A\mathrm{x} = \mathrm{k}\lambda \mathrm{x}$

  $\Rightarrow A(\mathrm{kx}) = \lambda(\mathrm{kx})$

  $\Rightarrow$ 若 x 是 eigenvector, kx 還是 eigenvector.

- if $A - \lambda I$ is nonsigular, 亦即 $\det(A - \lambda I) \neq 0$

  $\Rightarrow$ 有唯一解, 此解爲 x = 0

- if $A - \lambda I$ is sigular, 亦即 $\det(A - \lambda I) = 0$

  $\Rightarrow$ 有很多個解

  $\Rightarrow$ 有除了 x = 0 之外的解

- <u>Spectral radius</u>:

  $\rho(A) = \max \|\lambda\|$, where λ is an eigenvalue of A.

- In fact, $\|A\|_2 = \max\limits_{\|\mathrm{x}\|_2=1} \|A\mathrm{x}\|_2 = [\rho(A^t A)]^{\frac{1}{2}}$

- P.300 Example 3, 算 $\|A\|_\infty = 4$, $\|A\|_2 = 3.106$

- Convergent Matrix Equivalences

  (a) A is a convergent matrix

  (b) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm

  (c) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms

  (d) $\rho(A) < 1$ (P.301 Example 4, $\pi = \frac{1}{2}$)

  (e) $\lim_{n\to\infty} A^n\mathrm{x} = 0$, for every x

7.4 The Jacobi and Gauss-Seidel Methods

- Classic Methods used in problems where the matrix is large and has mostly zero entries

- Convergence and the Spectral Radius

  Ax=b

  $\Rightarrow (D - L - U)\mathrm{x} = b$

  $\Rightarrow D\mathrm{x} = (L + U)\mathrm{x} + b$

  $\mathrm{x} = D^{-1}(L + U)\mathrm{x} + D^{-1}b$

  $\Rightarrow \mathrm{x}^{(k)} = T\mathrm{x}^{(k-1)} + c$

  (converge to the unique solution iff $\rho(T) < 1$)

  P.304 Example 1. (Jacobi iterative method)

- A reordering of the equation is performed so that no $a_{ii} = 0$

- To speed convergence, the equations should be arranged so that a_{ii} is as large as possible.

- P.306 Example 2, Gauss-Seidel iterative technique

  (算過之 x_i^k, 取代 x_i^{k-1})

  比較 P.304 跟 P.307 的 $x_1^{(k)}, x_2^{(k)}, \ldots$

- <u>Gauss-Seidel Method</u>

  To write the Gauss-Seidel method in matrix form, multiply both sides of Eq.(7.2) by a_{ii} and collect all kth iterate terms, to give

  $a_{i1}x_1^k + a_{i2}x_2^k + .. + a_{ii}x_i^k = -a_{i,i+1}x_{i+1}^{k-1} - ... - a_{i,n}x_n^{k-1} + b_i$

  for each $i = 1, 2, ..., n$. Writing all n equations gives

$$
\begin{aligned}
a_{11}x_1^{(k)} &= -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1 \\
a_{21}x_1^{(k)} + a_{22}x_2^{(k)} &= -a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2 \\
&\vdots \\
a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{nn}x_n^{(k)} &= b_n
\end{aligned}
$$

  $A\text{x} = b$

  $(D - L - U)\text{x} = b$

  $(D - L)\text{x} = U\text{x} + b$

  $\Downarrow$ 假設 $(D - L)^{-1}$ 存在

  $\text{x} = (D - L)^{-1}U\text{x} + (D - L)^{-1}b = T_g\text{x} + c_g$

- Gauss-Seidel 一般來說比 Jacobi 好, 但可能有時 Jacobi converge 但 Gauss-Seidel 不 converge.

- 當 A 是 strictly diagonally dominant (nonsigular), 可保證此兩方法有唯一解.

7.5 The SOR Method (Successive Over-Relaxation)

- It uses a scaling factor to more rapidly reduce the approximation error.

- $x_i^{(k)} = (1-w)x_i^{(k-1)} + \frac{w}{a_{ii}}[b_i - \underbrace{\sum_{j=1}^{i-1} a_{ij}x_j^{(k)}}_{\text{之前更新過的}} - \underbrace{\sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}}_{\text{未更新過的}}]$

$w=1$	$\Rightarrow$	Gauss-Seidel
$0<w<1$	$\Rightarrow$	under-relaxation method (可用在一些 Gauss-Seidel 不收斂之情況)
$w>1$	$\Rightarrow$	over-relaxation method (用來加速收斂)

$\Rightarrow a_{ii}x_i^{(k)} + w\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} = (1-w)a_{ii}x_i^{(k-1)} - w\sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + wb_i$

$\Rightarrow (D-wL)\mathrm{x}^{(k)} = ((1-w)D + wU)\mathrm{x}^{(k-1)} + w\mathrm{b}$

$\Rightarrow \mathrm{x}^{(k)} = T_w\mathrm{x}^{(k-1)} + c_w$

where $T_w = (D-wL)^{-1}[(1-w)D + wU]$

$c_w = w(D-wL)^{-1}\mathrm{b}$

P.310 Ex1. SOR 比較好, 因爲在 $k=7$ 時已經很接近眞值.

- 如何選擇 w

(1) A 是正定 and $0<w<2 \Rightarrow$ 一定收斂

(2) A is tridiagonal,

此時 $\rho(T_g) = [\rho(T_j)]^2$, $w = \frac{2}{1+\sqrt{1-[\rho(T_j)]^2}} = 1 - \rho(T_w)$

其中 T_j 爲 Jacobi 之 transition matrix,

T_g 爲 Gauss-Seidel 之 transition matrix,

T_w 爲 SOR 之 transition matrix.

P.311 Example 2

7.6 Error Bounds and Ierative Refinement

- 怎麼去判斷所解的 x 是否爲好答案?
  一般都用 residual norm 來看誤差, 原本當 $\|b - A\tilde{x}\|$ 很小時, $\tilde{x} \to x$, 但未必如此
  P.314 Example 1

- Residual Vector Error Bounds
  $Ax = b \Rightarrow \tilde{x}$
  因爲 $A(x - \tilde{x}) = Ax - A\tilde{x} = b - A\tilde{x}$
  $\Rightarrow \|x - \tilde{x}\| \leq \|b - A\tilde{x}\|\|A^{-1}\|$ (絕對誤差)
  and $\frac{\|x-\tilde{x}\|}{\|Ax\|} \leq \|A^{-1}\|\frac{\|b-A\tilde{x}\|}{\|b\|}$ (相對誤差)
  $\Rightarrow \frac{\|x-\tilde{x}\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|b-A\tilde{x}\|}{\|b\|}$
  (where $\|A\|\|A^{-1}\|$ 通常$= 1$, 若比值很大, 表相對誤差很大)
  P.316 Example 2
  $k(A) = \|A\|_\infty\|A^{-1}\|_\infty = 6.002 > 1$

- iterative refinement (or iterative improvement) (從 residual 中再拿出一些有用的資訊)
  $Ax = b \Rightarrow \tilde{x}$
  $b - A\tilde{x} = r$
  $Ay = r \Rightarrow \tilde{y}$
  $\tilde{y} = A^{-1}r = A^{-1}(b - A\tilde{x}) = A^{-1}b - \tilde{x} = x - \tilde{x}$
  $\Rightarrow x \approx \tilde{x} + \tilde{y}$
  P.317 Example 3