數値分析

Chapter 4

Numerical Integration and Differentiation

4.2 Basic Quadrature Rules

- Newton interpolatory divided-difference formula

  $f(x) \approx P_{0,1,\cdots,n}(x)$

  $= f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots + f[x_0,x_1,\cdots,x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1}) + \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$

- $\int_a^b f(x)dx \approx \int_a^b P_0(x)dx = \int_a^b f[x_0]dx = f[x_0](b-a)$, 若選 $x_0 = \frac{a+b}{2}$, 可使 $f[x_0,x_1](x-x_0)$ 在 $[a,b]$ 之積分爲 0, 亦即使得 truncated error 有效降低.

  $\Rightarrow$ midpoint rule:

  $\int_a^b f(x)dx = (b-a)f(\frac{a+b}{2}) + \frac{f''(\xi)}{24}(b-a)^3$

- $\int_a^b f(x)dx \approx \int_a^b P_{0,1}(x)dx = \int_a^b [f[x_0] + f[x_0,x_1](x-x_0)]dx$, 若取 $x_0 = a$, $x_1 = b$

  $\Rightarrow f[x_0]x + \frac{f(b)-f(a)}{b-a}\frac{(x-a)^2}{2} \,|_a^b = (b-a)\frac{f(a)+f(b)}{2}$

  , 形成 trapezoidal rule:

  $\int_a^b f(x)dx = (b-a)\frac{f(a)+f(b)}{2} - \frac{f''(\xi)}{12}(b-a)^3$

- $\int_a^b f(x)dx \approx \int_a^b P_{0,1,2}(x)dx$

  $= \int_a^b [f(a) + f[a,\frac{a+b}{2}](x-a) + f[a,\frac{a+b}{2},b](x-a)(x-\frac{a+b}{2})]dx$

  , with $x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$

  $\Rightarrow$ Simpson's 1/3 rule:

  $\int_a^b f(x)dx = \frac{(b-a)}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] - \frac{f^{(4)}(\xi)}{2880}(b-a)^5$

  , P.113 Example 1.

  equally space $\Rightarrow$

  $\int_a^b f(x)dx = \frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)] - \frac{f^{(4)}(\xi)}{90}h^5$

, with $h = \frac{b-a}{2}$

- Simpson's 3/8 Rule

  $\int_a^b f(x)dx$

  $= \frac{3h}{8}[f(a) + 3f(a+h) + 3f(a+2h) + f(b)] - \frac{3}{80}f^{(4)}(\xi)h^5$

  , with $h = \frac{b-a}{3}$

4.3 Composite Quadrature Rules

- P.116 Example 1, (i) $0 \sim 2$ 用一次 Simpson's 1/3 rule, (ii) $0 \sim 1$, $1 \sim 2$ 各用一次 Simpson's 1/3 rule, (iii) $0 \sim 0.5$, $0.5 \sim 1$, $1 \sim 1.5$, $1.5 \sim 2$ 各用一次 Simpson's 1/3 rule.

- 取 n 爲偶數, 將 $[a,b]$ 分成 n 個 subinterval, 每兩個 interval 用一個 Simpson's 1/3 rule.

  $\int_a^b f(x)dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)dx$

  $= \sum_{j=1}^{n/2} \{\frac{h}{3}[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})] - \frac{f^{(4)}(\xi)}{90}h^5\}$

  $= \frac{h}{3}[f(x_0) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n)] - \frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi)$

- $\int_a^b f(x)dx$ 的誤差項爲 $E(f) = -\frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j)$

  Since $\min_{x\in[a,b]} f^{(4)}(x_j) \le f^{(4)}(\xi_j) \le \max_{x\in[a,b]} f^{(4)}(x_j)$

  we have $\min_{x\in[a,b]} f^{(4)}(x_j) \le \frac{2}{n}\sum_{j=1}^{n/2} f^{(4)}(\xi_j) \le \max_{x\in[a,b]} f^{(4)}(x_j)$

  $\Rightarrow \exists \mu \in (a,b), f^{(4)}(\mu) = \frac{2}{n}\sum_{j=1}^{n/2} f^{(4)}(\xi_j)$

Since $h = \frac{b-a}{n} \Rightarrow E(f) = -\frac{(b-a)h^5}{180} f^{(4)}(\mu)$
P.120 Ex 2 Composite Simpson's Rule 的 n 只要 ≥ 18 即可使誤差小於 0.00002, 而 Composite Trapezoidal rule 的 n 卻要 ≥ 360 才會使誤差小於 0.00002.

4.4 Romberg Integration

- Composite Trapezoidal + Richardson Extrapolation
  因爲爲了加速收斂, 常使用外插法.

- Exact value = M
  $M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + ...(*)$
  爲了更精確, $M = N(\frac{h}{2}) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + ...(**)$
  2(**)-(*) $\Rightarrow$
  $M = [N(\frac{h}{2}) + (N(\frac{h}{2}) - N(h))] + K_2(\frac{h^2}{2} - h^2) + K_3(\frac{h^3}{4} - h^3) + ...$
  Define $N_1(h) \equiv N(h)$ and $N_2(h) = N_1(\frac{h}{2}) + [N_1(\frac{h}{2}) - N_1(h)]$
  Then we have an $O(h^2)$ approximation formula for M:
  $M = N_2(h) - \frac{K_2}{2} h^2 - \frac{3K_3}{4} h^3 - ....$ (更接近 M)
  We have an $O(h^j)$ approximation of the form
  $N_j(h) = N_{j-1}(\frac{h}{2}) + \frac{N_{j-1}(\frac{h}{2}) - N_{j-1}(h)}{2^{j-1}-1}$ (通式)

- Composite Trapezoidal rule:
  切 2^{k-1} 等份得 $R_{k,1}$ approximation
  $= \frac{h_k}{2}[f(a) + f(b) + 2(\sum_{i=1}^{2^{k-1}-1} f(a + ih_k))]$

- recursive to generate $R_{k,1}$
  $R_{k,1} = \frac{1}{2}[R_{k-1,1} + h_{k-1} \sum_{i=1}^{2^{k-2}} f(a + (2i-1)h_k)]$

- $\int_a^b f(x)dx - R_{k,1} = \sum_{i=1}^{\infty} k_i h_k^{2i} = k_1 h_k^2 + \sum_{i=2}^{\infty} k_i h_k^{2i}$...(*)
  $\int_a^b f(x)dx - R_{k+1,1} = \sum_{i=1}^{\infty} k_i h_{k+1}^{2i} = \sum_{i=1}^{\infty} \frac{k_i h_k^{2i}}{2^{2i}} = \frac{k_1 h_k^2}{4} + \sum_{i=2}^{\infty} \frac{k_i h_k^{2i}}{4^i}$...(**)
  (只有偶數次之 error 是因爲, 原本之 Trapezoidal rule 中, error $= -\frac{f''(\xi)}{12}(b-a)^3 + \cdots + (b-a)^5$, 其中只有奇數次方, 但在 composite 後 error $= -\frac{f''(\mu)}{12}(b-a)h^2 + \cdots + (b-a)h^4 + \cdots$)

- 4(**)-(*)
  $\Rightarrow 3\int_a^b f(x)dx - 4R_{k+1,1} + R_{k,1} = 4\sum_{i=2}^{\infty} \frac{k_i h_k^{2i}}{4^i} - \sum_{i=2}^{\infty} k_i h_k^{2i}$
  $\Rightarrow \int_a^b f(x)dx = \frac{4R_{k+1,1} - R_{k,1}}{3} + \sum_{i=2}^{\infty} \frac{k_i}{3}(\frac{h_k^{2i}}{4^{i-1}} - h_k^{2i})$
  $\Rightarrow R_{k,2} = R_{k,1} + \frac{R_{k,1} - R_{k-1,1}}{3}$
  $\Rightarrow R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1}-1}$
  P.131 Table 4.4, Example 2, Table 4.5.

4.5 Gaussian Quadrature

- 在區間中, 找出點, 使求出答案, 不一定要端點.
  Fig 4.10 用 $[x_1, x_2]$ 會比用 $[a, b]$ 好

- 希望找到 (c_i, x_i), s.t. $\int_a^b f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$
  所以總共需要 $2n$ 個參數, 若 $f(x)$ 爲 polynomial, 且 $\deg(g) \leq 2n-1$, 可以算出完全正確的答案.

- P.136 Example 1.

- c_i 與 x_i 之選擇
  由 Legendre polynomial (orthogonal in $[-1,1]$) 之 roots 來求 x_i
  $P_0(x)=1, P_1(x)=x, P_2(x)=x^2-\frac{1}{3}, P_3(x)=x^3-\frac{3}{5}x, P_4(x)=x^4-\frac{6}{7}x^2+\frac{3}{35},\ldots$
  $c_i=\int_{-1}^{1}\pi_{j=1,j\neq i}^{n}\frac{x-x_j}{x_i-x_j}dx$
  P.137 Table 4.6

- 由 Table 4.6 可以發現 $\sum\limits_{i=1}^{n}c_{n,i}\approx 2 \Rightarrow c_{n,i}$ 可想成 $f(r_{n,i})$ 之寬度.

- 若是 $[a,b]$, 而非 $[-1,1]$, 可用 $t=\frac{2x-a-b}{b-a}$
  $\Rightarrow \int_a^b f(x)dx=\frac{b-a}{2}\sum\limits_{j=1}^{n}c_{n,j}f(\frac{(b-a)r_{n,j}+b+a}{2})$
  P.138 Example 2.

- P.138 Ex 2. n 越大, 越精確. 與 Gaussian 比較, Simpson 的誤差很大.

  (John Hull-Ch11)
  What is Wiener Process (Brownian Motion)?
  短暫時間的變化 (影響變化的因素只有一個: 時間)

  $x\sim N(0,\sigma_x^2)$
  $x+x\sim N(0,2\sigma_x^2)$
  $2x\sim N(0,4\sigma_x^2)$

- $\Delta Z = \varepsilon\sqrt{\Delta t}$
  因爲 $\varepsilon \sim N(0,1)$
  所以 $\Delta Z \sim N(0, \Delta t)$
  $E[\Delta Z] = 0$, $\text{Var}(\Delta Z) = \Delta t$
  (描述 ΔZ 的變動不是重點, 而是要知道累積的 ΔZ, 也就是說
  Z(T)-Z(0)=$\sum\limits_{i=1}^{N} \varepsilon_i\sqrt{\Delta t} = \sum\limits_{i=1}^{N} \Delta Z_i$)
  $Z(T) - Z(0)$ 是隨時間改變, ΔZ_i 的加總

- 目標:
  要知道平均位移:
  E[Z(T)-Z(0)]=0
  要知道平均 Var:
  Var(Z(T)-Z(0))=NΔt=T
  意思: 隨時間越久, Var 越大

- Generalized W Process
  $dx = a * dt + b * dz \sim N(adt, b^2 dt)$
  where adt is drift and $b^2 dt$ is volatility

- Itô Process(Difussion Process)
  假設: $dx = a(x,t)dt + b(x,t)dz$(**)[a, b 是函數]
  對股價做 $\frac{ds}{s} = \mu dt + \sigma dz$ 的成長率假設
  $\sim N(\mu dt, \sigma^2 dt)$
  (股價報酬率是 N 分布) (股價是 lognormal 分布)
  $\Rightarrow ds = \mu s dt + \sigma s dz$ (類似 (**) 的寫法)
  證明股價是 lognormal 分布 (其實並非正確的) 證明
  $\frac{dlns}{ds} = \frac{1}{s} \rightarrow dlns = \frac{ds}{s}$
  所以股價報酬率是 N 分布, 股價是 lognormal 分布

- Hull P.247 (12.22式)
  $C = e^{-rT}\widehat{E}(\max(S_T - k, 0))$
  ∧: 身處風險中立情況下
  [call的價值 = 期望後折現]
  爲什麼要有 "∧"
  (股票 vs 無風險)
  risk averse: 希望出像集中⇒現實中, 有風險的東西就有風險貼水⇒報酬率高
  risk loving: 希望出像分散⇒喜歡股票
  風險中立⇒股票跟無風險都一樣好⇒股票不需要超額報酬
  在風險中立下, $\frac{ds}{s} = rdt + \sigma dz$

- How to 處理 f(S_T), S_T之機率密度函數 (S_T 爲 lognormal distribution)?
  P.262
  $\ln V \sim N(m, s)$, 求 $E[\max(V - K, 0)]$
  引進 $Q = \frac{\ln V - m}{s}$ (變數轉換)
  where $Q \sim N(0, 1)$
  What's m? What's s?
  P.235 (12.3式) (一般世界, 所以 μ 要改成r) Itô Lemma (也就是泰勒展開式)
  (2 dimension)$f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{1}{2}(\frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(y - y_0)^2) + \ldots$
  If $\frac{ds}{s} = \mu dt + \sigma dz \Rightarrow df(s, t) =?$
  [$f(s, t)$ 在極短時間內的波動]
  $df = f(x, y) - f(x_0, y_0) = \frac{\partial f}{\partial s}(ds) + \frac{\partial f}{\partial t}(dt) + \frac{1}{2}(\frac{\partial^2 f}{\partial s^2}(ds)^2 + 2\frac{\partial^2 f}{\partial s \partial t}(ds)(dt) + \frac{\partial^2 f}{\partial t^2}(dt)^2) + \ldots$
  $\Rightarrow$

$df(s,t) = \frac{\partial f}{\partial s}(\mu s dt + \sigma s dz) + \frac{\partial f}{\partial t}dt + \frac{1}{2}(\frac{\partial^2 f}{\partial s^2}(\mu s dt + \sigma s dz)^2 + \ldots)$

$$\left\| \begin{array}{l} dz = \varepsilon\sqrt{dt} \\ \Rightarrow (dz)^2 = (\varepsilon\sqrt{dt})^2 \approx dt \\ \Rightarrow (\sigma s dz)^2 = \sigma^2 s^2 dt \end{array} \right.$$

(Rule: dt 很小$\Rightarrow$ $(dt)^2$ 更小 $\Rightarrow$ ignore)

(次方大於1.5都 ignore)

$\Rightarrow df(s,t) = (\frac{\partial f}{\partial s}\mu s + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2)dt + \frac{\partial f}{\partial s}\sigma s dz$

令 $f(s,t) = \ln s$

$\Rightarrow df(s,t) = (\frac{1}{s}\mu s - \frac{1}{2s^2}\sigma^2 s^2)dt + \frac{1}{s}\sigma s dz$

$d\ln s = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz$

$\Rightarrow \ln S(T) - \ln(S_0) \sim N((\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T)$

4.6 Adaptive Quadrature

- 萬一函數在某一 interval 較平滑, 某一 interval 變動較大, 則 equally-sized 會不適合, 因變動大的部分, error term 之微分值可能很大或很小, 造成整個的 error bound 上升.

- Fig 4.13
  在變動劇烈的地方$\Rightarrow$切割細
  在變動不劇烈的地方$\Rightarrow$切割不細
  $\int_a^b f(x)dx = S(a,b) - \frac{(b-a)^5}{2880}f^{(4)}(\xi)$
  (where $\frac{(b-a)^5}{2880}$ 改爲 $\frac{h^5}{90}$)
  ($S(a,b)$ 爲 Simpson's 1/3 rule)
  $\approx S(a,\frac{a+b}{2}) + S(\frac{a+b}{2},b) - \frac{1}{16}(\frac{h}{90})^5 f^{(4)}(\tilde{\xi})$
  ($S(a,\frac{a+b}{2}) + S(\frac{a+b}{2},b)$ 爲 composite Simpson's rule)
  假設 $f^{(4)}(\xi) = f^{(4)}(\tilde{\xi})$, 如果 ξ 與 $\tilde{\xi}$ 很接近, 則之後之推導會很精確.

$\Rightarrow \frac{h^5}{90} f^{(4)}(\xi) \approx \frac{16}{15}[S(a,b) - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b)]$

$\Rightarrow$

$| \int_a^b f(x)dx - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b) | \approx \frac{1}{15} | S(a,b) - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b) |$

where $| \int_a^b f(x)dx - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b) | \leq \varepsilon$

$| S(a,b) - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b) | \leq 15\varepsilon$(iterative error term)

- P.143 Example 1

  If 需要 error $\leq \varepsilon$

  (i) 先看 $S(a,b)$ 與 $S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b)$ 之差距, 如果夠小即可.

  (ii) 若不夠小, 看 $S(a, \frac{a+b}{2})$ 之 error 是否 $\leq \frac{\varepsilon}{2}$, $S(\frac{a+b}{2}, b)$ 之 error 是否 $\leq \frac{\varepsilon}{2}$

  (iii) 若有超過, 則 $S(x,y)$ 放入 (i) 繼續做.

- see Fig 4.13 (越陡峭, 割得越細)

  given $\varepsilon = 0.00001$

  if $S(1,3) - S(1,2) - S(2,3)$ 還不夠小$\Rightarrow$再切割

  $\Rightarrow S(1,2) - S(1,1.5) - S(1.5,2)$

  看看是否$< \frac{\varepsilon}{4}$

  $\Rightarrow$一直做到要求的精確度

  **此方法有問題: 不一定每個區間都要都要達目標, 最好是有 + 有 - 正好相消

4.8 Improper Integrals 瑕積分

- $\int_a^b \frac{1}{(x-a)^p} dx$ (as $x \to a$, $\frac{1}{(x-a)^p} \to \infty$)

  $f(x) = \frac{g(x)}{(x-a)^p}$,

  $[g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2 + ...]$

(where $g(a)+g'(a)(x-a)+\frac{g''(a)}{2}(x-a)^2+\ldots=p(x)$)

$$\Rightarrow \int_a^b f(x)dx = \underbrace{\int_a^b \frac{g(x)-P_4(x)}{(x-a)^p}dx}_{G(x)} + \underbrace{\int_a^b \frac{P_4(x)}{(x-a)^p}dx}_{\text{polynomial, 可直接積分}}$$

$\int_a^b G(x)dx$ 用 composite Simpson's rule 來算積分, $G(a)=0$ 當作一邊.

Ex1. 證明出 $\int_0^1 \frac{e^x}{\sqrt{x}}dx$的值並非不存在

(當 $x\to 0, \frac{e^x}{\sqrt{x}}$, 所以無法直接用 composite Simpson's rule)

- 如果 Singularity point 在右, 用右邊之 endpoint 作固定點
- 如果 singularity point 在中, 則 $\int_a^b = \int_a^c + \int_c^b$
- 如果 a 或 b 爲 ∞, 則用變數變換將 x 換掉.
- P.161 Example 2

4.9 Numerical Differentiation

- $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ (微分會有 h 要取多小的問題)

  $f(x_0+h) = f(x_0) + f'(x_0)(x-x_0) + \ldots$

  $\Rightarrow f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$

- (兩點) $x_0, x_1 = x_0+h$

  $f(x) = P_{0,1}(x) + \frac{(x-x_0)(x-x_1)}{2!}f''(\xi(x))$

  $\quad = f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0} + \frac{(x-x_0)(x-x_1)}{2!}f''(\xi(x))$

  (where $\frac{x-x_1}{x_0-x_1} = \frac{x-x_0-h}{-h}, \frac{x-x_0}{x_1-x_0} = \frac{x-x_0}{h}$)

  微分

$\Rightarrow f'(x) = -\frac{f(x_0)}{h} + \frac{f(x_0+h)}{h} + [\frac{(x-x_0)(x-x_1)}{2!}]' f''(\xi(x)) + [\frac{(x-x_0)(x-x_1)}{2!}] \frac{df''(\xi(x))}{dx}$

$\Rightarrow f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} + [\frac{(x-x_0)(x-x_0+x_0-x_1)}{2!}]' f''(\xi(x)) + 0$

$= \frac{f(x_0+h)-f(x_0)}{h} + [\frac{(x-x_0)^2-h(x-x_0)}{2!}]' f''(\xi(x))$

$= \frac{f(x_0+h)-f(x_0)}{h} - \frac{h}{2} f''(\xi(x))$

- 通式

  $f(x) = \sum_{j=0}^{n} f(x_j) L_j(x) + [\frac{(x-x_0)...(x-x_n)}{(n+1)!}] f^{(n+1)}(\xi(x))$

  where $\sum_{j=0}^{n} f(x_j) L_j(x)$: 用 polynomial Lagrange 做的,

  $[\frac{(x-x_0)...(x-x_n)}{(n+1)!}] f^{(n+1)}(\xi(x))$: error

  $x = x_j$時, $L_j(x) = 1$

  $f'(x) = \sum_{j=0}^{n} f(x_j) L'_j(x) + D_x[\,] f^{(n+1)}(\xi(x)) + [\,] D_x f^{(n+1)}(\xi(x))$

  $f'(x_k) = \sum_{j=0}^{n} f(x_j) L'_j(x_k) +$

  $D_x[\frac{(x-x_k+x_k-x_0)(x-x_k+x_k-x_1)...(x-x_k+x_k-x_k)...(x-x_k+x_k-x_n)}{(n+1)!}] f^{(n+1)}(\xi(x))$

  [ ]乘開 $= a(x-x_k)^n + b(x-x_k)^{n-1} + \ldots$
  $[\,]' = c(x-x_k)^{n-1} + d(x-x_k)^{n-2} + \ldots + e(x-x_k) + f$,
  當 $x = x_k$ 代入, 只剩下 f

  $= \sum_{j=0}^{n} f(x_j) L'_j(x_k) + \Pi_{j=0, j\neq k}^{n} (x_k - x_j) \frac{f^{(n+1)(\xi(x))}}{(n+1)!}$

- P.166 Three-Point Endpoint Formula

  (可用在 Spline boundary condition 的假設)

  $f'(x_0) = \frac{1}{2h}(-3f(x_0) + 4f(x_0+h) - f(x_0+2h)) + \frac{h^2}{3} f^3(\xi)$

  where ξ lies between x_0 and x_0+2h

  **用3點去做 Lagrange (x_0, x_0+h, x_0+2h)

  **用點越多越精準 (比較兩點跟三點的 error-三點, error 變小)

- 跟 Three-Point Endpoint Formula 比較
  Three-Point Midpoint Formula 比較好, error 較小
  (where Midpoint: $x_0 - h, x_0, x_0 + h$)
  $f'(x_0) = \frac{1}{2h}(f(x_0+h) - f(x_0-h)) - \frac{h^2}{6}f^3(\xi)$
  where ξ lies between $x_0 - h$ and $x_0 + h$
  **只要帶入兩次 f (endpoint 要代3次), 但 error 比 endpoint 小
  P.168 Ex2. $f(x) = xe^x$, five-point 比 three-point 好, Midpoint 比 endpoint 好

- 在積分時, $h \to 0$, truncation error 變小, 且雖然用大量之 f 值, 但均爲加總, rounf-off error 之變化約可抵銷.

- 在微分時, $h \to 0$, truncation error 變小, 但因 f 值之間爲相減, 不知 $f(x_0+h)$, $f(x_0)$, $f(x_0-h)$ 之間的差距, 是眞的差距, 還是 round-off error.

- $f'(x_0)\frac{f(x_0+h)-f(x_0-h)}{2h}$, $[\frac{\tilde{f}(x_0+h)-\tilde{f}(x_0-h)}{2h}$電腦值]
  因爲 $f(x_0+h) = \tilde{f}(x_0+h) + e(x_0+h)$
  所以 $\mid f'(x_0) - [\frac{f(x_0+h)-f(x_0-h)}{2h}] \mid = \mid f'(x_0) - [\frac{\tilde{f}(x_0+h)-\tilde{f}(x_0-h)}{2h}]$(電腦值)$- \frac{e(x_0+h)-e(x_0-h)}{2h} \mid$
  $\leq \mid f'(x_0) - \frac{\tilde{f}(x_0+h)-\tilde{f}(x_0-h)}{2h} \mid \leq \frac{\varepsilon}{h} + \frac{h^2}{6}f^{(3)}(\xi)$
  where $\frac{e(x_0+h)-e(x_0-h)}{2h} = \frac{2\varepsilon}{2h}$, round-off error
  $f^{(3)}(\xi)$:truncation error
  (h在分子, 也在分母⇒並不是越小越好)
  希望$\frac{\varepsilon}{h} + \frac{h^2}{6}M$ min ⇒對 h 微分 s.t.$(\frac{\varepsilon}{h} + \frac{h^2}{6}M)'_h = 0$
  $\Rightarrow h = \sqrt[3]{\frac{3\varepsilon}{M}}$

- Three-Point Midpoint Formula for Approximating f'' (二階微分)

  $f''(x_0) = \frac{1}{h^2}(f(x_0 - h) - 2f(x_0) + f(x_0 + h)) - \frac{h^2}{12}f^4(\varepsilon)$
  where ε lies between $x_0 - h$ and $x_0 + h$

  **由$\frac{\frac{f(x_0+h)-f(x_0)}{h} - \frac{f(x_0)-f(x_0-h)}{h}}{h}$而來

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	Advantage	Disadvantage
Midpoint rule Trapezoidal rule	Simple	需分割的夠多, 才能很精確
Simpson's 1/3 rule (n=2)	Simple and higher accuracy	分割偶數個 intervals
Simpson's 3/8 rule (n=3)	than trapezoidal	分割 intervals 爲3之倍數
composite quadrature	比一般 quadrature 來的準	分割偶數個 intervals (composite Simpson's) 可任意分割 equispace (composite trapezoidal) (亦即分割出多個梯形)
Roomberg Integration	可利用一些簡單算術, 將 composite trapezoidal 之 error 降低, 例從 $h^2 \to h^4$ $(\frac{1}{10})^2 \to (\frac{1}{10})^4$, 若從分的細著手 從$\frac{1}{10} \to (\frac{1}{100})^2$	需分割成 2^k個 intervals 且$k = 1, \ldots, n$ 都要做 然後才能用 Roomberg Integration
Gauss quadrature	More accuracy than general Simpson's rule	Data points are not equispaced
Adaptive quadrature	處理微分值變動大之函數, 因其變動大 $\Rightarrow$ error term 大 藉由將變動大之 subinterval之分割, 造成所需之 error bound subintervals都小於某個error bound	有可能永遠達不到需要之 criteria, 而無限切割, 另外眞正需求是 overall 之 error bound 的限制, 而並不一定要每個 subintervals 都小於某個 error bound
Newton-Cotes	General rule of simpson's rule	Higher-order formulas are not necessarily more accurate