數値分析

Chapter 3 .
Interpolation and Polynomial Approximation

- Interpolation: determine the values at intermedia points.

- 任一函數 f 若在 $[a, b]$ 間連續, 則一定可以找到一多項式函數 p 來 approximate f.

- Taylor polynomial. P.61 Figure 3.1

- 多項式函數之好處: 其微分與積分都還是多項式.

3.2 Lagrange Polynomials

- Given 3 pts $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$

  $y = ax^2 + bx + c$

  $$p(x) = f(x_0)\underbrace{\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}}_{L_{2,0}} + f(x_1)\underbrace{\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}}_{L_{2,1}} + f(x_2)\underbrace{\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}_{L_{2,2}}$$

  $$\left\| \begin{array}{l} p(x_0) = f(x_0), p(x_1) = f(x_1), p(x_2) = f(x_2) \\ L_{n,k}(x_k) = 1, L_{n,k}(x_i) = 0 \end{array} \right.$$

- Lagrange Polynomials

  Consider the construction of a polynomial of degree at most n that passes through the n+1 points

  $(x_0, f(x_0)), (x_1, f(x_1)), ..., (x_n, f(x_n))$

  In this case, we need to construct, for each $k = 0, 1, ..., n$, a function $L_{n,k}(x)$ with the property that $L_{n,k}(x_i) = 0$ when $i \neq k$ and $L_{n,k}(x_k) = 1$. To satisfy $L_{n,k}(x_i) = 0$ for each $i \neq k$ requires that the numerator of $L_{n,k}(x)$ contains the terms

  $(x - x_0)(x - x_1)...(x - x_{k-1})(x - x_{k+1})...(x - x_n)$

  To satisfy $L_{n,k}(x_k) = 1$, the denominator of $L_{n,k}(x)$ must be equal to this term evaluated at $x = x_k$. Thus,

$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)...(x-x_{k-1})(x-x_{k+1})...(x-x_n)}{(x_k-x_0)(x-x_1)...(x_k-x_{k-1})(x_k-x_{k+1})...(x_k-x_n)}$

- nth Lagrange Interpolation Polynomial

  $P_n(x) = f(x_0)L_{n,0}(x)+...+f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$

  where

  $L_{n,k}(x) = \frac{(x-x_0)(x-x_1)...(x-x_{k-1})(x-x_{k+1})...(x-x_n)}{(x_k-x_0)(x-x_1)...(x_k-x_{k-1})(x_k-x_{k+1})...(x_k-x_n)}$

  for each $k = 0, 1, ..., n$

  Ex 1. P.65 Using the numbers, or nodes, $x_0 = 2$, $x_1 = 2.5$ and $x_2 = 4$ to find the second interpolating polynomial for $f(x) = \frac{1}{x}$ require that we first determine the coefficient polynomials L_0, L_1 and L_2. In nested form they are

  $L_0(x) = \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} = (x - 6.5)x + 10$

  $L_1(x) = \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} = \frac{(-4x+24)x-32}{3}$

  $L_2(x) = \frac{(x-2)(x-2.5)}{(4-2)(4-2.5)} = \frac{(x-4.5)x+5}{3}$

  Since $f(x_0) = f(2) = 0.5$, $f(x_1) = f(2.5) = 0.4$, $f(x_2) = f(4) = 0.25$, we have

  $P_2(x) = \sum_{k=0}^{2} f(x_k)L_k(x) = (0.05x - 0.425)x + 1.15$

  An appoximation to $f(3) = \frac{1}{3}$ is

  $f(3) \approx P(3) = 0.325$

- Lagrange Polynomial Error Formula

  $f(x) = P_n(x) + \frac{f^{n+1}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)...(x - x_n)$

  for some number $\xi(x)$ between x_0 , x_1 ,..., x_n and x.

  P.67 Example 2, 用不同的點形成的 Lagrange 的 error 之分析.

- 未必考慮越多點 (越高階) 越好, 可能原因是 error term 中的 $\xi(x)$ 的可能範圍大, 意味可能誤差大, 尤其當加入那點的 $(x - x_k)$ 絕對值很大時 (意即新加入的點, 距離要求的 x 很遠時), error term 也會放大.

- Lagrange 之缺點, 不知需要用多少點之資訊, 即使用的越多也非最好. 且新加一點後需重算, 之前算的沒法 reuse.

- Recursively Generated Lagrange Polynomials

  Let f be defined at $x_0, x_1, ..., x_k$ and x_i, x_j, be two numbers in this set. 另一種 Lagrange 表示法:

  $P(x) = \frac{(x-x_j)P_{0,1,...,j-1,j+1,...,k}(x)-(x-x_i)P_{0,1,...,i-1,i+1,...,k}(x)}{(x_i-x_j)}$

  then $P(x)$ is the kth Lagrange Polynomial that interpolates f at the k+1 points $x_0, x_1, ... ,x_k$

  Let Q $\equiv P_{0,1,...,i-1,i+1,...,k}(x)$ and $\widehat{Q} \equiv P_{0,1,...,j-1,j+1,...,k}(x)$

  $\Rightarrow P(x) = \frac{(x-x_j)\hat{Q}(x)-(x-x_i)Q(x)}{(x_i-x_j)}$

	if $x_r \neq x_i, x_j$	$x_r = x_i$	$x_r = x_j$
$\widehat{Q}$	$\widehat{Q}(x_r)$	$f(x_i)$	0
Q	$Q(x_r)$	0	$f(x_j)$
P	$f(x_r)$	$f(x_i)$	$f(x_j)$

  Table 3.4 Neville's method

$$
\begin{array}{llllll}
x_0 & P_0 = Q_{0,0} = f(x_0) & & & & \\
x_1 & P_1 = Q_{1,0} = f(x_1) & P_{0,1} = Q_{1,1} & & & \\
x_2 & P_2 = Q_{2,0} = f(x_2) & P_{1,2} = Q_{2,1} & P_{0,1,2} = Q_{2,2} & & \\
x_3 & P_3 = Q_{3,0} = f(x_3) & P_{2,3} = Q_{3,1} & P_{1,2,3} = Q_{3,2} & P_{0,1,2,3} = Q_{3,3} & \\
x_4 & P_4 = Q_{4,0} = f(x_4) & P_{3,4} = Q_{4,1} & P_{2,3,4} = Q_{4,2} & P_{1,2,3,4} = Q_{4,3} & P_{0,1,2,3,4} = Q_{4,4}
\end{array}
$$

  P.70 Ex 4. Suppose that we want to use Neville's method to calculate the approximation to $f(1.5)$. If $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$, and $x_4 = 2.2$, then $f(1.0) = Q_{0.0}$, $f(1.3) = Q_{1.0}$, $f(1.6) = Q_{2.0}$, $f(1.9) = Q_{3.0}$ and $f(2.2) = Q_{4.0}$; so these are the five polynomials of degree

zero (constants) that approximate $f(1.5)$. Calculating the approximation $Q_{1,1}(1.5)$ gives

$Q_{1,1}(1.5) = \frac{(1.5-1.0)Q_{1.0}-(1.5-1.3)Q_{0.0}}{(1.3-1.0)} = \frac{0.5(0.6200860)-0.2(0.7651977)}{0.3} = 0.5233449$

Similarly,

$Q_{2,1}(1.5) = \frac{(1.5-1.3)(0.4554022)-(1.5-1.6)(0.6200860)}{(1.6-1.3)} = 0.5102968$

$Q_{3,1}(1.5)0.5132634$, $Q_{4,1}(1.5) = 0.5104270$

In a similarly manner, the approximation using quadrtic polynomials are given by

$Q_{2,2}(1.5) = \frac{(1.5-1.0)(0.5102968)-(1.5-1.6)(0.5233449)}{(1.6-1.0)} = 0.5124715$

$Q_{3,2}(1.5) = 0.5112857$, $Q_{4,2}(1.5) = 0.5137361$

The higher-degree approximations are generated in a similar manner and are shown in Table 3.5

- Divided Differences (差分)

  The zeroth divided difference of the function f with respect to x_i, $f[x_i]$, is simply the value of f at x_i:

  $f[x_i] = f(x_i)$

  The remaining divided differences are defined inductively. The first divided differences of f with respect to x_i and x_{i+1} is denoted $f[x_i, x_{i+1}]$ and is defined as

  $f[x_i, x_{i+1}] = \frac{f[x_{i+1}]-f[x_i]}{x_{i+1}-x_i}$

  After the (k-1)st divided differences,

  $f[x_i, x_{i+1}, x_{i+2}, ..., x_{i+k-1}]$ and $f[x_{i+1}, x_{i+2}, ..., x_{i+k-1}, x_{i+k}]$

  have been determined, the kth divided differences relative to $x_i, x_{i+1}, x_{i+2}, ..., x_{i+k}$ is

  $f[x_i, x_{i+1}, x_{i+2}, ..., x_{i+k}] = \frac{f[x_{i+1},x_{i+2},...,x_{i+k}]-f[x_i,x_{i+1},x_{i+2},...,x_{i+k-1}]}{x_{i+k}-x_i}$

  With this notation , it can be shown that

  $P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + ... + f[x_0, x_1, ..., x_n](x - x_0)(x - x_1)...(x - x_{n-1})$

(多個點之泰勒展開式, 因 $x_{i+k} - x_i$ 越除越大, 所以分母不用乘以 $n!$)

(f'' vs. second divided diffences)

$f(x_0 - h)$	$f(x_0)$	$f(x_0 + h)$
.	.	.
$x_0 - h$	x_0	$x_0 + h$

$$f'' \approx \frac{\frac{f(x_0+h)-f(x_0)}{h} - \frac{f(x_0)-f(x_0-h)}{h}}{\frac{x_0+x_0+h}{2} - \frac{x_0+x_0-h}{2}} = \frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}$$

divided difference: $x_0 = x_0 - h$, $x_1 = x_0$, $x_2 = x_0 + h$

$$f[x_0, x_1] = \frac{f(x_0)-f(x_0-h)}{h}$$

$$f[x_1, x_2] = \frac{f(x_0+h)-f(x_0)}{h}$$

$$\Rightarrow f[x_0, x_1, x_2] = \frac{\frac{f(x_0+h)-f(x_0)}{h} - \frac{f(x_0)-f(x_0-h)}{h}}{2h}$$

- Newton's Interpolatory Divided-Difference Formula

  $$P(x) = f[x_0] + \sum_{k+1}^{n} f[x_0, x_1, ..., x_k](x - x_0)...(x - x_{k-1})$$

  See P.75 Table 3.8, Example 1.

- Newton's interpolatory divided-difference formula has a simpler form when $x_0, x_1, ..., x_n$ are arranges consecutively with equal spacing. In this case, we introduce the notation $h = x_{i+1} - x_i$ for each $i = 0, 1, ..., n-1$ and let $x = x_0 + sh$. Then the difference $x - x_i$ can be written as $x - x_i = (s-i)h$, and the divided-difference formula becomes

  $$\begin{aligned} P_n(x) &= P_n(x_0 + sh) \\ &= f[x_0] + shf[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2] + \\ &\quad ... + s(s-1)...(s-n+1)h^n f[x_0, x_1, ..., x_n] \\ &= f[x_0] + \sum_{k=1}^{n} s(s-1)...(s-k+1)h^k f[x_0, x_1, ..., x_k] \end{aligned}$$

Using a generalization of the binomial-coefficient notation,

$$\left(\begin{array}{c} s \\ k \end{array} \right) = \frac{s(s-1)...(s-k+1)}{k!}$$

- Newton Forward Divided-Difference Formula

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \left(\begin{array}{c} s \\ k \end{array} \right) k! h^k f[x_0, x_1, ..., x_k]$$

$$= f[x_0] + \sum_{k=1}^{n} \left(\begin{array}{c} s \\ k \end{array} \right) \Delta^k f(x_0) \quad (\Delta f_i = f_{i+1} - f_i)$$

- backward Divided-Difference Formula. P.78

- P.78 Example 2. x 靠近 x_0, 用 forward divided difference, x 靠近 x_n, 用 backward divided difference, 若 x 在中間, 兩法都會有些誤差, 此時可改用 centered-difference formula.

3.4 Hermite Interpolation (只要有 f'的資訊, 即可用此法)

- Lagrange fits n+1 nodes ⇒ degree n, 但 Hermite 除了 fits n+1 nodes, 還要 fits 所有之 f' (共 n+1 個)⇒ 需多加 n+1 degree ⇒ 共 2n+1 degree.

- Hermite Polynomial

  Suppose that $f \in C^1[a, b]$ and that x_0,..., x_n in $[a, b]$ are distinct. The unique polynomial of least degree agreeing with f and f' at x_0,..., x_n is the polynomial of degree at most 2n+1 given by

  $H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \widehat{H}_{n,j}(x)$

where
$H_{n,j}(x) = [1 - 2(x - x_j)L'_{n,j}(x_j)]L^2_{n,j}(x)$
and
$\widehat{H}_{n,j}(x) = (x - x_j)L^2_{n,j}(x)$
Here, $L_{n,j}$ denotes the jth Lagrange coefficient polynomial of degree n, and $L'_{n,j}(x_j) = L'_{n,j}(x)|_{x=x_j}$

- $$\begin{cases} H_{n,j}(x_j) = 1 \\ H_{n,j}(x_i) = 0 \end{cases} \quad \begin{cases} \hat{H}_{n,j}(x_j) = 0 \\ \hat{H}_{n,j}(x_i) = 0 \end{cases}$$

  $$\begin{cases} H'_{n,j}(x_j) = 0 \\ H'_{n,j}(x_i) = 0 \end{cases} \quad \begin{cases} \hat{H}'_{n,j}(x_j) = 1 \\ \hat{H}'_{n,j}(x_i) = 0 \end{cases}$$

  $$\text{given} \begin{cases} L_{n,j}(x_j) = 1 \\ L_{n,j}(x_i) = 0 \end{cases}$$

- 但此法很麻煩, 除了要算 Lagrange 之外, 還要算 Lagrange 之微分, 此外, 通常很難知道每一點之微分值.

- Divided-Difference Form of the Hermite Polynomial.
  P.84 Table 3.11. P.85 Example 1.

- Hermite Polynomial Error Formula
  If $f \in C^{2n+2}[a, b]$, then
  $f(x) = H_{2n+1}(x) + \frac{f^{2n+2}(\xi(x))}{(2n+2)!}(x - x_0)^2...(x - x_n)^2$
  for some $\xi(x)$ with $a < \xi(x) < b$

3.5 Spline Interpolation

- 之前用的方法, 次方太高了, 有 oscillatory drawback.

- $[x_i, x_{i+1}]$ 若知道 $f(x_i)$, $f(x_{i+1})$ 與 $f'(x_i)$, $f'(x_{i+1})$ 則此段可用 cubic Hermite 來估:
  $H_3(x) = ax^3 + bx^2 + cx + d$ (但很常不知道 f')

- Cubic Spline Interpolation

  Given a function f defined on $[a, b]$ and a set of nodes, $a = x_0 < x_1 < ... < x_n = b$, a cubic spline interpolant, S, for f is a function that satisfies the following conditions:

  (a) For each $j = 0, 1, ..., n-1$, $S(x)$ is a cubic polynomial, denoted by $S_j(x)$, on the subinterval $[x_j, x_{j+1})$

  (b) $S_j(x_j) = f(x_j)$ for each $j = 0, 1, ..., n$

  (c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, ..., n-2$

  (d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, ..., n-2$

  (其實並不需要眞的知道 f', S' 也未不等於 f')

  (e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, ..., n-2$

  (其實並不需要眞的知道 f'', S'' 也未不等於 f'')

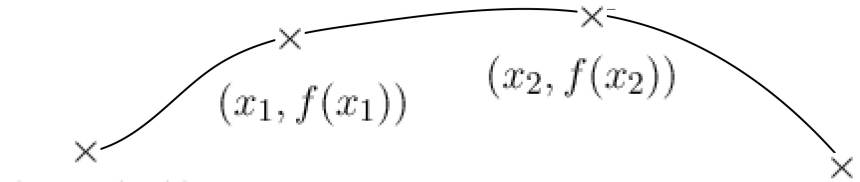
  (f) One of the following sets of boundary conditions is satisfied:

  (i) $S''(x_0) = S''(x_n) = 0$ (natural or free boundary)

  (ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$
  (clamped boundary)

$$y = a_1x^3 + b_1x^2 + c_1x + d_1$$

$$y = a_0x^3 + b_0x^2 + c_0x + d_0 \qquad y = a_2x^3 + b_2x^2 + c_2x + d_2$$



(i) $\begin{cases} f(x_0) = a_0x_0^3 + b_0x_0^2 + c_0x_0 + d_0 \\ f(x_1) = a_0x_1^3 + b_0x_1^2 + c_0x_1 + d_1 \\ f''(x_0) = 6a_0x_0 + 2b_0 = 0 \end{cases}$

(i)-(ii) $\begin{cases} f'(x_1)\text{相等} \Rightarrow 3a_0x_1^2 + 2b_0x_1 + c_0 = 3a_1x_1^2 + 2b_1x_1 + c_1 \\ f''(x_1)\ \text{相等} \Rightarrow 6a_0x_1 + 2b_0 = 6a_1x_1 + 2b_1 \end{cases}$

(ii) $\begin{cases} f(x_1) = a_1x_1^3 + b_1x_1^2 + c_1x_1 + d \\ f(x_2) = a_1x_2^3 + b_1x_2^2 + c_1x_2 + d \end{cases}$

(ii)-(iii) $\begin{cases} f'(x_2)\ \text{相等} \Rightarrow 3a_1x_2^2 + 2b_1x_2 + c_1 = 3a_2x_2^2 + 2b_2x_2 + c_2 \\ f''(x_2)\ \text{相等} \Rightarrow 6a_1x_2 + 2b_1 = 6a_2x_2 + 2b_2 \end{cases}$

(iii) $\begin{cases} f(x_2) = a_2x_2^3 + b_2x_2^2 + c_2x_2 + d_2 \\ f(x_3) = a_2x_3^3 + b_2x_3^2 + c_2x_3 + d_2 \\ f''(x_3) = 6a_2x_3 + 2b_2 = 0 \end{cases}$

$$\begin{bmatrix} x_0^3 & x_0^2 & x_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1^3 & x_1^2 & x_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6x_0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3x_1^2 & 2x_1 & 1 & 0 & -3x_1^2 & -2x_1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 6x_1 & 2 & 0 & 0 & -6x_1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}$$

$$= \begin{bmatrix} f(x_0) \\ f(x_1) \\ f''(x_0) = 0 \\ 0 \\ 0 \end{bmatrix}$$

3.6 Parametric Curves

- 因不是函數, 所以將 x 與 y 分開看, x, y 分別是參數 t 之函數.

- P.100 EX1. Construct a pair of Lagrange polynomials to approximate the curve shown in Figure 3.12, using the data points shown on the curve.

  There is flexibility in choosing the parameter, and we will choose the points t_i equally spaced in [0,1]. In this case, we have the data in Table 3.16

i	0	1	2	3	4
t_i	0	0.25	0.5	0.75	1
x_i	-1	0	1	0	1
y_i	0	1	0.5	0	-1

- $x(t)$ 與 $y(t)$ 可用 Lagrange polynomial 來表示.

- Piecewise cubic Hermite Polynomial

  We can simplify the process to one of determining a pair of cubic Hermite polynomials in the parameter t, where $t_0 = 0$, $t_1 = 1$, given the endpoint data $(x(0), y(0))$, and $(x(1), y(1))$ and the derivatives dy/dx (at $t = 0$) and dy/dx (at $t = 1$).

  $\frac{dy}{dx}$ (at $t = 0$) $= \frac{y'(0)}{x'(0)}$ and $\frac{dy}{dx}$ (at $t = 1$) $= \frac{y'(1)}{x'(1)}$

  因 $\frac{k_0 y'(0)}{k_0 x'(0)} = \frac{y'(0)}{x'(0)}$, $\frac{k_1 y'(1)}{k_1 x'(1)} = \frac{y'(1)}{x'(1)}$

  $\Rightarrow$ 需假設 desired tangent line (P.102 Figure 3.14, 3.15)

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	Advantages	Disadvantage
Lagrange	Convenient form; Easy to program	Hard to calculate by hand; 二邊之波動變化大, 不知需用多少點資訊, 用的多也未必好 (fit 未必好, 且太複雜) $\Rightarrow$ 避免
Newton (Divided Differences)	計算量 $<$ Lagrange 且有新的 data 點時, 原table 可更新再使用	A difference or divided difference table must be prepares; 要估的點要在這些點的中間最好; 微分值會越來越小, round off 要注意
Hermite	很精確, 因爲多考慮了 f'	需要 f'
Cubic spline	當資料點多時, 不會形成高階之函數; fit 很好; 且一階, 二階微分連續	需要解聯立方程組; 需假設 boundary condition
Parametric Curves	可用於非函數之曲線 一段動, 其他段不動 或只動一點 (piecewise cubic Hermite 也可達到類似效果)	