數值分析

Chapter 2

Solutions of Equations of One Variable

2.2 The Bisection Method

- To begin the Bisection method, set $a_1 = a$ and $b_1 = b$, as shown in Figure 2.1, and let p_1 be the midpoint of the interval $[a, b]$:

  (1) define $p_1 = a_1 + \frac{b_1 - a_1}{2}$

  (2) If $f(p_1)$=0, then the root p is given by $p = p_1$;

  if $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.

  (3) If $f(a_1)$ and $f(p_1)$ have the same sign, then p is in the interval (p_1, b_1), and we set $a_2 = p_1$ and $b_2 = b_1$

  (4) If, on the other hand, $f(p_1)$ and $f(a_1)$ have opposite signs, then p is in the interval (a_1, p_1), and we set $a_2 = a_1$ and $b_2 = p_1$

  We rapply the process to the interval $[a_2, b_2]$, and continue forming $[a_3, b_3]$, $[a_4, b_4]$,...

  (P.31 Figure 2.1) (P.32 $\sim$ P.34) Example 1.

- Bisection Method

  An interval $[a_{i+1}, b_{i+1}]$ containing an approximation to a root of $f(x) = 0$ is constructed from an interval $[a_i, b_i]$ containing the root by first letting

  $p_i = a_i + \frac{b_i - a_i}{2}$

  Then set

  $a_{i+1} = a_i$ and $b_{i+1} = p_i$ if $f(a_i)f(p_i) < 0$

  and

  $a_{i+1} = p_i$ and $b_{i+1} = b_i$ otherwise

2.3 The Secant Method

- (Figure 2.4) Setting $p_0 = a$ and $p_1 = b$. The equation of the secant line through $(p_0, f(p_0))$ and $(p_1, f(p_1))$ is

  $y = f(p_1) + \frac{f(p_1)-f(p_0)}{p_1-p_0}(x - p_1)$

  The x-intercept (p_2,0) of this satisfies

  $0 = f(p_1) + \frac{f(p_1)-f(p_0)}{p_1-p_0}(p_2 - p_1)$

  and solving for p_2 gives

  $p_2 = p_1 - \frac{f(p_1)(p_1-p_0)}{f(p_1)-f(p_0)}$

- <u>Secant Method</u>

  The approximation p_{n+1}, for n >1, to a root of $f(x) = 0$ is computed from the approximation p_n and p_{n-1} using the equation

  $p_{n+1} = p_n - \frac{f(p_n)(p_n-p_{n-1})}{f(p_n)-f(p_{n-1})}$

  P.37 Figure 2.4

  P.38 Example 1

- <u>Method of False Position</u>

  (結合 Secant Method 與 Bisection Method)

  An interval $[a_{n+1}, b_{n+1}]$ for $n > 1$, containing an approximation to a root of $f(x) = 0$ is found from an interval $[a_n, b_n]$ containing the root by first computing

  $p_{n+1} = a_n - \frac{f(a_n)(b_n-a_n)}{f(b_n)-f(a_n)}$. Then set

  $a_{n+1} = a_n$ and $b_{n+1} = p_{n+1}$ if $f(a_n)f(p_{n+1}) < 0$

  $(\Rightarrow [a_n, p_{n+1}])$

  and

  $a_{n+1} = p_{n+1}$ and $b_{n+1} = b_n$ otherwise $(\Rightarrow [p_{n+1}, b_n])$

  P.39 Figure 2.5 Secant mwthod vs. False position

2.4 Newton's Method

- Newton's Method

  The approximation p_{n+1} to a root of $f(x)=0$ is computed from the approximation p_n using the equation

  $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$

  $$\left\| \begin{array}{l} y - f(p_0) = f'(p_0)(x-p_0) \\ 0 - f(p_0) = f'(p_0)(p_1 - p_0) \\ \Rightarrow p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} \end{array} \right.$$

  Expanding f in the first Taylor polynomial at p_n and evaluating at $x=p$ gives

  $0 = f(p) = f(p_n) + f'(p_n)(p-p_n) + \frac{f''(\xi)}{2}(p-p_n)^2$

  where ξ lies between p_n and p. Consequently, if $f'(p_n) \neq 0$, we have

  $p - p_n + \frac{f(p_n)}{f'(p_n)} = -\frac{f''(\xi)}{2f'(p_n)}(p-p_n)^2$

  Since

  $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$

  this implies that

  $p - p_{n+1} = -\frac{f''(\xi)}{2f'(p_n)}(p-p_n)^2$

  若 f'' is bounded by M$\Rightarrow \mid p - p_{n+1} \mid \leq -\frac{M}{2|f'(p_n)|} \mid p - p_n \mid^2 \Rightarrow$ converge quadratically. (因每次的 error 會比之前 error 之平方還小) P.49 $\sim$ 50 Example 1

- 此法之限制

  (i) $f'(p) \neq 0$, 則根的地方一階微分爲 0, 還是可收斂, 但收斂速度會慢很多. P.46 Figure 2.7

  (ii) 可能不收斂. 例: 反曲點.

2.5 Error Analysis and Accelerating Convergence

- Aitken's Δ^2 Method (用來加速 linearly convergent)

  $\widehat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$

  (because $\frac{p_{n+1}-p}{p_n-p} \approx \frac{p_{n+2}-p}{p_{n+1}-p}$, 非鋸齒型逼近才有此性質)

  P.51 Example 2 與 Table 2.8

2.6 Müller's Method

- Müller's Method: (二次之 Secant method) 有機會找到所有的根 (包括實數, 複數), 之前的方法除非一開始猜複數根, 不然是無法找出複數根的.

  Given initial approximations p_0, p_1 and p_2, generate

  $p_3 = p_2 - \frac{2c}{b+\text{sgn}(b)\sqrt{b^2-4ac}}$

  where

  $c = f(p_2)$

  $b = \frac{(p_0-p_2)^2[f(p_1)-f(p_2)]-(p_1-p_2)^2[f(p_0)-f(p_2)]}{(p_0-p_2)(p_1-p_2)(p_0-p_1)}$

  and

  $a = \frac{(p_1-p_2)[f(p_0)-f(p_2)]-(p_0-p_2)[f(p_1)-f(p_2)]}{(p_0-p_2)(p_1-p_2)(p_0-p_1)}$

  sgn(b) 之原因: 原本應有兩個解爲 0, 取與 b 同號使得分母不會有幾乎爲 0 之情況, 使得 p_3 接近 p_2.

  Then continue the iteration, with p_1, p_2 and p_3 replacing p_0, p_1 and p_2

  P.55 Example 1

	Initial interval containing the root	Continuity of f'
Bisection	Y	N
Secant	N	Y
False Position	Y	Y
Newton's	N	Y
Müller's	N	N

	Others
Bisection	Robust; need a good initial interval; singulatority may be caught as if were a root; 一定收斂, 但速度很慢, 因只用到 sgn(f), 而沒用到 f 之値.
Secant	未必收斂, 但如果收斂速度比 Bisection 快, 因爲其爲 Newton's 一種變形.
False Position	未必比 Bisection 快.
Newton's	need a good initial guess; iterative convergence rate is high; 可用於 complex roots, if initial variable is a complex number.
Müller's	適合來 approximation polynomial 之 roots, 因其 roots 通常是 complex number; need three initial points; 且 initial points 是 real number 依然可找到 complex roots; 但 three initial points 之選擇依然是問題; 效率比 Newton's 差, 但比 Secant 好.

- 高階多根時, 每找到一個根, 就降階, 再繼續找下個根. (如此可以減少計算)

- 最後再將找到的所有根, 當作 Newton's Method 之 initial guess, 放回原式中, 找到精準的根. (如此可去掉降階之誤差)

- BS formula, Hull, Ch 12.

- Implied Volatility.