7.4 Applications of Eigenvalues and Eigenvectors

- Model population growth using an age transition matrix and an age distribution vector, and find a stable age distribution vector.
- Use a matrix equation to solve a system of first-order linear differential equations.
- Find the matrix of a quadratic form and use the Principal Axes Theorem to perform a rotation of axes for a conic and a quadric surface.

POPULATION GROWTH

Matrices can be used to form models for population growth. The first step in this process is to group the population into age classes of equal duration. For instance, if the maximum life span of a member is L years, then the following n intervals represent the age classes.

$$\left[0, \frac{L}{n}\right) \qquad \text{First age class}$$

$$\left[\frac{L}{n}, \frac{2L}{n}\right) \qquad \text{Second age class}$$

$$\vdots$$

$$\left[\frac{(n-1)L}{n}, L\right] \qquad n\text{th age class}$$

The **age distribution vector x** represents the number of population members in each age class, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \qquad \begin{matrix} \text{Number in first age class} \\ \text{Number in second age class} \\ \vdots \\ \text{Number in } n\text{th age class} \end{matrix}$$

Over a period of L/n years, the *probability* that a member of the ith age class will survive to become a member of the $(i+1)$th age class is given by p_i, where

$$0 \leq p_i \leq 1, \quad i = 1, 2, \ldots, n-1.$$

The *average number* of offspring produced by a member of the ith age class is given by b_i, where

$$0 \leq b_i, \quad i = 1, 2, \ldots, n.$$

These numbers can be written in matrix form, as follows.

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ p_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & 0 \end{bmatrix}$$

Multiplying this **age transition matrix** by the age distribution vector for a specific time period produces the age distribution vector for the next time period. That is,

$$A\mathbf{x}_i = \mathbf{x}_{i+1}.$$

Example 1 illustrates this procedure.

EXAMPLE 1 A Population Growth Model

A population of rabbits has the following characteristics.

a. Half of the rabbits survive their first year. Of those, half survive their second year. The maximum life span is 3 years.

b. During the first year, the rabbits produce no offspring. The average number of offspring is 6 during the second year and 8 during the third year.

The population now consists of 24 rabbits in the first age class, 24 in the second, and 20 in the third. How many rabbits will there be in each age class in 1 year?

SOLUTION

The current age distribution vector is

$$\mathbf{x}_1 = \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} \quad \begin{array}{l} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{array}$$

and the age transition matrix is

$$A = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

After 1 year, the age distribution vector will be

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} = \begin{bmatrix} 304 \\ 12 \\ 12 \end{bmatrix}. \quad \begin{array}{l} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{array}$$

REMARK

If the pattern of growth in Example 1 continued for another year, then the rabbit population would be

$$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 168 \\ 152 \\ 6 \end{bmatrix}.$$

From the age distribution vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, you can see that the percent of rabbits in each of the three age classes changes each year. To obtain a stable growth pattern, one in which the percent in each age class remains the same each year, the $(n + 1)$th age distribution vector must be a scalar multiple of the nth age distribution vector. That is, $\mathbf{x}_{n+1} = A\mathbf{x}_n = \lambda\mathbf{x}_n$. Example 2 shows how to solve this problem.

EXAMPLE 2 Finding a Stable Age Distribution Vector

Find a stable age distribution vector for the population in Example 1.

SOLUTION

To solve this problem, find an eigenvalue λ and a corresponding eigenvector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The characteristic polynomial of A is

$$|\lambda I - A| = (\lambda + 1)^2(\lambda - 2)$$

(check this), which implies that the eigenvalues are -1 and 2. Choosing the positive value, let $\lambda = 2$. Verify that the corresponding eigenvectors are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix} = t\begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}.$$

For instance, if $t = 2$, then the initial age distribution vector would be

$$\mathbf{x}_1 = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} \quad \begin{array}{l} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{array}$$

and the age distribution vector for the next year would be

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 64 \\ 16 \\ 4 \end{bmatrix}. \quad \begin{array}{l} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{array}$$

Notice that the ratio of the three age classes is still 16 : 4 : 1, and so the percent of the population in each age class remains the same.

Simulation

Explore this concept further with an electronic simulation available at *www.cengagebrain.com*.

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS (CALCULUS)

A **system of first-order linear differential equations** has the form

$$\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n\\
y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n\\
&\ \vdots\\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n
\end{aligned}$$

where each y_i is a function of t and $y_i' = \dfrac{dy_i}{dt}$. If you let

$$\mathbf{y} = \begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_n \end{bmatrix} \quad \text{and} \quad \mathbf{y}' = \begin{bmatrix} y_1'\\ y_2'\\ \vdots\\ y_n' \end{bmatrix}$$

then the system can be written in matrix form as

$$\mathbf{y}' = A\mathbf{y}.$$

EXAMPLE 3 Solving a System of Linear Differential Equations

Solve the system of linear differential equations.

$$\begin{aligned}
y_1' &= 4y_1\\
y_2' &= -y_2\\
y_3' &= 2y_3
\end{aligned}$$

SOLUTION

From calculus, you know that the solution of the differential equation $y' = ky$ is

$$y = Ce^{kt}.$$

So, the solution of the system is

$$\begin{aligned}
y_1 &= C_1e^{4t}\\
y_2 &= C_2e^{-t}\\
y_3 &= C_3e^{2t}.
\end{aligned}$$

The matrix form of the system of linear differential equations in Example 3 is $\mathbf{y}' = A\mathbf{y}$, or

$$\begin{bmatrix} y_1'\\ y_2'\\ y_3' \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1\\ y_2\\ y_3 \end{bmatrix}.$$

So, the coefficients of t in the solutions $y_i = C_i e^{\lambda_i t}$ are given by the *eigenvalues* of the matrix A.

If A is a *diagonal* matrix, then the solution of

$$\mathbf{y}' = A\mathbf{y}$$

can be obtained immediately, as in Example 3. If A is *not* diagonal, then the solution requires more work. First, attempt to find a matrix P that diagonalizes A. Then, the change of variables $\mathbf{y} = P\mathbf{w}$ and $\mathbf{y}' = P\mathbf{w}'$ produces

$$P\mathbf{w}' = \mathbf{y}' = A\mathbf{y} = AP\mathbf{w} \quad \longrightarrow \quad \mathbf{w}' = P^{-1}AP\mathbf{w}$$

where $P^{-1}AP$ is a diagonal matrix. Example 4 demonstrates this procedure.

EXAMPLE 4 Solving a System of Linear Differential Equations

Solve the system of linear differential equations.

$$\begin{aligned} y_1' &= 3y_1 + 2y_2 \\ y_2' &= 6y_1 - y_2 \end{aligned}$$

SOLUTION

First, find a matrix P that diagonalizes $A = \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 5$, with corresponding eigenvectors $\mathbf{p}_1 = [1 \quad -3]^T$ and $\mathbf{p}_2 = [1 \quad 1]^T$. Diagonalize A using the matrix P whose columns consist of $\mathbf{p}_1$ and $\mathbf{p}_2$ to obtain

$$P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}, \quad \text{and} \quad P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}.$$

The system $\mathbf{w}' = P^{-1}AP\mathbf{w}$ has the following form.

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \rightarrow \quad \begin{aligned} w_1' &= -3w_1 \\ w_2' &= 5w_2 \end{aligned}$$

The solution of this system of equations is

$$\begin{aligned} w_1 &= C_1e^{-3t} \\ w_2 &= C_2e^{5t}. \end{aligned}$$

To return to the original variables y_1 and y_2, use the substitution $\mathbf{y} = P\mathbf{w}$ and write

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

which implies that the solution is

$$\begin{aligned} y_1 &= w_1 + w_2 = C_1e^{-3t} + C_2e^{5t} \\ y_2 &= -3w_1 + w_2 = -3C_1e^{-3t} + C_2e^{5t}. \end{aligned}$$

If A has eigenvalues with multiplicity greater than 1 or if A has complex eigenvalues, then the technique for solving the system must be modified.

1. *Eigenvalues with multiplicity greater than 1:* The coefficient matrix of the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -4y_1 + 4y_2 \end{aligned} \quad \text{is} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}.$$

   The only eigenvalue of A is $\lambda = 2$, and the solution of the system is

$$\begin{aligned} y_1 &= C_1e^{2t} + C_2te^{2t} \\ y_2 &= (2C_1 + C_2)e^{2t} + 2C_2te^{2t}. \end{aligned}$$

2. *Complex eigenvalues:* The coefficient matrix of the system

$$\begin{aligned} y_1' &= -y_2 \\ y_2' &= y_1 \end{aligned} \quad \text{is} \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

   The eigenvalues of A are $\lambda_1 = i$ and $\lambda_2 = -i$, and the solution of the system is

$$\begin{aligned} y_1 &= C_1 \cos t + C_2 \sin t \\ y_2 &= -C_2 \cos t + C_1 \sin t. \end{aligned}$$

Try checking these solutions by differentiating and substituting into the original systems of equations.

QUADRATIC FORMS

Eigenvalues and eigenvectors can be used to solve the rotation of axes problem introduced in Section 4.8. Recall that classifying the graph of the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$ Quadratic equation

is fairly straightforward as long as the equation has no xy-term (that is, $b = 0$). If the equation has an xy-term, however, then the classification is accomplished most easily by first performing a rotation of axes that eliminates the xy-term. The resulting equation (relative to the new $x'y'$-axes) will then be of the form

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0.$$

You will see that the coefficients a' and c' are eigenvalues of the matrix

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}.$$

The expression

$$ax^2 + bxy + cy^2$$ Quadratic form

is called the **quadratic form** associated with the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

and the matrix A is called the **matrix of the quadratic form.** Note that the matrix A *is symmetric*. Moreover, the matrix A will be diagonal if and only if its corresponding quadratic form has no xy-term, as illustrated in Example 5.

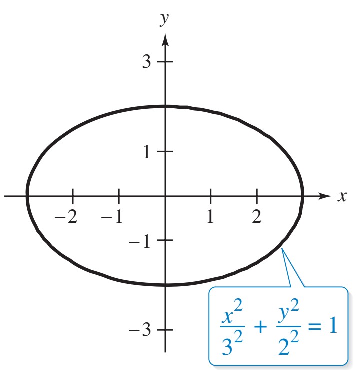


Figure 7.3

EXAMPLE 5 Finding the Matrix of a Quadratic Form

Find the matrix of the quadratic form associated with each quadratic equation.

a. $4x^2 + 9y^2 - 36 = 0$ **b.** $13x^2 - 10xy + 13y^2 - 72 = 0$

SOLUTION

a. Because $a = 4$, $b = 0$, and $c = 9$, the matrix is

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$ Diagonal matrix (no xy-term)

b. Because $a = 13$, $b = -10$, and $c = 13$, the matrix is

$$A = \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}.$$ Nondiagonal matrix (xy-term)

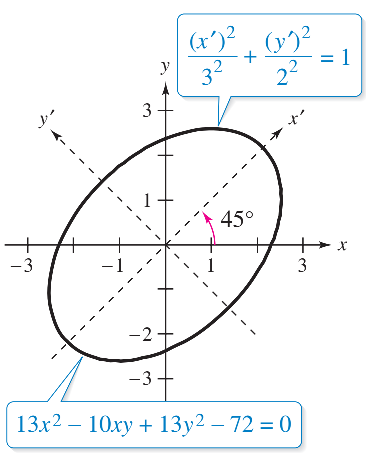


Figure 7.4

In standard form, the equation $4x^2 + 9y^2 - 36 = 0$ is

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

which is the equation of the ellipse shown in Figure 7.3. Although it is not apparent by inspection, the graph of the equation $13x^2 - 10xy + 13y^2 - 72 = 0$ is similar. In fact, when you rotate the x- and y-axes counterclockwise 45° to form a new $x'y'$-coordinate system, this equation takes the form

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

which is the equation of the ellipse shown in Figure 7.4.

To see how to use the matrix of a quadratic form to perform a rotation of axes, let

$$X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then the quadratic expression $ax^2 + bxy + cy^2 + dx + ey + f$ can be written in matrix form as follows.

$$X^TAX + [d \quad e]X + f = [x \quad y]\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [d \quad e]\begin{bmatrix} x \\ y \end{bmatrix} + f$$
$$= ax^2 + bxy + cy^2 + dx + ey + f$$

If $b = 0$, then no rotation is necessary. But if $b \neq 0$, then because A is symmetric, you can apply Theorem 7.10 to conclude that there exists an orthogonal matrix P such that $P^TAP = D$ is diagonal. So, if you let

$$P^TX = X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

then it follows that $X = PX'$, and $X^TAX = (PX')^TA(PX') = (X')^TP^TAPX' = (X')^TDX'$.

The choice of the matrix P must be made with care. Because P is orthogonal, its determinant will be ± 1. It can be shown (see Exercise 65) that if P is chosen so that $|P| = 1$, then P will be of the form

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ gives the angle of rotation of the conic measured from the positive x-axis to the positive x'-axis. This leads to the **Principal Axes Theorem.**

REMARK

Note that the matrix product $[d \quad e]PX'$ has the form

$$(d\cos\theta + e\sin\theta)x' + (-d\sin\theta + e\cos\theta)y'.$$

Principal Axes Theorem

For a conic whose equation is $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the rotation given by $X = PX'$ eliminates the xy-term when P is an orthogonal matrix, with $|P| = 1$, that diagonalizes A. That is,

$$P^TAP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 are eigenvalues of A. The equation of the rotated conic is given by

$$\lambda_1(x')^2 + \lambda_2(y')^2 + [d \quad e]PX' + f = 0.$$

EXAMPLE 6 Rotation of a Conic

Perform a rotation of axes to eliminate the xy-term in the quadratic equation

$$13x^2 - 10xy + 13y^2 - 72 = 0.$$

SOLUTION

The matrix of the quadratic form associated with this equation is

$$A = \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}.$$

Because the characteristic polynomial of A is $(\lambda - 8)(\lambda - 18)$ (check this), it follows that the eigenvalues of A are $\lambda_1 = 8$ and $\lambda_2 = 18$. So, the equation of the rotated conic is

$$8(x')^2 + 18(y')^2 - 72 = 0$$

which, when written in the standard form

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

is the equation of an ellipse. (See Figure 7.4.)

In Example 6, the eigenvectors of the matrix A are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which you can normalize to form the columns of P, as follows.

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note first that $|P| = 1$, which implies that P is a rotation. Moreover, because $\cos 45° = 1/\sqrt{2} = \sin 45°$, the angle of rotation is $45°$, as shown in Figure 7.4.

The orthogonal matrix P specified in the Principal Axes Theorem is not unique. Its entries depend on the ordering of the eigenvalues λ_1 and λ_2 *and* on the subsequent choice of eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. For instance, in the solution of Example 6, any of the following choices of P would have worked.

$$\begin{array}{c} \begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array} \\ \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\ \lambda_1 = 8,\ \lambda_2 = 18 \\ \theta = 225° \end{array} \qquad \begin{array}{c} \begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array} \\ \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\ \lambda_1 = 18,\ \lambda_2 = 8 \\ \theta = 135° \end{array} \qquad \begin{array}{c} \begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array} \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ \lambda_1 = 18,\ \lambda_2 = 8 \\ \theta = 315° \end{array}$$

For any of these choices of P, the graph of the rotated conic will, of course, be the same. (See Figure 7.5.)

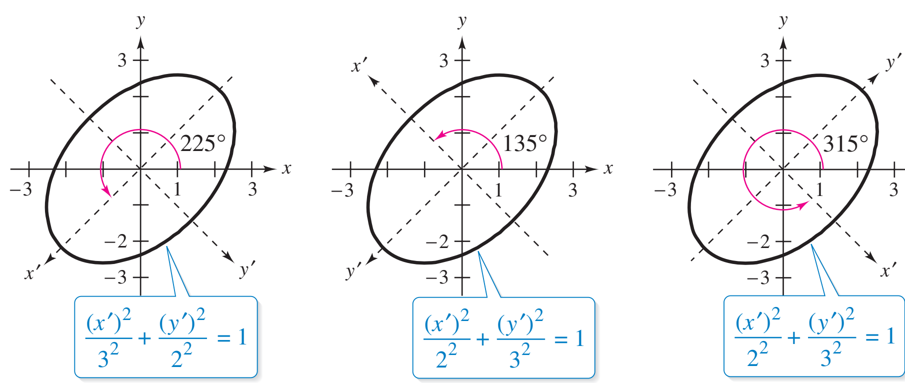


Figure 7.5

The following summarizes the steps used to apply the Principal Axes Theorem.

1. Form the matrix A and find its eigenvalues λ_1 and λ_2.
2. Find eigenvectors corresponding to λ_1 and λ_2. Normalize these eigenvectors to form the columns of P.
3. If $|P| = -1$, then multiply one of the columns of P by -1 to obtain a matrix of the form

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

4. The angle θ represents the angle of rotation of the conic.
5. The equation of the rotated conic is $\lambda_1(x')^2 + \lambda_2(y')^2 + [d \quad e]PX' + f = 0$.

Example 7 shows how to apply the Principal Axes Theorem to rotate a conic whose center has been translated away from the origin.

EXAMPLE 7 Rotation of a Conic

Perform a rotation of axes to eliminate the xy-term in the quadratic equation

$$3x^2 - 10xy + 3y^2 + 16\sqrt{2}x - 32 = 0.$$

SOLUTION

The matrix of the quadratic form associated with this equation is

$$A = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}.$$

The eigenvalues of A are

$$\lambda_1 = 8 \quad \text{and} \quad \lambda_2 = -2$$

with corresponding eigenvectors of

$$\mathbf{x}_1 = (-1, 1) \quad \text{and} \quad \mathbf{x}_2 = (-1, -1).$$

This implies that the matrix P is

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ where } |P| = 1.$$

Because $\cos 135° = -1/\sqrt{2}$ and $\sin 135° = 1/\sqrt{2}$, the angle of rotation is 135°. Finally, from the matrix product

$$[d \quad e]PX' = [16\sqrt{2} \quad 0]\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= -16x' - 16y'$$

the equation of the rotated conic is

$$8(x')^2 - 2(y')^2 - 16x' - 16y' - 32 = 0.$$

In standard form, the equation

$$\frac{(x'-1)^2}{1^2} - \frac{(y'+4)^2}{2^2} = 1$$

is the equation of a hyperbola. Its graph is shown in Figure 7.6.

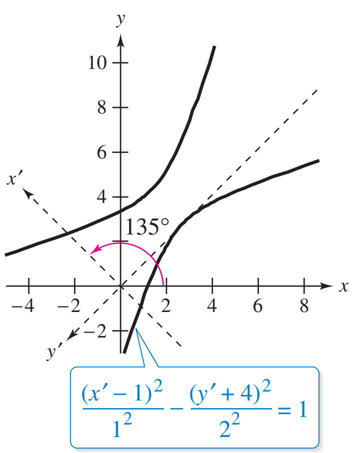
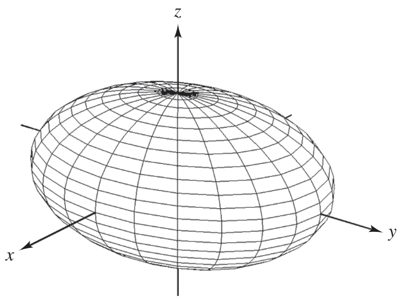


Figure 7.6

Quadratic forms can also be used to analyze equations of quadric surfaces in R^3, which are the three-dimensional analogs of conic sections. The equation of a quadric surface in R^3 is a second-degree polynomial of the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0.$$

There are six basic types of quadric surfaces: ellipsoids, hyperboloids of one sheet, hyperboloids of two sheets, elliptic cones, elliptic paraboloids, and hyperbolic paraboloids. The intersection of a surface with a plane, called the **trace** of the surface in the plane, is useful to help visualize the graph of the surface in R^3. The six basic types of quadric surfaces, together with their traces, are shown on the next two pages.

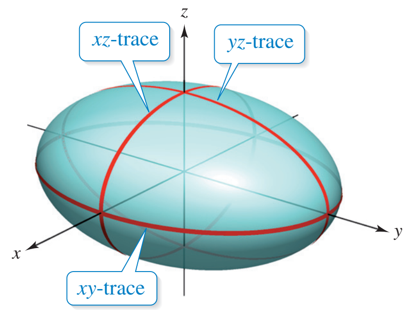


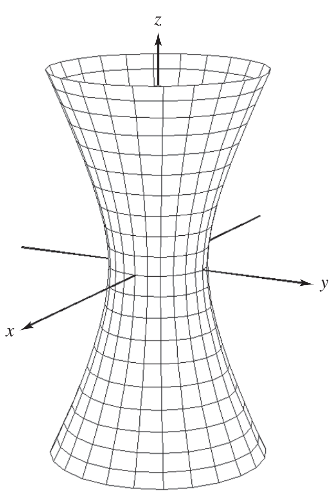
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Ellipse	Parallel to xz-plane
Ellipse	Parallel to yz-plane

The surface is a sphere when $a = b = c \neq 0$.



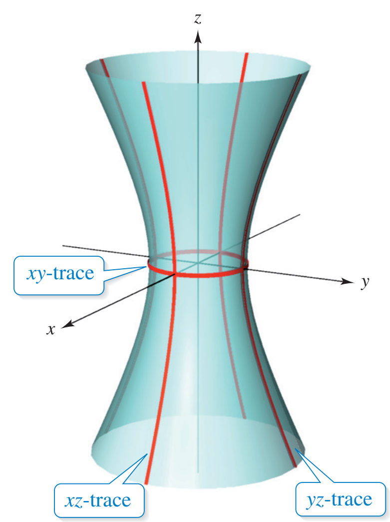


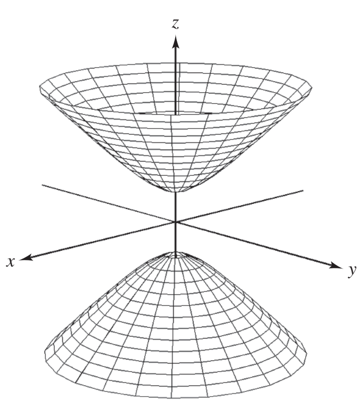
Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



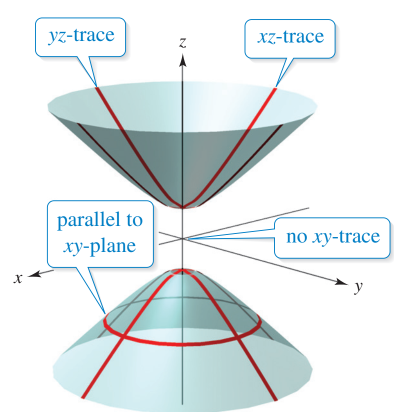


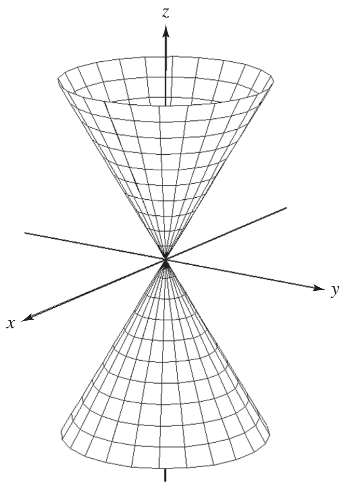
Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



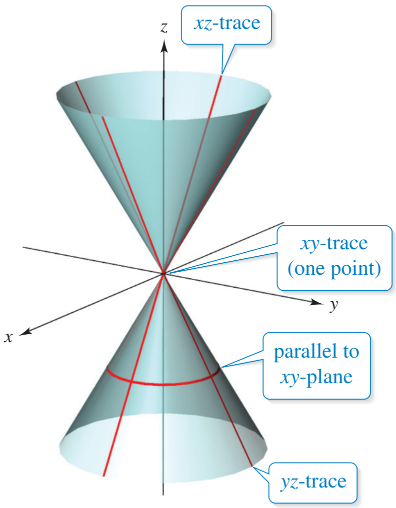


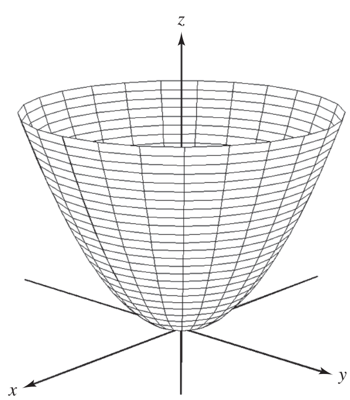
Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.



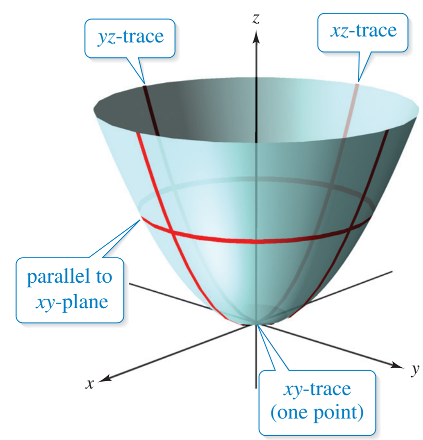


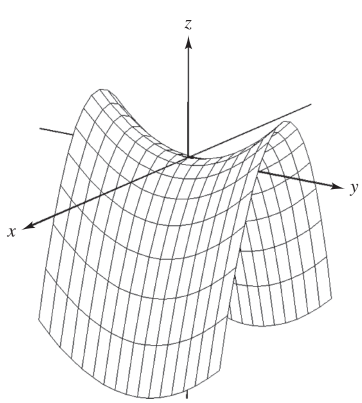
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



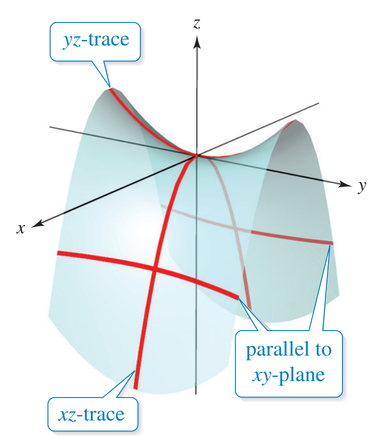


Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



LINEAR ALGEBRA APPLIED

Some of the world's most unusual architecture makes use of quadric surfaces. For instance, *Catedral Metropolitana Nossa Senhora Aparecida,* a cathedral located in Brasilia, Brazil, is in the shape of a hyperboloid of one sheet. It was designed by Pritzker Prize winning architect Oscar Niemeyer, and dedicated in 1970. The sixteen identical curved steel columns, weighing 90 tons each, are intended to represent two hands reaching up to the sky. Pieced together between the columns, in the 10-meter-wide and 30-meter-high triangular gaps formed by the columns, is semitransparent stained glass, which allows light inside for nearly the entire height of the columns.

The quadratic form of the equation

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0 \qquad \text{Quadric surface}$$

is defined as

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz. \qquad \text{Quadratic form}$$

The corresponding matrix is

$$A = \begin{bmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{bmatrix}.$$

In its three-dimensional version, the Principal Axes Theorem relates the eigenvalues and eigenvectors of A to the equation of the rotated surface, as shown in Example 8.

EXAMPLE 8 Rotation of a Quadric Surface

Perform a rotation of axes to eliminate the xz-term in the quadratic equation

$$5x^2 + 4y^2 + 5z^2 + 8xz - 36 = 0.$$

SOLUTION

The matrix A associated with this quadratic equation is

$$A = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix}$$

which has eigenvalues of $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 9$. So, in the rotated $x'y'z'$-system, the quadratic equation is $(x')^2 + 4(y')^2 + 9(z')^2 - 36 = 0$, which in standard form is

$$\frac{(x')^2}{6^2} + \frac{(y')^2}{3^2} + \frac{(z')^2}{2^2} = 1.$$

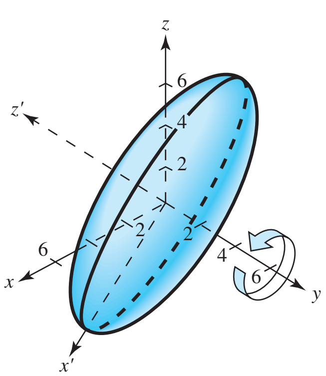


Figure 7.7

The graph of this equation is an ellipsoid. As shown in Figure 7.7, the $x'y'z'$-axes represent a counterclockwise rotation of 45° about the y-axis. Moreover, the orthogonal matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

whose columns are the eigenvectors of A, has the property that P^TAP is diagonal.