5.5 Applications of Inner Product Spaces

- Find the cross product of two vectors in R^3.
- Find the linear or quadratic least squares approximation of a function.
- Find the nth-order Fourier approximation of a function.

THE CROSS PRODUCT OF TWO VECTORS IN R^3

Here you will look at a vector product that yields a vector in R^3 orthogonal to two vectors. This vector product is called the **cross product,** and it is most conveniently defined and calculated with vectors written in standard unit vector form

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

REMARK

The cross product is defined only for vectors in R^3. The cross product of two vectors in R^n, $n \neq 3$, is not defined here.

Definition of the Cross Product of Two Vectors

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in R^3. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

A convenient way to remember the formula for the cross product $\mathbf{u} \times \mathbf{v}$ is to use the following determinant form.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \begin{matrix} \\ \leftarrow \text{Components of } \mathbf{u} \\ \leftarrow \text{Components of } \mathbf{v} \end{matrix}$$

Technically this is not a determinant because it represents a vector and not a real number. Nevertheless, it is useful because it can help you remember the cross product formula. Using cofactor expansion in the first row produces

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\end{aligned}$$

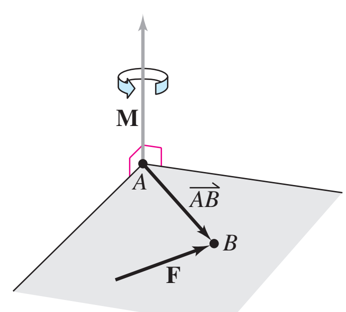
which yields the formula in the definition. Be sure to note that the **j**-component is preceded by a minus sign.

LINEAR ALGEBRA APPLIED

In physics, the cross product can be used to measure *torque*—the moment **M** of a force **F** about a point A, as shown in the figure below. When the point of application of the force is B, the moment of **F** about A is given by

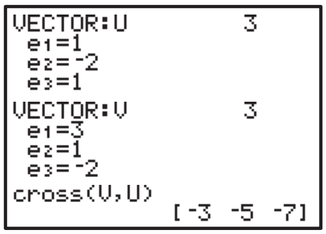
$$\mathbf{M} = \overrightarrow{AB} \times \mathbf{F}$$

where $\overrightarrow{AB}$ represents the vector whose initial point is A and whose terminal point is B. The magnitude of the moment **M** measures the tendency of $\overrightarrow{AB}$ to rotate counterclockwise about an axis directed along the vector **M**.



TECHNOLOGY

Many graphing utilities and software programs can find a cross product. For instance, if you use a graphing utility to verify the result of Example 1(b), then you may see something similar to the following.



Simulation

Explore this concept further with an electronic simulation, and for syntax regarding specific programs involving Example 1, please visit *www.cengagebrain.com*. Similar exercises and projects are also available on the website.

EXAMPLE 1 Finding the Cross Product of Two Vectors

Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Find each cross product.

a. $\mathbf{u} \times \mathbf{v}$ **b.** $\mathbf{v} \times \mathbf{u}$ **c.** $\mathbf{v} \times \mathbf{v}$

SOLUTION

a. $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}\mathbf{k}$$

$$= 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

b. $$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$

$$= -3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

Note that this result is the negative of that in part (a).

c. $$\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

The results obtained in Example 1 suggest some interesting *algebraic* properties of the cross product. For instance,

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \quad \text{and} \quad \mathbf{v} \times \mathbf{v} = \mathbf{0}.$$

These properties, along with several others, are stated in Theorem 5.17.

THEOREM 5.17 Algebraic Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^3 and c is a scalar, then the following properties are true.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = c\mathbf{u} \times \mathbf{v} = \mathbf{u} \times c\mathbf{v}$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

PROOF

The proof of the first property is given here. The proofs of the other properties are left to you. (See Exercises 53–57.) Let $\mathbf{u}$ and $\mathbf{v}$ be

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

and

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Then $\mathbf{u} \times \mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

and $\mathbf{v} \times \mathbf{u}$ is

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$
$$= (v_2u_3 - v_3u_2)\mathbf{i} - (v_1u_3 - v_3u_1)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k}$$
$$= -(u_2v_3 - u_3v_2)\mathbf{i} + (u_1v_3 - u_3v_1)\mathbf{j} - (u_1v_2 - u_2v_1)\mathbf{k}$$
$$= -(\mathbf{v} \times \mathbf{u}).$$

Property 1 of Theorem 5.17 tells you that the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but opposite directions. The geometric implication of this will be discussed after establishing some geometric properties of the cross product of two vectors.

THEOREM 5.18 Geometric Properties of the Cross Product

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^3, then the following properties are true.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta$.
3. $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
4. The parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides has an area of $\|\mathbf{u} \times \mathbf{v}\|$.

PROOF

The proof of Property 4 follows. The proofs of the other properties are left to you. (See Exercises 58–60.) Let $\mathbf{u}$ and $\mathbf{v}$ represent adjacent sides of a parallelogram, as shown in Figure 5.28. By Property 2, the area of the parallelogram is

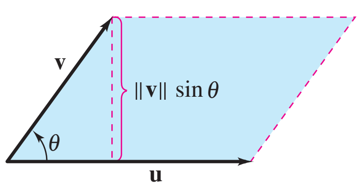


Figure 5.28

$$\text{Area} = \underbrace{\|\mathbf{u}\|}_{\text{Base}}\,\underbrace{\|\mathbf{v}\| \sin \theta}_{\text{Height}} = \|\mathbf{u} \times \mathbf{v}\|.$$

Property 1 states that the vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. This implies that $\mathbf{u} \times \mathbf{v}$ (and $\mathbf{v} \times \mathbf{u}$) is orthogonal to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. One way to remember the orientation of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is to compare them with the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, as shown in Figure 5.29. The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a *right-handed system,* whereas the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} \times \mathbf{u}$ form a *left-handed system.*

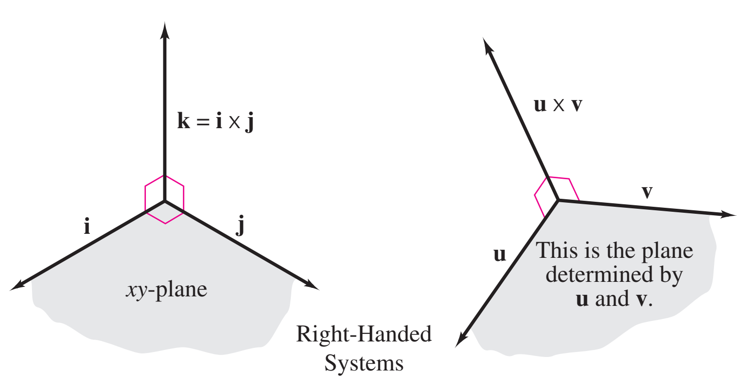


Figure 5.29

EXAMPLE 2 Finding a Vector Orthogonal to Two Given Vectors

Find a unit vector orthogonal to both

$$\mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}.$$

SOLUTION

From Property 1 of Theorem 5.18, you know that the cross product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ 2 & 3 & 0 \end{vmatrix} = -3\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}$$

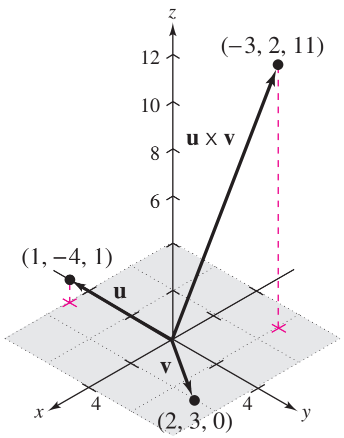


Figure 5.30

is orthogonal to both **u** and **v**, as shown in Figure 5.30. Then, by dividing by the length of **u** × **v**,

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 11^2} = \sqrt{134}$$

you obtain the unit vector

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{3}{\sqrt{134}}\mathbf{i} + \frac{2}{\sqrt{134}}\mathbf{j} + \frac{11}{\sqrt{134}}\mathbf{k}$$

which is orthogonal to both **u** and **v**, because

$$\left(-\frac{3}{\sqrt{134}}, \frac{2}{\sqrt{134}}, \frac{11}{\sqrt{134}}\right) \cdot (1, -4, 1) = 0$$

and

$$\left(-\frac{3}{\sqrt{134}}, \frac{2}{\sqrt{134}}, \frac{11}{\sqrt{134}}\right) \cdot (2, 3, 0) = 0.$$

EXAMPLE 3 Finding the Area of a Parallelogram

Find the area of the parallelogram that has

$$\mathbf{u} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{v} = -2\mathbf{j} + 6\mathbf{k}$$

as adjacent sides, as shown in Figure 5.31.

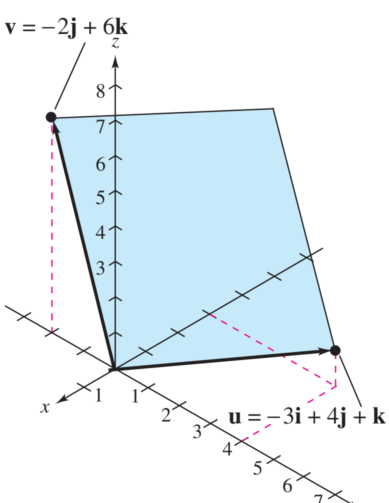
SOLUTION

From Property 4 of Theorem 5.18, you know that the area of this parallelogram is $\|\mathbf{u} \times \mathbf{v}\|$. Because

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} = 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$$

the area of the parallelogram is

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{26^2 + 18^2 + 6^2} = \sqrt{1036} \approx 32.19 \text{ square units.}$$



The area of the parallelogram is $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{1036}$.

Figure 5.31

LEAST SQUARES APPROXIMATIONS (CALCULUS)

Many problems in the physical sciences and engineering involve an approximation of a function f by another function g. If f is in $C[a, b]$ (the inner product space of all continuous functions on $[a, b]$), then g is usually chosen from a subspace W of $C[a, b]$. For instance, to approximate the function

$$f(x) = e^x, \quad 0 \le x \le 1$$

you could choose one of the following forms of g.

1. $g(x) = a_0 + a_1x, \quad 0 \le x \le 1$ — Linear
2. $g(x) = a_0 + a_1x + a_2x^2, \quad 0 \le x \le 1$ — Quadratic
3. $g(x) = a_0 + a_1 \cos x + a_2 \sin x, \quad 0 \le x \le 1$ — Trigonometric

Before discussing ways of finding the function g, you must define how one function can "best" approximate another function. One natural way would require the area bounded by the graphs of f and g on the interval $[a, b]$,

$$\text{Area} = \int_a^b |f(x) - g(x)|\, dx$$

to be a minimum with respect to other functions in the subspace W, as shown in Figure 5.32.

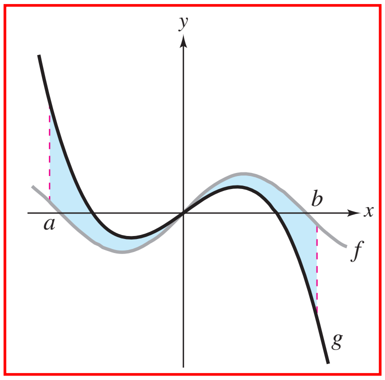


Figure 5.32

Because integrands involving absolute value are often difficult to evaluate, however, it is more common to square the integrand to obtain

$$\int_a^b [f(x) - g(x)]^2\, dx.$$

With this criterion, the function g is called the **least squares approximation** of f with respect to the inner product space W.

Definition of Least Squares Approximation

Let f be continuous on $[a, b]$, and let W be a subspace of $C[a, b]$. A function g in W is called a **least squares approximation** of f with respect to W when the value of

$$I = \int_a^b [f(x) - g(x)]^2\, dx$$

is a minimum with respect to all other functions in W.

Note that if the subspace W in this definition is the entire space $C[a, b]$, then $g(x) = f(x)$, which implies that $I = 0$.

EXAMPLE 4 Finding a Least Squares Approximation

Find the least squares approximation $g(x) = a_0 + a_1 x$ of

$$f(x) = e^x, \quad 0 \le x \le 1.$$

SOLUTION

For this approximation you need to find the constants a_0 and a_1 that minimize the value of

$$\begin{aligned} I &= \int_0^1 [f(x) - g(x)]^2 \, dx \\ &= \int_0^1 (e^x - a_0 - a_1 x)^2 \, dx. \end{aligned}$$

Evaluating this integral, you have

$$\begin{aligned} I &= \int_0^1 (e^x - a_0 - a_1 x)^2 \, dx \\ &= \int_0^1 (e^{2x} - 2a_0 e^x - 2a_1 x e^x + a_0^2 + 2a_0 a_1 x + a_1^2 x^2) \, dx \\ &= \left[\frac{1}{2} e^{2x} - 2a_0 e^x - 2a_1 e^x (x - 1) + a_0^2 x + a_0 a_1 x^2 + a_1^2 \frac{x^3}{3} \right]_0^1 \\ &= \frac{1}{2}(e^2 - 1) - 2a_0(e - 1) - 2a_1 + a_0^2 + a_0 a_1 + \frac{1}{3} a_1^2. \end{aligned}$$

Now, considering I to be a function of the variables a_0 and a_1, use calculus to determine the values of a_0 and a_1 that minimize I. Specifically, by setting the partial derivatives

$$\begin{aligned} \frac{\partial I}{\partial a_0} &= 2a_0 - 2e + 2 + a_1 \\ \frac{\partial I}{\partial a_1} &= a_0 + \frac{2}{3} a_1 - 2 \end{aligned}$$

equal to zero, you obtain the following two linear equations in a_0 and a_1.

$$\begin{aligned} 2a_0 + a_1 &= 2(e - 1) \\ 3a_0 + 2a_1 &= 6 \end{aligned}$$

The solution of this system is

$$a_0 = 4e - 10 \approx 0.873 \quad \text{and} \quad a_1 = 18 - 6e \approx 1.690.$$

(Verify this.) So, the best *linear approximation* of $f(x) = e^x$ on the interval $[0, 1]$ is

$$g(x) = 4e - 10 + (18 - 6e)x \approx 0.873 + 1.690x.$$

Figure 5.33 shows the graphs of f and g on $[0, 1]$.

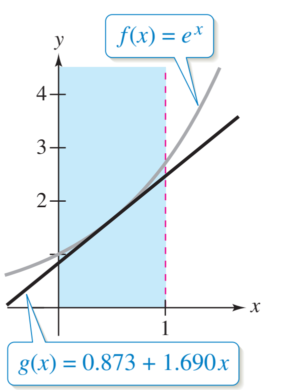


Figure 5.33

Of course, whether the approximation obtained in Example 4 is the best approximation depends on the definition of the best approximation. For instance, if the definition of the best approximation had been the *Taylor polynomial of degree 1* centered at 0.5, then the approximating function g would have been

$$\begin{aligned} g(x) &= f(0.5) + f'(0.5)(x - 0.5) \\ &= e^{0.5} + e^{0.5}(x - 0.5) \\ &\approx 0.824 + 1.649x. \end{aligned}$$

Moreover, the function g obtained in Example 4 is only the best *linear* approximation of f (according to the least squares criterion). In Example 5 you will find the best *quadratic* approximation.

EXAMPLE 5 Finding a Least Squares Approximation

Find the least squares approximation $g(x) = a_0 + a_1x + a_2x^2$ of $f(x) = e^x$, $0 \le x \le 1$.

SOLUTION

For this approximation you need to find the values of a_0, a_1, and a_2 that minimize the value of

$$\begin{aligned} I &= \int_0^1 [f(x) - g(x)]^2\,dx \\ &= \int_0^1 (e^x - a_0 - a_1x - a_2x^2)^2\,dx \\ &= \frac{1}{2}(e^2 - 1) + 2a_0(1 - e) + 2a_2(2 - e) \\ &\qquad + a_0^2 + a_0a_1 + \frac{2}{3}a_0a_2 + \frac{1}{2}a_1a_2 + \frac{1}{3}a_1^2 + \frac{1}{5}a_2^2 - 2a_1. \end{aligned}$$

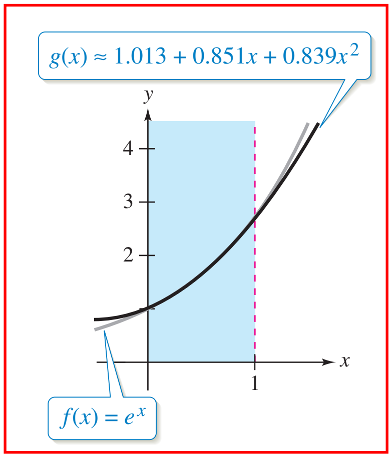


Figure 5.34

Setting the partial derivatives of I (with respect to a_0, a_1, and a_2) equal to zero produces the following system of linear equations.

$$\begin{aligned} 6a_0 + 3a_1 + 2a_2 &= 6(e - 1) \\ 6a_0 + 4a_1 + 3a_2 &= 12 \\ 20a_0 + 15a_1 + 12a_2 &= 60(e - 2) \end{aligned}$$

The solution of this system is

$$\begin{aligned} a_0 &= -105 + 39e \approx 1.013 \\ a_1 &= 588 - 216e \approx 0.851 \\ a_2 &= -570 + 210e \approx 0.839. \end{aligned}$$

(Verify this.) So, the approximating function g is $g(x) \approx 1.013 + 0.851x + 0.839x^2$. Figure 5.34 shows the graphs of f and g.

The integral I (given in the definition of the least squares approximation) can be expressed in vector form. To do this, use the inner product defined in Example 5 in Section 5.2:

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$

With this inner product you have

$$I = \int_a^b [f(x) - g(x)]^2\,dx = \langle f - g, f - g\rangle = \|f - g\|^2.$$

This means that the least squares approximating function g is the function that minimizes $\|f - g\|^2$ or, equivalently, minimizes $\|f - g\|$. In other words, the least squares approximation of a function f is the function g (in the subspace W) closest to f in terms of the inner product $\langle f, g\rangle$. The next theorem gives you a way of determining the function g.

THEOREM 5.19 Least Squares Approximation

Let f be continuous on $[a, b]$, and let W be a finite-dimensional subspace of $C[a, b]$. The least squares approximating function of f with respect to W is given by

$$g = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 + \cdots + \langle f, \mathbf{w}_n\rangle\mathbf{w}_n$$

where $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for W.

PROOF

To show that g is the least squares approximating function of f, prove that the inequality $\|f - g\| \leq \|f - \mathbf{w}\|$ is true for any vector $\mathbf{w}$ in W. By writing $f - g$ as

$$f - g = f - \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 - \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 - \cdots - \langle f, \mathbf{w}_n\rangle\mathbf{w}_n$$

you can see that $f - g$ is orthogonal to each $\mathbf{w}_i$, which in turn implies that it is orthogonal to each vector in W. In particular, $f - g$ is orthogonal to $g - \mathbf{w}$. This allows you to apply the Pythagorean Theorem to the vector sum $f - \mathbf{w} = (f - g) + (g - \mathbf{w})$ to conclude that $\|f - \mathbf{w}\|^2 = \|f - g\|^2 + \|g - \mathbf{w}\|^2$. So, it follows that $\|f - g\|^2 \leq \|f - \mathbf{w}\|^2$, which then implies that $\|f - g\| \leq \|f - \mathbf{w}\|$.

Now observe how Theorem 5.19 can be used to produce the least squares approximation obtained in Example 4. First apply the Gram-Schmidt orthonormalization process to the standard basis $\{1, x\}$ to obtain the orthonormal basis $B = \{1, \sqrt{3}(2x - 1)\}$. (Verify this.) Then, by Theorem 5.19, the least squares approximation of e^x in the subspace of all linear functions is

$$\begin{aligned} g(x) &= \langle e^x, 1\rangle(1) + \langle e^x, \sqrt{3}(2x - 1)\rangle\sqrt{3}(2x - 1) \\ &= \int_0^1 e^x\,dx + \sqrt{3}(2x - 1)\int_0^1 \sqrt{3}e^x(2x - 1)\,dx \\ &= \int_0^1 e^x\,dx + 3(2x - 1)\int_0^1 e^x(2x - 1)\,dx \\ &= 4e - 10 + (18 - 6e)x \end{aligned}$$

which agrees with the result obtained in Example 4.

EXAMPLE 6 Finding a Least Squares Approximation

Find the least squares approximation of $f(x) = \sin x$, $0 \leq x \leq \pi$, with respect to the subspace W of polynomial functions of degree 2 or less.

SOLUTION

To use Theorem 5.19, apply the Gram-Schmidt orthonormalization process to the standard basis for W, $\{1, x, x^2\}$, to obtain the orthonormal basis

$$B = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \left\{\frac{1}{\sqrt{\pi}}, \frac{\sqrt{3}}{\pi\sqrt{\pi}}(2x - \pi), \frac{\sqrt{5}}{\pi^2\sqrt{\pi}}(6x^2 - 6\pi x + \pi^2)\right\}.$$

(Verify this.) The least squares approximating function g is

$$g(x) = \langle f, \mathbf{w}_1\rangle\mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle\mathbf{w}_2 + \langle f, \mathbf{w}_3\rangle\mathbf{w}_3$$

and you have

$$\langle f, \mathbf{w}_1\rangle = \frac{1}{\sqrt{\pi}}\int_0^\pi \sin x\,dx = \frac{2}{\sqrt{\pi}}$$

$$\langle f, \mathbf{w}_2\rangle = \frac{\sqrt{3}}{\pi\sqrt{\pi}}\int_0^\pi \sin x(2x - \pi)\,dx = 0$$

$$\langle f, \mathbf{w}_3\rangle = \frac{\sqrt{5}}{\pi^2\sqrt{\pi}}\int_0^\pi \sin x(6x^2 - 6\pi x + \pi^2)\,dx = \frac{2\sqrt{5}}{\pi^2\sqrt{\pi}}(\pi^2 - 12).$$

So, g is

$$g(x) = \frac{2}{\pi} + \frac{10(\pi^2 - 12)}{\pi^5}(6x^2 - 6\pi x + \pi^2) \approx -0.4177x^2 + 1.3122x - 0.0505.$$

Figure 5.35 shows the graphs of f and g.

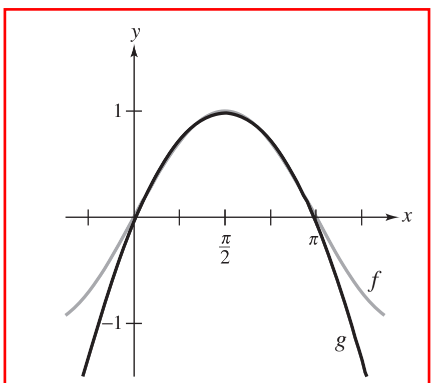


Figure 5.35

FOURIER APPROXIMATIONS (CALCULUS)

You will now look at a special type of least squares approximation called a **Fourier approximation.** For this approximation, consider functions of the form

$$g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$$

in the subspace W of

$$C[0, 2\pi]$$

spanned by the basis

$$S = \{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}.$$

These $2n + 1$ vectors are orthogonal in the inner product space $C[0, 2\pi]$ because

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\, dx = 0, \quad f \neq g$$

as demonstrated in Example 3 in Section 5.3. Moreover, by normalizing each function in this basis, you obtain the orthonormal basis

$$\begin{aligned} B &= \{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_n, \mathbf{w}_{n+1}, \ldots, \mathbf{w}_{2n}\} \\ &= \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \ldots, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin x, \ldots, \frac{1}{\sqrt{\pi}} \sin nx\right\}. \end{aligned}$$

With this orthonormal basis, you can apply Theorem 5.19 to write

$$g(x) = \langle f, \mathbf{w}_0 \rangle \mathbf{w}_0 + \langle f, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle f, \mathbf{w}_{2n} \rangle \mathbf{w}_{2n}.$$

The coefficients

$$a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$$

for $g(x)$ in the equation

$$g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$$

are given by the following integrals.

$$\begin{aligned}
a_0 &= \langle f, \mathbf{w}_0 \rangle \frac{2}{\sqrt{2\pi}} = \frac{2}{\sqrt{2\pi}} \int_0^{2\pi} f(x) \frac{1}{\sqrt{2\pi}}\, dx = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx \\
a_1 &= \langle f, \mathbf{w}_1 \rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \cos x\, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x\, dx \\
&\vdots \\
a_n &= \langle f, \mathbf{w}_n \rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \cos nx\, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx \\
b_1 &= \langle f, \mathbf{w}_{n+1} \rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \sin x\, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin x\, dx \\
&\vdots \\
b_n &= \langle f, \mathbf{w}_{2n} \rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \sin nx\, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx
\end{aligned}$$

The function $g(x)$ is called the ***n*th-order Fourier approximation** of f on the interval $[0, 2\pi]$. Like Fourier coefficients, this function is named after the French mathematician Jean-Baptiste Joseph Fourier (1768–1830). This brings you to Theorem 5.20.

THEOREM 5.20 Fourier Approximation

On the interval $[0, 2\pi]$, the least squares approximation of a continuous function f with respect to the vector space spanned by

$$\{1, \cos x, \ldots, \cos nx, \sin x, \ldots, \sin nx\}$$

is

$$g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$$

where the **Fourier coefficients** $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ are

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_j = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos jx\,dx, \quad j = 1, 2, \ldots, n$$

$$b_j = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin jx\,dx, \quad j = 1, 2, \ldots, n.$$

EXAMPLE 7 Finding a Fourier Approximation

Find the third-order Fourier approximation of $f(x) = x$, $0 \le x \le 2\pi$.

SOLUTION

Using Theorem 5.20, you have

$$g(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}2\pi^2 = 2\pi$$

$$a_j = \frac{1}{\pi}\int_0^{2\pi} x\cos jx\,dx = \left[\frac{1}{\pi j^2}\cos jx + \frac{x}{\pi j}\sin jx\right]_0^{2\pi} = 0$$

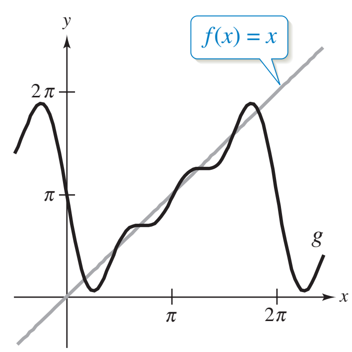
$$b_j = \frac{1}{\pi}\int_0^{2\pi} x\sin jx\,dx = \left[\frac{1}{\pi j^2}\sin jx - \frac{x}{\pi j}\cos jx\right]_0^{2\pi} = -\frac{2}{j}.$$

This implies that $a_0 = 2\pi$, $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $b_1 = -2$, $b_2 = -\frac{2}{2} = -1$, and $b_3 = -\frac{2}{3}$. So, you have

$$g(x) = \frac{2\pi}{2} - 2\sin x - \sin 2x - \frac{2}{3}\sin 3x$$

$$= \pi - 2\sin x - \sin 2x - \frac{2}{3}\sin 3x.$$

Figure 5.36 compares the graphs of f and g.



Third-Order Fourier Approximation
Figure 5.36

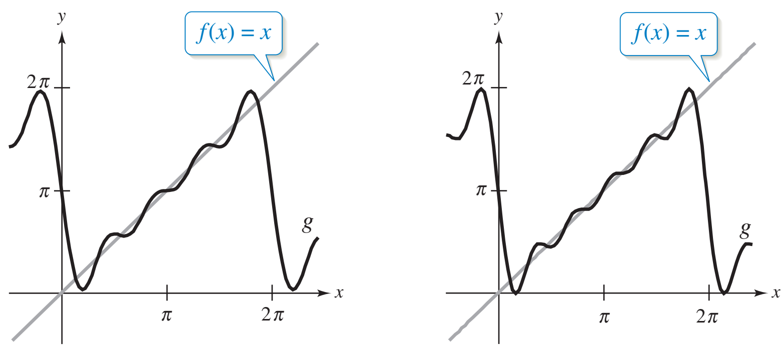
In Example 7, the pattern for the Fourier coefficients appears to be $a_0 = 2\pi$, $a_1 = a_2 = \cdot \cdot \cdot = a_n = 0$, and

$$b_1 = -\frac{2}{1},\ b_2 = -\frac{2}{2},\ \ldots,\ b_n = -\frac{2}{n}.$$

The nth-order Fourier approximation of $f(x) = x$ is

$$g(x) = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdot \cdot \cdot + \frac{1}{n}\sin nx\right).$$

As n increases, the Fourier approximation improves. For instance, Figure 5.37 shows the fourth- and fifth-order Fourier approximations of $f(x) = x$, $0 \leq x \leq 2\pi$.



Fourth-Order Fourier Approximation Fifth-Order Fourier Approximation

Figure 5.37

In advanced courses it is shown that as $n \to \infty$, the approximation error $\|f - g\|$ approaches zero for all x in the interval $(0, 2\pi)$. The infinite *series* for $g(x)$ is called a *Fourier series*.

EXAMPLE 8 Finding a Fourier Approximation

Find the fourth-order Fourier approximation of $f(x) = |x - \pi|$, $0 \leq x \leq 2\pi$.

SOLUTION

Using Theorem 5.20, find the Fourier coefficients as follows.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} |x - \pi|\, dx = \pi$$

$$a_j = \frac{1}{\pi}\int_0^{2\pi} |x - \pi| \cos jx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x) \cos jx\, dx$$

$$= \frac{2}{\pi j^2}(1 - \cos j\pi)$$

$$b_j = \frac{1}{\pi}\int_0^{2\pi} |x - \pi| \sin jx\, dx = 0$$

So, $a_0 = \pi$, $a_1 = 4/\pi$, $a_2 = 0$, $a_3 = 4/9\pi$, $a_4 = 0$, $b_1 = 0$, $b_2 = 0$, $b_3 = 0$, and $b_4 = 0$, which means that the fourth-order Fourier approximation of f is

$$g(x) = \frac{\pi}{2} + \frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x.$$

Figure 5.38 compares the graphs of f and g.

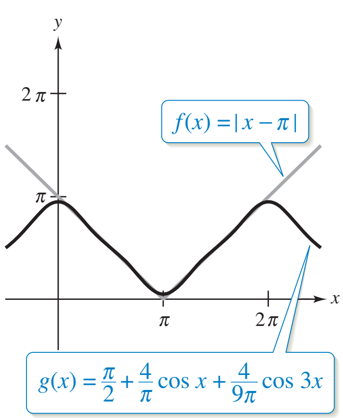


Figure 5.38