4.8 Applications of Vector Spaces

- Use the Wronskian to test a set of solutions of a linear homogeneous differential equation for linear independence.
- Identify and sketch the graph of a conic section and perform a rotation of axes.

LINEAR DIFFERENTIAL EQUATIONS (CALCULUS)

A **linear differential equation of order *n*** is of the form

$$y^{(n)} + g_{n-1}(x)y^{(n-1)} + \cdots + g_1(x)y' + g_0(x)y = f(x)$$

where $g_0, g_1, \ldots, g_{n-1}$ and f are functions of x with a common domain. If $f(x) = 0$, then the equation is **homogeneous.** Otherwise it is **nonhomogeneous.** A function y is called a **solution** of the linear differential equation if the equation is satisfied when y and its first n derivatives are substituted into the equation.

EXAMPLE 1 A Second-Order Linear Differential Equation

Show that both $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the second-order linear differential equation $y'' - y = 0$.

SOLUTION

For the function $y_1 = e^x$, you have $y_1' = e^x$ and $y_1'' = e^x$. So,

$$y_1'' - y_1 = e^x - e^x = 0$$

which means that $y_1 = e^x$ is a solution of the differential equation. Similarly, for $y_2 = e^{-x}$, you have

$$y_2' = -e^{-x} \quad \text{and} \quad y_2'' = e^{-x}.$$

This implies that

$$y_2'' - y_2 = e^{-x} - e^{-x} = 0.$$

So, $y_2 = e^{-x}$ is also a solution of the linear differential equation.

There are two important observations you can make about Example 1. The first is that in the vector space $C''(-\infty, \infty)$ of all twice differentiable functions defined on the entire real line, the two solutions $y_1 = e^x$ and $y_2 = e^{-x}$ are *linearly independent*. This means that the only solution of

$$C_1y_1 + C_2y_2 = 0$$

that is valid for all x is $C_1 = C_2 = 0$. The second observation is that every *linear combination* of y_1 and y_2 is also a solution of the linear differential equation. To see this, let $y = C_1y_1 + C_2y_2$. Then

$$\begin{aligned} y &= C_1e^x + C_2e^{-x} \\ y' &= C_1e^x - C_2e^{-x} \\ y'' &= C_1e^x + C_2e^{-x}. \end{aligned}$$

Substituting into the differential equation $y'' - y = 0$ produces

$$y'' - y = (C_1e^x + C_2e^{-x}) - (C_1e^x + C_2e^{-x}) = 0.$$

So, $y = C_1e^x + C_2e^{-x}$ is a solution.

The next theorem, which is stated without proof, generalizes these observations.

REMARK

The solution

$y = C_1y_1 + C_2y_2 + \cdot\cdot\cdot + C_ny_n$

is called the **general solution** of the given differential equation.

Solutions of a Linear Homogeneous Differential Equation

Every nth-order linear homogeneous differential equation

$$y^{(n)} + g_{n-1}(x)y^{(n-1)} + \cdot\cdot\cdot + g_1(x)y' + g_0(x)y = 0$$

has n linearly independent solutions. Moreover, if $\{y_1, y_2, \dots, y_n\}$ is a set of linearly independent solutions, then every solution is of the form

$$y = C_1y_1 + C_2y_2 + \cdot\cdot\cdot + C_ny_n$$

where $C_1, C_2, \dots, C_n$ are real numbers.

In light of the preceding theorem, you can see the importance of being able to determine whether a set of solutions is linearly independent. Before describing a way of testing for linear independence, you are given the following preliminary definition.

REMARK

The Wronskian of a set of functions is named after the Polish mathematician Josef Maria Wronski (1778–1853).

Definition of the Wronskian of a Set of Functions

Let $\{y_1, y_2, \dots, y_n\}$ be a set of functions, each of which has $n - 1$ derivatives on an interval I. The determinant

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of the given set of functions.

EXAMPLE 2 Finding the Wronskian of a Set of Functions

a. The Wronskian of the set $\{1 - x, 1 + x, 2 - x\}$ is

$$W = \begin{vmatrix} 1 - x & 1 + x & 2 - x \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

b. The Wronskian of the set $\{x, x^2, x^3\}$ is

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3.$$

The Wronskian in part (a) of Example 2 is said to be **identically equal to zero,** because it is zero for any value of x. The Wronskian in part (b) is not identically equal to zero because values of x exist for which this Wronskian is nonzero.

The next theorem shows how the Wronskian of a set of functions can be used to test for linear independence.

REMARK

This test does *not* apply to an arbitrary set of functions. Each of the functions $y_1, y_2, \dots,$ and y_n must be a solution of the same linear homogeneous differential equation of order n.

Wronskian Test for Linear Independence

Let $\{y_1, y_2, \dots, y_n\}$ be a set of n solutions of an nth-order linear homogeneous differential equation. This set is linearly independent if and only if the Wronskian is not identically equal to zero.

The proof of this theorem for the case where $n = 2$ is left as an exercise. (See Exercise 40.)

EXAMPLE 3 **Testing a Set of Solutions for Linear Independence**

Determine whether $\{1, \cos x, \sin x\}$ is a set of linearly independent solutions of the linear homogeneous differential equation

$$y''' + y' = 0.$$

SOLUTION

Begin by observing that each of the functions is a solution of $y''' + y' = 0$. (Try checking this.) Next, testing for linear independence produces the Wronskian of the three functions, as follows.

$$\begin{aligned} W &= \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} \\ &= \sin^2 x + \cos^2 x = 1 \end{aligned}$$

Because W is not identically equal to zero, the set

$$\{1, \cos x, \sin x\}$$

is linearly independent. Moreover, because this set consists of three linearly independent solutions of a third-order linear homogeneous differential equation, the general solution is

$$y = C_1 + C_2 \cos x + C_3 \sin x$$

where C_1, C_2, and C_3 are real numbers.

EXAMPLE 4 **Testing a Set of Solutions for Linear Independence**

Determine whether $\{e^x, xe^x, (x + 1)e^x\}$ is a set of linearly independent solutions of the linear homogeneous differential equation

$$y''' - 3y'' + 3y' - y = 0.$$

SOLUTION

As in Example 3, begin by verifying that each of the functions is actually a solution of $y''' - 3y'' + 3y' - y = 0$. (This verification is left to you.) Testing for linear independence produces the Wronskian of the three functions, as follows.

$$W = \begin{vmatrix} e^x & xe^x & (x+1)e^x \\ e^x & (x+1)e^x & (x+2)e^x \\ e^x & (x+2)e^x & (x+3)e^x \end{vmatrix} = 0$$

So, the set $\{e^x, xe^x, (x + 1)e^x\}$ is linearly dependent.

In Example 4, the Wronskian is used to determine that the set

$$\{e^x, xe^x, (x + 1)e^x\}$$

is linearly dependent. Another way to determine the linear dependence of this set is to observe that the third function is a linear combination of the first two. That is,

$$(x + 1)e^x = e^x + xe^x.$$

Try showing that a different set, $\{e^x, xe^x, x^2e^x\}$, forms a linearly independent set of solutions of the differential equation

$$y''' - 3y'' + 3y' - y = 0.$$

CONIC SECTIONS AND ROTATION

Every conic section in the xy-plane has an equation that can be written in the form

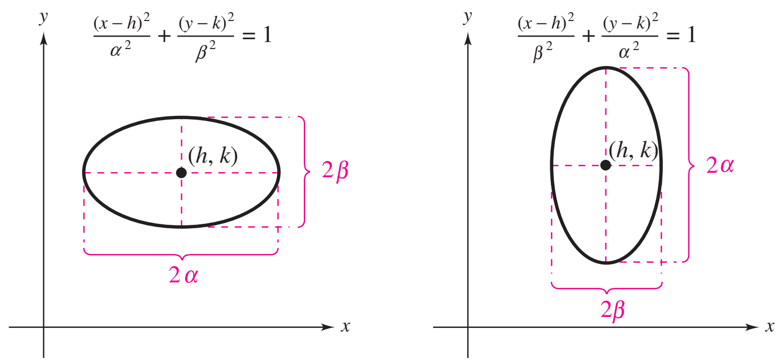
$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Identifying the graph of this equation is fairly simple as long as b, the coefficient of the xy-term, is zero. When b is zero, the conic axes are parallel to the coordinate axes, and the identification is accomplished by writing the equation in standard (completed square) form. The standard forms of the equations of the four basic conics are given in the following summary. For circles, ellipses, and hyperbolas, the point (h, k) is the center. For parabolas, the point (h, k) is the vertex.

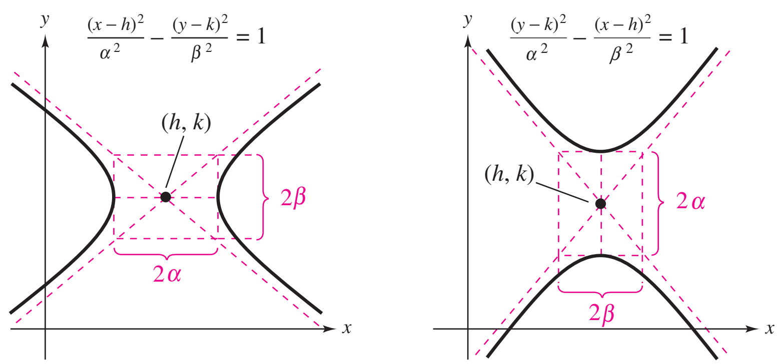
Standard Forms of Equations of Conics

Circle (r = radius): $(x - h)^2 + (y - k)^2 = r^2$

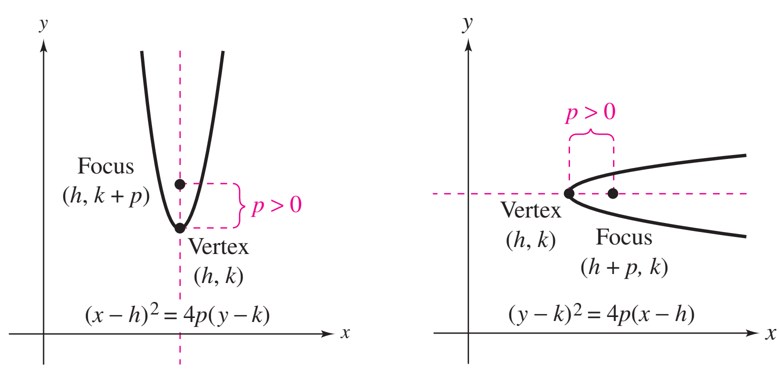
Ellipse (2α = major axis length, 2β = minor axis length):



Hyperbola (2α = transverse axis length, 2β = conjugate axis length):



Parabola (p = directed distance from vertex to focus):



EXAMPLE 5 **Identifying Conic Sections**

a. The standard form of $x^2 - 2x + 4y - 3 = 0$ is

$$(x-1)^2 = 4(-1)(y-1).$$

The graph of this equation is a parabola with the vertex at $(h, k) = (1, 1)$. The axis of the parabola is vertical. Because $p = -1$, the focus is the point $(1, 0)$. Finally, because the focus lies below the vertex, the parabola opens downward, as shown in Figure 4.20(a).

b. The standard form of $x^2 + 4y^2 + 6x - 8y + 9 = 0$ is

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{1} = 1.$$

The graph of this equation is an ellipse with its center at $(h, k) = (-3, 1)$. The major axis is horizontal, and its length is $2\alpha = 4$. The length of the minor axis is $2\beta = 2$. The vertices of this ellipse occur at $(-5, 1)$ and $(-1, 1)$, and the endpoints of the minor axis occur at $(-3, 2)$ and $(-3, 0)$, as shown in Figure 4.20(b).

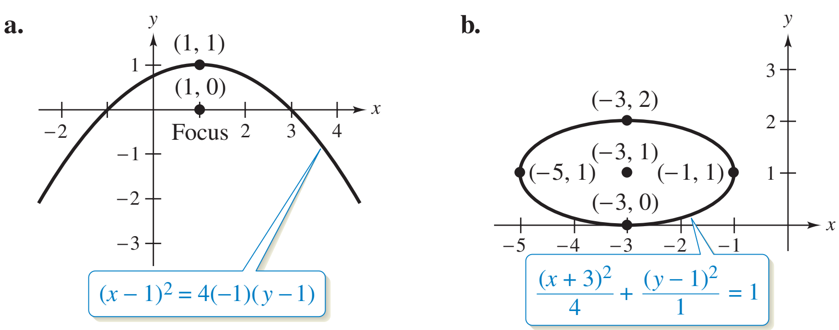


Figure 4.20

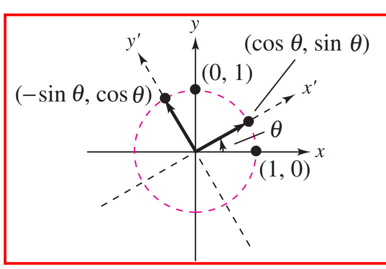
The equations of the conics in Example 5 have no xy-term. Consequently, the axes of the graphs of these conics are parallel to the coordinate axes. For second-degree equations that have an xy-term, the axes of the graphs of the corresponding conics are not parallel to the coordinate axes. In such cases it is helpful to *rotate* the standard axes to form a new x'-axis and y'-axis. The required rotation angle θ (measured counterclockwise) is $\cot 2\theta = (a - c)/b$. With this rotation, the standard basis for R^2,

$$B = \{(1, 0), (0, 1)\}$$

is rotated to form the new basis

$$B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$$

as shown in Figure 4.21.



If b is nonzero, it is helpful for identifying the graph of a conic section by rotating the coordinate axes such that the conic section is parallel to the new axes

Figure 4.21

To find the coordinates of a point (x, y) relative to this new basis, you can use a transition matrix, as demonstrated in Example 6.

EXAMPLE 6 A Transition Matrix for Rotation in R^2

Find the coordinates of a point (x, y) in R^2 relative to the basis

$$B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}.$$

SOLUTION

By Theorem 4.21 you have

target basis original basis

$$[B' \ B] = \begin{bmatrix} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{bmatrix}.$$

Because B is the standard basis for R^2, P^{-1} is represented by $(B')^{-1}$. You can use the formula given in Section 2.3 (page 66) for the inverse of a 2×2 matrix to find $(B')^{-1}$. This results in

[B'|B] → [I | P^(-1)] vs.
[B'| I] → [I | B'^(-1)]

$$[I \ \ P^{-1}] = \begin{bmatrix} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{bmatrix}.$$

By letting (x', y') be the coordinates of (x, y) relative to B', you can use the transition matrix P^{-1} as follows.

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

P^(-1) * [v]_B = [v]_B'

The x'- and y'-coordinates are $\boxed{x' = x\cos\theta + y\sin\theta}$ and $\boxed{y' = -x\sin\theta + y\cos\theta}$.

The last two equations in Example 6 give the $x'y'$-coordinates in terms of the xy-coordinates. To perform a rotation of axes for a general second-degree equation, it is helpful to express the xy-coordinates in terms of the $x'y'$-coordinates. To do this, solve the last two equations in Example 6 for x and y to obtain

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta.$$

Substituting these expressions for x and y into the given second-degree equation produces a second-degree equation in x' and y' that has no $x'y'$-term.

Rotation of Axes

The general second-degree equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be written in the form

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$$

by rotating the coordinate axes counterclockwise through the angle θ, where θ is defined by $\cot 2\theta = \dfrac{a-c}{b}$. The coefficients of the new equation are obtained from the substitutions

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta.$$

The proof of the above result is left to you. (See Exercise 82.)

LINEAR ALGEBRA APPLIED

A satellite dish is an antenna that is designed to transmit or receive signals of a specific type. A standard satellite dish consists of a bowl-shaped surface and a *feed horn* that is aimed toward the surface. The bowl-shaped surface is typically in the shape of an elliptic paraboloid. (See Section 7.4.) The cross section of the surface is typically in the shape of a rotated parabola.

Chris H. Galbraith/Shutterstock.com

Example 7 demonstrates how to identify the graph of a second-degree equation by rotating the coordinate axes.

EXAMPLE 7 Rotation of a Conic Section

Perform a rotation of axes to eliminate the xy-term in

$$5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$$

and sketch the graph of the resulting equation in the $x'y'$-plane.

SOLUTION

The angle of rotation is given by

$$\cot 2\theta = \frac{a-c}{b} = \frac{5-5}{-6} = 0.$$

This implies that $\theta = \pi/4$. So,

$$\sin\theta = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos\theta = \frac{1}{\sqrt{2}}.$$

By substituting

$$x = x'\cos\theta - y'\sin\theta = \frac{1}{\sqrt{2}}(x' - y')$$

and

$$y = x'\sin\theta + y'\cos\theta = \frac{1}{\sqrt{2}}(x' + y')$$

into the original equation and simplifying, you obtain

$$(x')^2 + 4(y')^2 + 6x' - 8y' + 9 = 0.$$

Finally, by completing the square, you find the standard form of this equation to be

$$\frac{(x'+3)^2}{2^2} + \frac{(y'-1)^2}{1^2} = \frac{(x'+3)^2}{4} + \frac{(y'-1)^2}{1} = 1$$

which is the equation of an ellipse, as shown in Figure 4.22.

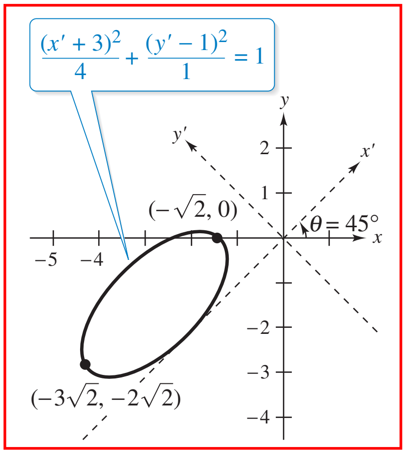


Figure 4.22

In Example 7 the new (rotated) basis for R^2 is

$$B' = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$$

and the coordinates of the vertices of the ellipse relative to B' are

$$\begin{bmatrix}-5\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}-1\\1\end{bmatrix}.$$

Two vertex points on the major axis (長軸) in the new basis

To find the coordinates of the vertices relative to the standard basis $B = \{(1, 0), (0, 1)\}$, use the equations

$$x = \frac{1}{\sqrt{2}}(x' - y')$$

and

$$y = \frac{1}{\sqrt{2}}(x' + y')$$

Two vertex points on the major axis in the original basis

to obtain $(-3\sqrt{2}, -2\sqrt{2})$ and $(-\sqrt{2}, 0)$, as shown in Figure 4.22.