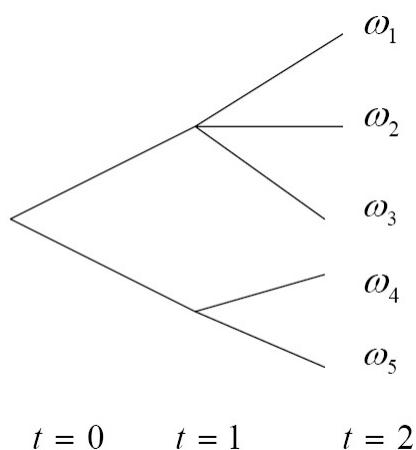


## Ch7. Multiperiod Securities Markets I: Equilibrium Valuation

- 在兩期的模型，只有 $t = 0$ 有交易，但多期之模型需考慮之後也可交易之可能性，因此均衡中需包括個人對未來價格的期望，如此之均衡也可稱為 rational expectation equilibrium (事實上，此期望之價格將來會實現)
- 在多期模型下，要 complete market 所需之證券的數目，一般(遠)小於 state of nature 之數目，因為 state contingent claims 可經由 long-lived complex securities 的 dynamically trading 而製造出來
- 因為 state contingent claims 的可被製造，市場 complete，且均衡之分配為 Pareto optimal ( $\frac{\phi_{at}}{\phi_0}$  與  $i$  無關)，因此，可類似 Ch5 建立一個 representative agent economy 來方便計算均衡價格
- $\text{sign}\{\text{risk premium}\} = \text{sign}\{\text{Cov}(\tilde{r}, \tilde{C})\}$ ，並不像兩期的模型，只有在很特殊的情況下， $\tilde{C}$  才會等於  $\tilde{M}$
- There are trading dates  $t = 0, 1, \dots, T$  for a single perishable consumption good. Any possibly complete history of the exogenous uncertain environment from 0 to  $T$  is a state of nature and denoted by  $\omega$ .
- Event tree or information structure



- Information set

$$t = 0, \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}\} = \{\Omega\} = \mathcal{F}_0$$

$$t = 1, \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}\} = \mathcal{F}_1$$

$$t = 2, \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}\} = \mathcal{F}_2$$

(the true state of nature is completely revealed)

- An event is a subset of  $\Omega$ , for example,  $a_t = \{\omega_1, \omega_2, \omega_3\}$  is an event.
- A partition of  $\Omega$  is a collection of events  $\{a_1, a_2, \dots, a_n\}$  such that the union of these events is equal to  $\Omega$  and pairwise intersections of these events are empty.
- The information revolution represented by an event tree can be described by a family of partitions of  $\Omega$ , indexed by time, that become finer and finer.
- $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \dots, T\}$ : information structure or filtration, where each  $\mathcal{F}_t$  is a partition of  $\Omega$  and  $\mathcal{F}_t$  is finer than  $\mathcal{F}_s$  if  $t \geq s$
- A time-event contingent claim is a security that pays one unit of consumption at trading date  $t \geq 1$  in an event  $a_t \in \mathcal{F}_t$  and nothing otherwise (走到 tree 上的某個 node (相對於某個 time event 發生), 就得 1 單位 consumption, 到其他 node 則沒有)

- 假設  $\left\{ \begin{array}{l} \text{homogeneous beliefs } \pi_w (\pi_{at} \equiv \sum_{w \in a_t} \pi_w) \\ \text{(7.3 到 7.14 節之結果, 對 heterogeneous beliefs } \pi_w^i > 0 \text{ 也成立)} \\ \text{time-additive and state independent von Neumann-Morgenstern utility functions: } u_{i0}(z_0) + \sum_{t=1}^T \sum_{a_t \in \mathcal{F}_t} \pi_{at} u_{it}(z_{at}) (u'_i \geq 0, u''_i < 0) \end{array} \right.$

- 假設  $\{c_0^i, c_{a_t}^i, a_t \in \mathcal{F}_t, t = 1, \dots, T, i = 1, \dots, I\}$  為 Pareto optimal allocation for a pure exchange economy with complete time-event contingent claims ( $c_0^i$  為 time-0 consumption and  $c_{a_t}^i$  為  $a_t$ -contingent claim 之張數, 決定了  $\{c_0^i, c_{a_t}^i\}$ , 即決定了一個多期的 consumption plan)

- $$\max_{\left\{ \begin{array}{l} z_0^i, z_{a_t}^i, a_t \in \mathcal{F}_t \\ t=1, \dots, T, i=1, \dots, I \end{array} \right\}} \sum_{i=1}^I \lambda_i (u_{i0}(z_0^i) + \sum_{t=1}^T \sum_{a_t \in \mathcal{F}_t} \pi_{a_t} u_{it}(z_{a_t}^i))$$

$$\text{s.t. } \sum_{i=1}^I z_0^i = \sum_{i=1}^I e_0^i$$

$$\sum_{i=1}^I z_{a_t}^i = \sum_{i=1}^I e_{a_t}^i \quad \forall a_t \in \mathcal{F}_t \text{ for } \forall t$$

(其中  $\{e_0^i, e_{a_t}^i, a_t \in \mathcal{F}_t, t = 1, \dots, T\}$  為 endowment, 其中  $e_0^i$  的單位為 time-0 consumption, 而  $e_{a_t}^i$  為有幾張  $a_t$ -contingent claim)(避免退化, 假設至少有一個  $e_{a_t}^i > 0$ )

- FOC 加上假設  $\{c_0^i, c_{a_t}^i\}$  為上述問題之解

$$\Rightarrow \begin{cases} \lambda_i u'_{i0}(c_0^i) = \phi_0 & \forall i \\ \lambda_i \pi_{a_t} u'_{it}(c_{a_t}^i) = \phi_{a_t} & \forall i, \forall a_t \in \mathcal{F}_t \text{ for } \forall t \end{cases}$$

(其中  $\phi_0$  與  $\phi_{a_t}$  為 Lagrangian multipliers)

$$\text{兩式相除} \Rightarrow \frac{\pi_{a_t} u'_{it}(c_{a_t}^i)}{u'_{i0}(c_0^i)} = \frac{\phi_{a_t}}{\phi_0} \quad \forall i, \forall a_t \in \mathcal{F}_t \text{ for } \forall t$$

(意即, 在 PO 時, the ratios of marginal utilities between  $a_t$ -consumption and time-0 consumption are equal across individuals)

- 證明 complete market competitive equilibrium (CMCE)  $\Rightarrow$  PO (with a spot markets of time-0 consumption and complete time-event contingent claims at time 0)

$$\begin{aligned} & \max_{\{z_0^i, z_{a_t}^i\}} u_{i0}(z_0^i) + \sum_{t=1}^T \sum_{a_t \in \mathcal{F}_t} \pi_{a_t} u_{it}(z_{a_t}^i) \\ \text{s.t. } & \phi_0 z_0^i + \sum_{t=1}^T \sum_{a_t \in \mathcal{F}_t} \phi_{a_t} z_{a_t}^i = \phi_0 e_0^i + \sum_{t=1}^T \sum_{a_t \in \mathcal{F}_t} \phi_{a_t} e_{a_t}^i \end{aligned}$$

$$\text{FOC} \Rightarrow \begin{cases} u'_{i0}(c_0^i) = \gamma_i \phi_0 \\ \pi_{a_t} u'_{it}(c_{a_t}^i) = \gamma_i \phi_{a_t} \quad \forall a_t \in \mathcal{F}_t \text{ for any } t \end{cases}$$

(其中  $\{c_0^i, c_{a_t}^i\}$  為 optimal solution,  $\gamma_i$  is the Lagrangian multiplier,  $\phi_0$  與  $\phi_{a_t}$  為今日之 consumption 與 time-event contingent claims 在今日 spot market 之價格)

In equilibrium, 因為 market clear, 所以  $\sum_{i=1}^I c_0^i = \sum_{i=1}^I e_0^i$ ,  $\sum_{i=1}^I c_{a_t}^i = \sum_{i=1}^I e_{a_t}^i$

(今天整體之消費=整體 time-0 consumption 之 endowment;  $a_t$ -event 時, 整體之消費=今日整體  $a_t$  time-event contingent claim 之 endowment 有幾張)

加上 FOC 兩式相除  $\Rightarrow \frac{\pi_{a_t} u'_{it}(c_{a_t}^i)}{u'_{i0}(c_0^i)} = \frac{\phi_{a_t}}{\phi_0}$ , 因此可得此種均衡 (CMCE)=PO

- 若 markets for the time-event contingent claims reopen at each trading date, 則均衡中除了  $\{c_0^i, c_{a_t}^i\}$  之外, 還需 individuals' expectation about prices for time-event contingent claims at each trading date  $\phi_{a_s}(a_t)$  (從  $a_t$  之 time-event, 看  $a_s$  time-event contingent claim 之價格, if  $t < s$  and  $a_s \subseteq a_t$ )
- 此種考慮市場可 reopen 之均衡稱為 rational expectations equilibrium, 之後會證明 CMREE=CMCE, 且即使市場 reopen, there is no trading in equilibrium after time 0

- For  $t < s$ , define  $\pi_{a_s}(a_t)$  to be the probabilities of  $a_s$  at the event  $a_t$ .

$$\pi_{a_s}(a_t) = \begin{cases} 0 & \text{if } a_s \not\subseteq a_t \\ \frac{\pi_{a_s}}{\pi_{a_t}} & \text{if } a_s \subseteq a_t \end{cases}$$

- Define  $\phi_{a_s}(a_t)$  to be the ex-dividend price at time  $t$  in the event  $a_t$  for the time-event contingent claim paying off at time  $s$  in event  $a_s$ .

$$\phi_{a_s}(a_t) = \begin{cases} 0 & \text{if } t \geq s \\ 0 & \text{if } t < s \text{ and } a_s \not\subseteq a_t \\ \frac{\phi_{a_s}}{\phi_{a_t}} & \text{if } t < s \text{ and } a_s \subseteq a_t \end{cases}$$

$$(\Rightarrow \phi_{a_t}(a_t) = 1)$$

(Individuals have rational expectation in the sense that they believe the prices and probabilities will evolve according to  $\pi_{a_s}(a_t)$  and  $\phi_{a_s}(a_t)$ )

- 證明 CMREE=CMCE, 亦即證明 CMCE 中的  $\{c_{a_s}^i, a_s \in \mathcal{F}_s, a_s \subseteq a_t, s \geq t\}$  為下面問題之解: For any  $a_t \in \mathcal{F}_t$

$$\max_{\substack{\{z_{a_s}^i, a_s \in \mathcal{F}_s\} \\ a_s \subseteq a_t, s \geq t}} u_{it}(z_{a_t}^i) + \sum_{s=t+1}^T \sum_{\substack{a_s \in \mathcal{F}_s \\ a_s \subseteq a_t}} \pi_{a_s}(a_t) u_{is}(z_{a_s}^i)$$

$$\text{s.t. } z_{a_t}^i + \sum_{s=t+1}^T \sum_{\substack{a_s \in \mathcal{F}_s \\ a_s \subseteq a_t}} \phi_{a_s}(a_t) z_{a_s}^i = c_{a_t}^i + \sum_{s=t+1}^T \sum_{\substack{a_s \in \mathcal{F}_s \\ a_s \subseteq a_t}} \phi_{a_s}(a_t) c_{a_s}^i$$

(即在時間  $t$ , 在 given  $\pi_{a_s}(a_t)$  與  $\phi_{a_s}(a_t)$  下, 個人會傾向持有與 CMCE 相同之 consumption plan)

$$\text{FOC} \Rightarrow \begin{cases} u'_{it}(c_{a_t}^i) = \gamma_{a_t}^i \\ \pi_{a_s}(a_t) u'_{is}(c_{a_s}^i) = \gamma_{a_t}^i \phi_{a_s}(a_t) \end{cases}$$

if let  $\gamma_{a_t}^i = \gamma_i \frac{\phi_{a_t}}{\pi_{a_t}}$ , 加上 CMCE 的  $\{c_{a_s}^i\}$  原本就滿足上述之預算限制式 ((7.5.4)vs.(7.3.5))  
 $\Rightarrow$  CMREE 之 FOC 會等於 CMCE 之 FOC, 意即 CMCE 之  $\{c_{a_s}^i\}$  亦為 CMREE 之解 (此時  $\pi_{a_s}(a_t) u'_{is}(c_{a_s}^i) = \gamma_{a_t}^i \phi_{a_s}(a_t) \Rightarrow \frac{\pi_{a_s}}{\pi_{a_t}} u'_{is}(c_{a_s}^i) = \gamma_i \frac{\phi_{a_t}}{\pi_{a_t}} \frac{\phi_{a_s}}{\pi_{a_t}} \Rightarrow \pi_{a_s} \cdot u'_{is}(c_{a_s}^i) = \gamma_i \phi_{a_s}$ )

- 上述之建構證明了存在一個 CMREE=CMCE, 因此在此種 CMREE 中雖然有 re-trade 之機會, 個人會在 time=0 交易 time-0 consumption 與 time-event contingent claims, 以獲得 consumption plan  $\{c_0^i, c_{a_t}^i\}$ , 且之後的最佳 consumption allocation 也會等於  $\{c_{a_t}^i\}$ , 故個人在  $t = 0$  之後將不再需要交易
- 但當沒有 complete set of time-event contingent claims, retrade 之機會的價值將可顯現, the market can be completed by dynamically managing a portfolio of long-lived securities, whose number is far fewer than the number of time-events.
  - A complex security is composed of time-0 consumption good and a bundle of time-event contingent claims and is represented by  $x = \{x_0, x_{a_t}\}$ , where  $x_0$  and  $x_{a_t}$  are dividends paid at time 0 and at event  $a_t$
  - A long-lived security is a complex security that is available for trading at all trading dates

- $\bar{\theta}_j(0)$  denotes the number of shares of security  $j$  with which individual is endowed, and  $\sum_{i=1}^I \bar{\theta}_j(0) = 1$ , for  $j = 0, 1, \dots, N$  ( $N$  遠小於 time event 之數目)
- $S_j(t)$  denotes the ex-dividend price of security  $j$  at time  $t$ , and represents the value of the future dividend process  $x_j$  at  $t$  ( $S_j(T) = 0$ )  
 $S_j(t)$ 's realizations do not vary across states in an event  $a_t \in \mathcal{F}_t$  ( $S_j(t)$  只與 event  $a_t$  的 evolution 有關, 並沒有辦法分辨  $a_t$  中的各個 state), and this property is termed that  $S_j(t)$  is measurable with respect to  $\mathcal{F}_t$ 
  - A process is a collection of random variables indexed by  $t$   
 If a process is measurable with respect to  $\mathcal{F}_t$  for all  $t$ , we say that the process is adapted to  $\mathbb{F}$ . (For example, a long-lived security is represented by a dividend process adapted to  $\mathbb{F}$ )

- A trading strategy  $\theta$  is an  $(N + 1)$ -dimensional process

$$\theta = \{\theta_j(t); j = 0, 1, \dots, T\}$$

(其中  $\theta_j(t)$  為持有 security  $j$  in the time period  $[t - 1, t]$ ), 意即  $\theta_j(t)$  是在 time  $t - 1$  決定, 此時 process  $\theta_j$  is said to be measurable with respect to  $\mathcal{F}_{t+1}$  or predictable with respect to  $\mathbb{F}$ )

$$\theta(t) = \begin{bmatrix} \theta_0(t) \\ \theta_1(t) \\ \vdots \\ \theta_N(t) \end{bmatrix}$$

- Individual  $i$ 's problem

$$\max_{\theta} u_{i0}(c(0)) + E\left[\sum_{t=1}^T u_{it}(c(t))\right]$$

$$\text{s.t. } \theta(0) = \bar{\theta}^i(0)$$

$$\theta(t+1)^T S(t) = \theta(t)^T (S(t) + X(t)) - c(t) \quad \text{for } t = 0, 1, \dots, T-1$$

$$0 = \theta(T)^T X(T) - c(T) \quad (\text{一般都只寫上式, 並假設 } t = T \text{ 時, 因 } S(T) = 0, \text{ 等號左邊為 } 0, \text{ 即為此式})$$

(其中  $S(t) = (S_1(t), \dots, S_N(t))^T$ ,  $X(t) = (x_0(t), \dots, x_N(t))^T$  為 dividend process)

(上述的限制式表示: The consumption plan  $c(t)$  is “financed” by the “admissible” strategy  $\theta(t)$ .)

- A rational expectations equilibrium for the securities markets (SMREE) is  $\{\theta^i(t), c^i(t)\}$  for  $i = 1, \dots, I$  and the price process  $\{S(t)\}$ , and markets clear at all times:

$$\sum_{i=1}^I \theta_j^i(t) = 1$$

$$\sum_{i=1}^I c^i(t) = \sum_{j=0}^N x_j(t) \quad (\text{at time } t, \text{ 整個經濟體的總消費} = \text{整個經濟體總dividend})$$

- 證明即使之後有 retrade 之機會, 依然會按照  $t = 0$  之 optimal strategy 進行, 並沒有一個新的 consumption plan 可以增加所有個人之 welfare
- 反證法: 假設存在一種可 improve expected utility 之策略  $\theta'(t)$  at time  $t > 1$  (在  $t$  之前之交易策略  $\theta' = \theta^i$ ), 希望證明 the expected utility of “life-time” consumption at time 0 也被 improve, 如此則得到矛盾 (因在 time 0 所求得之  $\theta^i(t)$  與  $c^i(t)$  已經極大化  $0 \sim T$  之 expected utility 總合)

$$\theta'(t+1)^T S(t) = \theta^i(t)^T (S(t) + X(t)) - c'(t)$$

$$\theta'(t+k+1)^T S(t+k) = \theta'(t+k)^T (S(t+k) + X(t+k)) - c'(t+k) \\ \text{for } k = 1, \dots, T-t$$

使得  $\sum_{s=t}^T \sum_{a_s \leq a_t} \pi_{a_s}(a_t) u_{is}(c'_{a_s}) > \sum_{s=t}^T \sum_{a_s \leq a_t} \pi_{a_s}(a_t) u_{is}(c^i_{a_s})$

Define a trading strategy and a consumption plan starting at time 0:

$$\hat{\theta}(s) = \begin{cases} \theta^i(s) & \text{if } s \leq t \\ \theta^i(s) & \text{if } s > t \text{ and in the event } a_s \not\subseteq a_t \\ \theta'(s) & \text{if } s > t \text{ and in the event } a_s \subseteq a_t \end{cases}$$

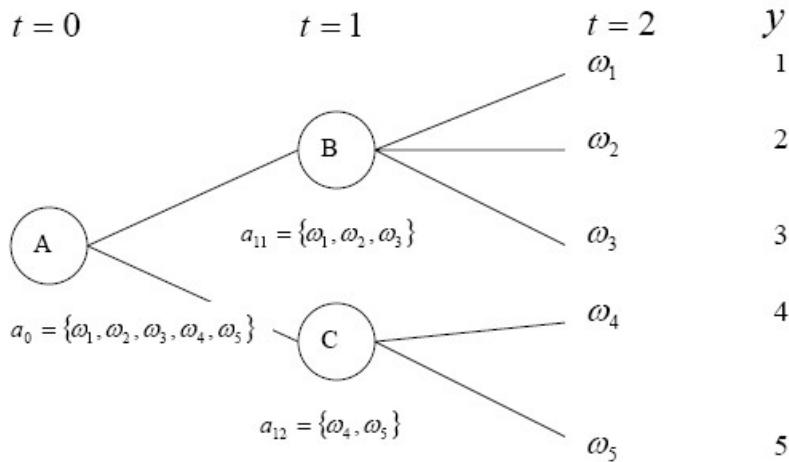
$$\hat{c}(s) = \begin{cases} c^i(s) & \text{if } s < t \\ c^i(s) & \text{if } s \geq t \text{ and in the event } a_s \not\subseteq a_t \\ c'(s) & \text{if } s \geq t \text{ and in the event } a_s \subseteq a_t \end{cases}$$

$$\text{由 } \sum_{s=t}^T \sum_{a_s \leq a_t} \pi_{a_s}(a_t) u_{is}(c'_{a_s}) > \sum_{s=t}^T \sum_{a_s \leq a_t} \pi_{a_s}(a_t) u_{is}(c^i_{a_s})$$

$$\Rightarrow E\left[\sum_{s=t}^T u_{is}(\hat{c}(s)) \mid \mathcal{F}_t\right] \geq E\left[\sum_{s=t}^T u_{is}(c^i(s)) \mid \mathcal{F}_t\right]$$

(上述不等式, 在 $a_t$ 時是 $>$ , 但在 $\mathcal{F}_t$ 中的其它 events, 則是=)

假設  $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$  等機率出現  $\Rightarrow \pi_{\{\omega_1\}} = \pi_{\{\omega_2\}} = \dots = \pi_{\{\omega_5\}} = \frac{1}{5}$



$$(i) \text{ for node A: } \pi_{a11}(a_0) = \frac{\pi_{a11}}{\pi_{a_0}} = \frac{3/5}{1} = 3/5$$

$$\pi_{a12}(a_0) = \frac{2}{5}$$

$$\pi_{\{\omega_1\}}(a_0) = \dots = \pi_{\{\omega_5\}}(a_0) = \frac{1}{5}$$

$$\text{for node B: } \pi_{\{\omega_1\}}(a_{11}) = \frac{\pi_{\{\omega_1\}}}{\pi_{a_{11}}} = \frac{1/5}{3/5} = \frac{1}{3}$$

$$\pi_{\{\omega_2\}}(a_{11}) = \pi_{\{\omega_3\}}(a_{11}) = \frac{1}{3}$$

$$\text{for node C: } \pi_{\{\omega_4\}}(a_{12}) = \pi_{\{\omega_5\}}(a_{12}) = \frac{1}{2}$$

$$(ii) \text{ for node B: } E[y|\mathcal{F}_1] = \frac{(1/5) \cdot 1 + (1/5) \cdot 2 + (1/5) \cdot 3}{3/5} = 2$$

$$(\text{or } (1/3) \cdot 1 + (1/3) \cdot 2 + (1/3) \cdot 3 = 2)$$

$$\text{for node C: } E[y|\mathcal{F}_1] = \frac{(1/5) \cdot 4 + (1/5) \cdot 5}{2/5} = 4.5$$

$$(\text{or } (1/2) \cdot 4 + (1/2) \cdot 5 = 4.5)$$

(conditional expectation is at time  $t$  a random variable measurable with respect to  $\mathcal{F}_t$ )

$$\text{for node A: } E[y|\mathcal{F}_0] = E[y] = (3/5) \cdot 2 + (2/5) \cdot 4.5 = 3$$

$$(\text{or } (1/5) \cdot 1 + (1/5) \cdot 2 + (1/5) \cdot 3 + (1/5) \cdot 4 + (1/5) \cdot 5 = 3)$$

$$\Rightarrow E[E[\sum_{s=t}^T u_{is}(\hat{c}(s))|\mathcal{F}_t]] > E[E[\sum_{s=t}^T u_{is}(c^i(s))|\mathcal{F}_t]]$$

(因有的 event 是  $>$ , 有些 event 是  $=$ , 取期望值 (可想成 weighted average) 後, 由  $\geq$  變成  $>$ )

$$\Rightarrow E[\sum_{s=t}^T u_{is}(\hat{c}(s))] > E[\sum_{s=t}^T u_{is}(c^i(s))]$$

$$\left\| \begin{array}{l} \text{By definition} \\ E[\sum_{s=0}^{t-1} u_{is}(\hat{c}(s))] = E[\sum_{s=0}^{t-1} u_{is}(c^i(s))] \end{array} \right.$$

$$\Rightarrow E[\sum_{s=0}^T u_{is}(\hat{c}(s))] > E[\sum_{s=0}^T u_{is}(c^i(s))] (\rightarrow \leftarrow)$$

(Therefore, in a rational expectations equilibrium, there is no incentive for individuals to deviate from the strategies chosen at time 0)

- Dynamic programming

(i) at  $T - 1$ , 假設  $\theta(T - 1), S(T - 1), X(T - 1)$  已知 ( $S(T - 1), X(T - 1)$

反應不同之  $a_{T-1} \in \mathcal{F}_{T-1}$ ,  $\theta(T - 1)$  為 augmented state variable)

(用  $\theta(T - 1), S(T - 1), X(T - 1)$  展出三度空間)

$$\begin{aligned} & \max_{\{c(T-1), \theta(T)\}} u_{T-1}(c(T - 1)) + E[u_T(c(T)) | \mathcal{F}_{T-1}] \\ \text{s.t. } & c(T - 1) + \theta(T)^T S(T - 1) = \theta(T - 1)^T (S(T - 1) + X(T - 1)) \\ & c(T) = \theta(T)^T X(T) \end{aligned}$$

(其中期望值是對  $X(T)$  做, 且  $c(T - 1)$  and  $\theta(T)$  are random variables measurable with respect to  $\mathcal{F}_{T-1}$  ( $\theta(T)$  為  $\theta(T - 1), S(T - 1), X(T - 1)$  之函數, 同時 value function 亦為  $\theta(T - 1), S(T - 1), X(T - 1)$  之函數)

(ii) at  $T - 2$  假設  $\theta(T - 2), S(T - 2), X(T - 2)$  已知 ( $S(T - 2), X(T - 2)$  反應不  
同之  $a_{T-2} \in \mathcal{F}_{T-2}$ ,  $\theta(T - 2)$  為 augmented state variable)

$$\begin{aligned} & \max_{\left\{ \begin{array}{l} c(T-1), \theta(T) \\ c(T-2), \theta(T-1) \end{array} \right\}} u_{T-2}(C(T - 2)) + E\left[ \sum_{t=T-1}^T u_t(C(t)) | \mathcal{F}_{T-2} \right] \\ \text{s.t. } & c(t - 1) + \theta(t)^T S(t - 1) = \theta(t - 1)^T (S(t - 1) + X(t - 1)), t = T - 1, T \\ & c(T) = \theta(T)^T X(T) \end{aligned}$$

等同於

$$\begin{aligned} & \max_{\{c(T-2), \theta(T-1)\}} u_{T-2}(c(T - 2)) + E[v(\theta(T - 1); \mathcal{F}_{T-1}) | \mathcal{F}_{T-2}] \\ \text{s.t. } & c(T - 2) + \theta(T - 1)^T S(T - 2) = \theta(T - 2)^T (S(T - 2) + X(T - 2)) \end{aligned}$$

(其中  $v(\theta(T - 1); \mathcal{F}_{T-1})$  為  $T - 1$  之 value function, 期望值為在 given  $S(T - 2), X(T - 2)$  下, 對  $S(T - 1)$  與  $X(T - 1)$  做)

- for  $t = 0, 1, \dots, T - 1$

the dynamic programming problem is equivalent to a sequence of two-period problems

$$\begin{aligned} & \max_{\{c(t), \theta(t+1)\}} u_t(c(t)) + E[v(\theta(t+1); \mathcal{F}_{t+1}) | \mathcal{F}_t] \\ \text{s.t. } & c(t) + \theta(t+1)^T S(t) = \theta(t)^T (S(t) + X(t)) = W(t) \end{aligned}$$

個人在 time  $t$  時, 選擇  $c(t)$  與  $\theta(t+1)$  或者說是  $W(t+1) = \theta(t+1)^T (S(t+1) + X(t+1))$  來極大化其 expected utility. 此外, 因為在 rational expectations equilibrium 中, based on  $\mathcal{F}_t$ , 所有人都對  $S(t+1)$  有正確的期望, 加上  $S(t+1)$  與  $X(t+1)$  的 evolution 都隱藏在  $\mathcal{F}_{t+1}$  中, 所以根據未來之財富  $W(t+1)$  所得到之 value function 中, 看似與  $S(t+1)$  和  $X(t+1)$  無關只與  $\theta(t+1)$  有關

- (7.10節) 證明  $c(t)$  與  $W(t)$  存在一個對映之關係

由上述之兩期的 problem  $\Rightarrow V(W(t); \mathcal{F}_t) = u_t(c(t)) + E[V(W(t+1); \mathcal{F}_{t+1}) | \mathcal{F}_t]$   
(其中  $c(t) + \theta(t+1)^T S(t) = \theta(t)^T (S(t) + X(t)) = W(t)$ )

加上  $V_w \geq 0$  and  $V_{ww} < 0$

則上述式子兩邊對  $W(t)$  微分可得  $V_W(W(t); F_t) = u'_t(c(t))$

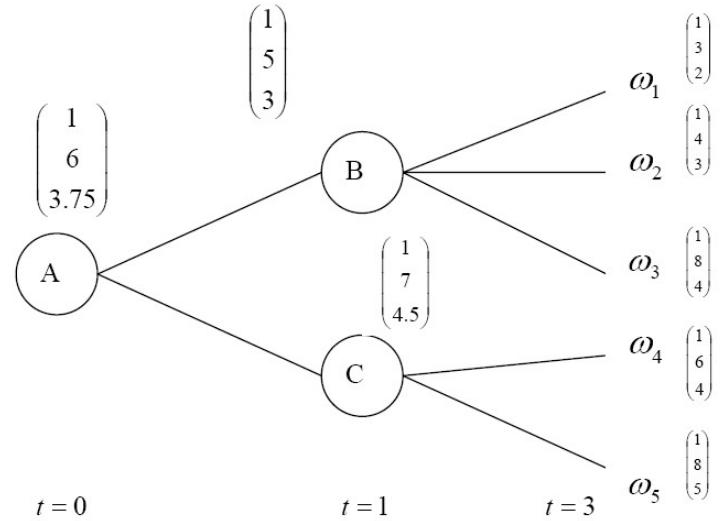
(花掉一單位財富之 utility 損失 = 消費一單位所帶來之 utility 增加, 此稱為 envelope condition)

若  $V_{ww} < 0$  和  $u''_t < 0$ , 則  $W(t) \uparrow, c(t) \uparrow$ , 且存在一 strictly increasing function  $g$  such that  $c(t) = g(W(t); \mathcal{F}_t)$

(值得注意的是, 就像 conditional on  $\mathcal{F}_t$  之 expectation,  $c(t)$  亦為隨機的函數 on  $W(t)$ )

- (7.11)~(7.13) 用數字例子證明只要 securities number > max branch number, 市場可由 dynamically trading 來 complete, 使得即使市場上 securities number(3) 遠小於 tiem event 之 number(7), 依然可達到 PO

- An example of a securities markets economy



三種securities, 且都在 $t = 2$ 時才發 dividend, 所以圖中 $t = 0$ 與 $t = 1$ 的 ( ) 中, 都是 securities prices,  $t = 2$ 的 ( ) 中, 都是 dividend

- 證明 any time-event contingent claim can be created by dynamically managing a portfolio of the three long-lived securities (例: 如何做出 $\phi_{\omega_1}$ )
  - for node B, 投資 $(x, y_1, y_2)$ 到第0, 1, 2個 securities

$$\begin{cases} x + 3y_1 + 2y_2 = 1 \\ x + 4y_1 + 3y_2 = 0 \\ x + 8y_1 + 4y_2 = 0 \end{cases} \Rightarrow (x, y_1, y_2) = \left(\frac{8}{3}, \frac{1}{3}, -\frac{4}{3}\right)$$

(此時, 建構投資組合的成本為  $\frac{8}{3} \cdot 1 + \frac{1}{3} \cdot 5 - \frac{4}{3} \cdot 3 = \frac{1}{3}$ )

- for node C,

不做事, 成本亦為0

- for node A,

$$\begin{cases} x + 5y_1 + 3y_2 = \frac{1}{3} \\ x + 7y_1 + 4.5y_2 = 0 \end{cases} \Rightarrow \text{有無限多解, 其中一個為 } (x, y_1, y_2) = \left(\frac{7}{6}, -\frac{1}{6}, 0\right)$$

(此時建構此投資組合的成本為  $\frac{7}{6} \cdot 1 - \frac{1}{6} \cdot 6 = \frac{1}{6} = \phi_{\omega_1}$ , 其實無論那個解, 成本均相同)

- the number of long-lived securities  $\geq$  the number of max branches is a necessary but not sufficient condition, 例如萬一 securities prices 在某個 node 形成 linear dependent, 則會使得 dynamical strategy 解不出來 (多期模型與單期模型之不同地方也在此, 單期模型只需 payoff 為線性獨立即可, 而多期模型還需內生決定的 long-lived securities 之價格亦為線性獨立, 才能動態 complete market)
- In CMREE, long-lived securities 之價格可由其 payoff 與 prices of time-event contingent claim 之價格得出

$$S_x(a_t, t) = \sum_{s=t+1}^T \sum_{\substack{a_s \in \mathcal{F}_s \\ a_s \subseteq a_t}} \phi_{a_s}(a_t) x_{a_s}$$

$$\left\| \begin{array}{l} \text{由(7.5.5) 式} \\ \left\{ \begin{array}{l} u'_{it}(c_{a_t}^i) = r_{a_t}^i \\ \pi_{a_s}(a_t) u'_{is}(c_{a_s}^i) = r_{a_t}^i \phi_{a_s}(a_t) \end{array} \right. \Rightarrow \phi_{a_s}(a_t) = \frac{\pi_{a_s}(a_t) u'_{is}(c_{a_s}^i)}{u'_{it}(c_{a_t}^i)} \end{array} \right.$$

$$\begin{aligned} &= \sum_{s=t+1}^T \sum_{\substack{a_s \in \mathcal{F}_s \\ a_s \subseteq a_t}} \frac{\pi_{a_s}(a_t) u'_{is}(c_{a_s}^i)}{u'_{it}(c_{a_t}^i)} x_{a_s} \\ &= \sum_{\substack{a_{t+1} \in \mathcal{F}_{t+1} \\ a_{t+1} \subseteq a_t}} \frac{\pi_{a_s}(a_t) u'_{is}(c_{a_s}^i)}{u'_{it}(c_{a_t}^i)} (x_{a_{t+1}} + S_x(a_{t+1}, t+1)) \\ \Rightarrow S_x(t) &= E \left[ \sum_{s=t+1}^T \frac{u'_{is}(c^i(s))}{u'_{it}(c^i(t))} x(s) | \mathcal{F}_t \right] \\ &= E \left[ \frac{u'_{i,t+1}(c^i(t+1))}{u'_{it}(c^i(t))} (x(t+1) + S(t+1)) | \mathcal{F}_t \right] \end{aligned}$$

(期望值都是對未來可能之 events  $a_s$  或是  $a_{t+1}$  之發生機率來做)

- SMREE implements CMREE if
  - SMREE 中的  $S(t)$  要與 CMREE 中用  $x_{a_s}$  與  $\phi_{a_s}(a_t)$  漲出來的一致
  - SMREE 可 dynamically complete market
  - SMREE 之 optimal allocation  $c_{a_t}^i$  與 CMREE 中的一致

- Fix a CMREE, 則我們可 random 選取  $N$  個 securities, 並假設其 securities prices 符合 CMREE 所求得的, 則此  $N$  個 securities 可 dynamically complete markets, 且 optimal allocation  $c_{a_t}^i$  也會與 CMREE 中的一致 (CMREE can be implemented in SMREE by randomly selecting  $N$  long-lived seurities)
- 對每個可 dynamically complete market 之 SMREE 而言, 因為每個 time-event contingent claims 都可由 dynamically trading 得出, 所以一定對映到某個 CM-REE (The proof for the above generic implementability is rather technical, and it can be referred to Kreps(1982)for complete details.)
- (7.15節) 資訊 reveal 慢 (branch 數較少), 需較少之 securities, 即可 dynamically complete market, 而資訊 reveal 很快時, 需較多之 securities 來 dynamically complete market (the minimum number of long-lived securities is determined by the temporal resolution of uncertainty)
- 如同 5.10~5.19 節 (homogeneous belief + time-additive utility  $\Rightarrow$  PO sharing rule  $\Rightarrow$  aggregate consumption 為唯一之 aggregate shock  $\Rightarrow$  不需考慮所有 state of nature, 從  $\Omega \rightarrow \mathbb{R}^+$  即可), 在 multiperiod economy 也可以證明 PO 之 individual consumption 為 aggregate consumption 之 strictly increasing function (即存在 PO sharing rule  $f_{i,a_t}$  且  $f'_{i,a_t} > 0$ ), 因此只要 SMREE 可 dynamically complete with the “aggregate consumption events”, 則可達到一個 PO
- 如同 5.21~5.23 節 (homogeneous belief+time-additive utility, 存在一個 representative agent, 消費所有 aggregate endowment, 可得到同樣之均衡價格), SMREE 也存在一個 representative agent endowed with all long-lived securities, 且其 optimal trading strategy and consumption plan 為永遠持有 long-lived securities, 並消費每個 time event 之 aggregate dividends

- 此時, securities 之均衡價格可由 representative agent 之 utility 與 aggregate consumption 得出

$$S_j(t-1) = E\left[\frac{u'_t(\tilde{C}_t)}{u'_{t-1}(\tilde{C}_{t-1})}(x_j(t) + S_j(t))|\mathcal{F}_{t-1}\right], \quad (7.16.2\text{式})(\text{由 (7.14.5) 而來})$$

$$\text{where } \tilde{C}_t \equiv \sum_{i=1}^I c^i(t) = \sum_{j=0}^N x_j(t)$$

$$\begin{cases} \text{representative agent 之 utility 需滿足} \\ u_t(z) \equiv \max_{(z_i)_{i=1}^I} \sum_{i=1}^I \lambda_i u_{it}(z_i) \\ \text{s.t. } \sum_{i=1}^I z_i = z \end{cases}$$

- Define  $\tilde{r}_{jt} \equiv \frac{x_j(t) + S_j(t)}{S_j(t-1)} - 1$

$$\begin{aligned} \Rightarrow 1 &= E\left[\frac{u'_t(\tilde{C}_t)}{u'_{t-1}(\tilde{C}_{t-1})}(1 + \tilde{r}_{jt})|\mathcal{F}_{t-1}\right] \\ &= \text{Cov}\left(\frac{u'_t(\tilde{C}_t)}{u'_{t-1}(\tilde{C}_{t-1})}, \tilde{r}_{jt}|\mathcal{F}_{t-1}\right) + E\left[\frac{u'_t(\tilde{C}_t)}{u'_{t-1}(\tilde{C}_{t-1})}|\mathcal{F}_{t-1}\right]E[1 + \tilde{r}_{jt}|\mathcal{F}_{t-1}] \end{aligned}$$

$$\begin{cases} \text{假設0-th long-lived riskless security 之初始價格為1, 且一直到} T \text{時才} \\ \text{一次還本利和} x_0(T) = \prod_{s=1}^T (1 + r_{f_s}), \text{ 則其 price at } t < T \text{ 為} \\ S_0(t) = \prod_{s=1}^t (1 + r_{f_s}) \\ \text{代入 (7.16.2)} \\ \Rightarrow E\left[\frac{u'_t(\tilde{C}_t)}{u'_{t-1}(\tilde{C}_{t-1})}|\mathcal{F}_{t-1}\right] = \frac{1}{1+r_{f_t}} \end{cases}$$

$$\Rightarrow E[\tilde{r}_{jt}|\mathcal{F}_{t-1}] - r_{f_t} = -(1 + r_{f_t})\text{Cov}_{t-1}(\tilde{r}_{jt}, \frac{u'_t(\tilde{C}_t)}{u'_{t-1}(\tilde{C}_{t-1})}|\mathcal{F}_{t-1}) \quad (7.16.5)$$

$(\tilde{C}_t \uparrow, u'_t(\tilde{C}_t) \downarrow$ , 故  $\text{Cov}(\tilde{r}_{jt}, \tilde{C}_t) > 0 \Rightarrow \text{risk premium} > 0$ , 意即報酬與消費同向變動之 security, 因其不能用來 smooth consumption, 所以一般人不喜歡, 故價格下降, risk premium  $\uparrow$ )

- (7.17) 與 (7.18) 節用兩種方式導 consumption CAPM (CCAPM), (7.17) 節是假設 quadratic utility, 而 (7.18) 節是導 Breeden (1979) 之結果, 將  $t-1 \rightarrow t$ , 轉成  $t-\Delta \rightarrow t$  (local approximation argument), 且是用加總的方式求出 aggregate consumption, 而非從 representative agent 出發

- 假設 representative agent 之 utility 為  $u_t(z) = \rho^t(a_t z - \frac{b_t}{2} z^2)$ , 則 (7.16.5) 式變成

$$\begin{aligned} E[\tilde{\gamma}_{jt} | \mathcal{F}_{t-1}] - r_{f_t} &= -(1 + r_{f_t}) \text{Cov}_{t-1}(\tilde{r}_{jt}, \frac{\rho^t(a_t - b_t \tilde{C}_t)}{\rho^{t-1}(a_{t-1} - b_{t-1} \tilde{C}_{t-1})}) \\ &= -(1 + r_{f_t}) \frac{-\rho b_t}{a_{t-1} - b_{t-1} \tilde{C}_{t-1}} \cdot \text{Cov}_{t-1}(\tilde{r}_{jt}, \tilde{C}_t) \end{aligned}$$

一期模型很容易找到 market portfolio 與  $\tilde{C}$  完全相關, 之後便能計算出與市場相關之  $\beta$ , 但在多期之模型沒有 portfolio 之 payoff 與 aggregate consumption 完全相關 (因在均衡中, market portfolio 所產生每期之 aggregate dividends 與 aggregate consumption 相等, 亦即 market portfolio 之 dividends payoff 與 aggregate consumption 是完全正相關, 但可能因  $r_{mt}$  除了與 aggregate dividends 有關, 與下期之價格也有關, 所以  $r_{mt}$  與 aggregate consumption 可能並非完全正相關), 因此假設  $\tilde{r}_{ct}$  為一 dynamically rebalanced portfolio, 與  $\tilde{C}_t$  有最高之 correlation

$$E[\tilde{r}_{ct} | \mathcal{F}_{t-1}] - r_{f_t} = -(1 + r_{f_t}) \frac{-\rho b_t}{a_{t-1} - b_{t-1} \tilde{C}_{t-1}} \text{Cov}_{t-1}(\tilde{r}_{ct}, \tilde{C}_t)$$

$$\Rightarrow -(1 + r_{f_t}) \frac{-\rho b_t}{a_{t-1} - b_{t-1} \tilde{C}_{t-1}} = \frac{E[\tilde{r}_{ct} | \mathcal{F}_{t-1}] - r_{f_t}}{\text{Cov}_{t-1}(\tilde{r}_{ct}, \tilde{C}_t)}$$

$$\begin{aligned} \Rightarrow E[\tilde{r}_{jt} | \mathcal{F}_{t-1}] - r_{f_t} &= \frac{\text{Cov}_{t-1}(\tilde{r}_{jt}, \tilde{C}_t)}{\text{Cov}_{t-1}(\tilde{r}_{ct}, \tilde{C}_t)} (E[\tilde{r}_{ct} | \mathcal{F}_{t-1}] - r_{f_t}) \\ &= \frac{\text{Cov}_{t-1}(\tilde{r}_{jt}, \tilde{C}_t) / \text{Var}_{t-1}(\tilde{C}_t)}{\text{Cov}_{t-1}(\tilde{r}_{ct}, \tilde{C}_t) / \text{Var}_{t-1}(\tilde{C}_t)} (E[\tilde{r}_{ct} | \mathcal{F}_{t-1}] - r_{f_t}) \\ &= \frac{\beta_{jc_{t-1}}}{\beta_{cc_{t-1}}} (E[\tilde{r}_{ct} | \mathcal{F}_{t-1}] - r_{f_t}) (\text{Consumption CAPM or CCAPM}) \end{aligned}$$

- Rubinstein (1976) 導出單期之 Consumption-based asset pricing model. Breeden and Litzenberger (1978) 導出 CCAPM in CMCE. Breeden (1979) 證明 CCAPM 在 continuous time securities market economy 也成立 (when local changes of prices of long-lived securities and the optimal individual consumption are small)

- (Breeden (1979)) 由 Taylor expansion series 可得

$$u'_{it}(c^i(t)) \approx u'_{it}(c^i(t - \Delta)) + u''_{it}(c^i(t - \Delta))\Delta c^i(t),$$

$$\text{where } \Delta c^i(t) \equiv c^i(t) - c^i(t - \Delta t)$$

希望由(7.14.5) 式加上類似 (7.16.5) 式之導證後的個人的均衡價格關係式,  
也可導出類似CCPAM 之式子

$$\begin{aligned} E[\tilde{r}_{jt}|\mathcal{F}_{t-\Delta}] - r_{f_t} &= -(1 + r_{f_t}) \text{Cov}_{t-\Delta}(\tilde{r}_{jt}, \frac{u'_{it}(c^i(t))}{u'_{it-\Delta}(c^i(t-\Delta))}) \\ &\approx -(1 + r_{f_t}) \text{Cov}_{t-\Delta}(\tilde{r}_{jt}, \frac{u'_{it}(c^i(t-\Delta)) + u''_{it}(c^i(t-\Delta))\Delta c^i(t-\Delta)}{u'_{it-\Delta}(c^i(t-\Delta))}) \\ &= -(1 + r_{f_t}) \frac{u''_{it}(c^i(t-\Delta))}{u'_{it-\Delta}(c^i(t-\Delta))} \text{Cov}_{t-\Delta}(\tilde{r}_{jt}, \Delta c^i(t)) \end{aligned}$$

$$\text{令 } -\frac{u''_{it}(c^i(t-\Delta))}{u'_{it-\Delta}(c^i(t-\Delta))} = \theta_i$$

$$\Rightarrow \theta_i^{-1}(E[\tilde{r}_{jt}|\mathcal{F}_{t-\Delta}] - r_{f_t}) \approx (1 + r_{f_t}) \text{Cov}_{t-\Delta}(\tilde{r}_{jt}, \Delta c^i(t))$$

$$\sum_i^I \theta_i^{-1}(E[\tilde{r}_{jt}|\mathcal{F}_{t-\Delta}] - r_{f_t}) \approx (1 + r_{f_t}) \text{Cov}_{t-\Delta}(\tilde{r}_{jt}, \Delta c_t)$$

$$\Rightarrow E[\tilde{r}_{jt}|\mathcal{F}_{t-\Delta}] - r_{f_t} \approx (\sum_{i=1}^I \theta_i^{-1})^{-1}(1 + r_{f_t}) \text{Cov}_{t-\Delta}(\tilde{r}_{jt}, \Delta \tilde{C}_t)$$

$$= (\sum_{i=1}^I \theta_i^{-1})^{-1}(1 + r_{f_t}) \text{Cov}_{t-\Delta}(\tilde{r}_{jt}, \tilde{C}_t)$$

$$\text{where } \Delta \tilde{C}_t \equiv \sum_{i=1}^I \Delta c^i(t) = \tilde{C}_t - \tilde{C}_{t-\Delta}$$

$$E[\tilde{r}_{ct}|\mathcal{F}_{t-\Delta}] - r_{f_t} \approx (\sum_{i=1}^I \theta_i^{-1})^{-1}(1 + r_{f_t}) \text{Cov}_{t-\Delta}(\tilde{r}_{ct}, \tilde{C}_t)$$

代回  $\tilde{r}_{jt}$  公式, 可得

$$E[\tilde{r}_{jt}|\mathcal{F}_{t-\Delta}] - r_{f_t} \approx \frac{\beta_{jct-\Delta}}{\beta_{cc_{t-\Delta}}} (E[\tilde{r}_{ct}|\mathcal{F}_{t-\Delta}] - r_{f_t})$$

$\Rightarrow$  CCAPM 在很短的時間  $[t - \Delta, t]$  間 approximately hold

(實際上 Breeden (1979) 是證明上述式子 exactly hold, 因其使用 stochastic calculus, 其中有  $(dt)^2 \rightarrow 0$  之幫助, 使得一開始之 Taylor expansion “=” 成立)

- 在一期模型中,  $\tilde{C}_1 = \tilde{M}$ , 但在多期模型中, 只有最後一期  $\tilde{C}_T = \tilde{M}_T$ , 對其餘時間點而言,  $\tilde{M}_t = \sum_{i=1}^I W_{it} = \sum_{j=0}^N x_j$  (or  $\sum_{i=1}^I c^i(t)$ ) +  $\sum_{j=0}^N S_j(t)$ , 意即每期 representative agent 之消費, 只佔 aggregate wealth (market) 的一部份, 加上之前的 dynamic programming 所得之個人之 consumption 與 wealth 存在關係  $c(t) = g(W(t); \mathcal{F}_t)$ , 可推得  $\tilde{C}_t$  與  $\tilde{M}_t$  中也存在著關係,  $\tilde{C}_t = g(\tilde{M}_t; \mathcal{F}_t)$ , 但因  $g$  為 stochastic function, 且無法確認  $\tilde{C}_t$  與  $\tilde{M}_t$  為一對一對映, 所以下面式子中的 risk premium 的正負號, 沒法由  $\text{Cov}_{t-1}(\tilde{r}_{jt}, \tilde{M}_t)$  之正負號求得

$$E[\tilde{r}_{jt} | \mathcal{F}_{t-1}] - r_{f_t} = -(1 + r_{f_t}) \text{Cov}_{t-1}(\tilde{r}_{jt}, \frac{u'_t(g(\tilde{M}_t; \mathcal{F}_t))}{u'_{t-1}(g(\tilde{M}_t; \mathcal{F}_t))})$$

(7.22) 節會提出充份條件, 使得上述式子再度變成 market-based asset pricing model, 即 risk premium 之正負與大小, 與  $\text{Cov}(\tilde{r}_{jt}, \tilde{M}_t)$  之正負與大小有關

- 如果 aggregate consumption 與 aggregate wealth 之關係為 nonstochastic and one-to-one, 則可形成與一期模型類似之結果, 即 risk premium 取求於  $\tilde{r}_j$  與  $\tilde{r}_m$  之隨機性質, 而非如同目前的 asset pricing relations for  $\tilde{r}_j$  是與 aggregate consumption 有關
- (7.20) 節引進 Markov state variable, (7.21) 節導出 Merton (1973) 之 multi-beta asset pricing model (不過將 continuous time 改成 discrete time), (7.22) 節說明當 state variables 之 evolution 為 independent 時, 可以得到  $\tilde{C}_t$  與  $\tilde{M}_t$  為 nonstochastic 且 one-to-one 之關係
- Let  $Y(t)$  denote an M-vector of random variables that are observable at time  $t$ . (假設觀察  $Y(t)$  之 realization 可知道那個 state of nature  $\omega$  還有可能, 那些  $\omega$  已經不可能發生, 則  $Y(t)$  稱為 vector of state variables at time  $t$ , 且 all the possible realizations of  $Y$  from time 0 to any time  $t$  generate a partition  $\mathcal{F}_t$ )

- Markov property(or memorylessness property): 知道  $\{Y(0), \dots, Y(t)\}$  等同於知道  $Y(t)$ , 即過去的 historical realizations 無關

$$E[z|\mathcal{F}_t] = E[z|Y(t)]$$

- 對於 representative agent 而言, the aggregate consumption

$$C(Y(t), t) = \sum_{j=0}^N x_j(Y(t), t)$$

$$\begin{aligned} \Rightarrow S_j(Y(t), t) &\equiv E\left[\sum_{s=t+1}^T \frac{u'_s(C(Y(s), s))}{u'_t(C(Y(t), t))} x_j(Y(s), s) | \mathcal{F}_t\right] \\ &= E\left[\sum_{s=t+1}^T \frac{u'_s(C(Y(s), s))}{u'_t(C(Y(t), t))} x_j(Y(s), s) | Y(t)\right] \end{aligned}$$

- 在 given  $Y(t)$  為 state variables 且為 Markov 之情況下, (7.18) 節 Breeden(1979) 的 local approximation argument 可以重導一次, 得出一個新的 linear asset pricing relation (holds approximately), 其中有多個 beta.

- 假設  $C(Y(t), t)$  對  $Y(t)$  與  $t$  可微，並考慮 representative agent 之  $u_t(\cdot)$

$$\begin{cases} C(t) \approx C(t - \Delta) + C_Y(t - \Delta)^T \Delta Y(t) + C_{t-\Delta}(t - \Delta) \Delta t \\ u'_t(C(t)) \approx u'_t(C(t - \Delta)) + u''_t(C(t - \Delta)) \Delta C(t) \end{cases}$$

where  $\Delta Y(t) \equiv Y(t) - Y(t - \Delta)$  ( $\Delta Y(t)$  為  $M \times 1$  vector)

$$\Delta C(t) \equiv C(t) - C(t - \Delta)$$

將  $\Delta C(t)$  代換掉

$$\Rightarrow u'_t(C(t)) \approx u'_t(C(t - \Delta)) + u''_t(C(t - \Delta))(C_Y(t - \Delta)^T \Delta Y(t) + C_{t-\Delta}(t - \Delta) \Delta t)$$

代入 (7.16.5) 式 + Markov property + 換掉 (7.16.5) 式中的  $u'_t(\tilde{c}_t)$

$$\Rightarrow E[\tilde{r}_t | Y(t - \Delta)] - r_{f_t} \approx -(1 + r_{f_t}) \frac{u''_t(C(t - \Delta)) C_Y(t - \Delta)^T}{u'_t(C(t - \Delta))} \text{Cov}_{t-\Delta}(\tilde{r}_{jt}, \Delta Y(t)) \quad \forall_j$$

用矩陣表示  $N$  個 securities 之 linear pricing relation

$$\Rightarrow E[\tilde{\mathbf{r}}_t | Y(t - \Delta)] - r_{f_t} \mathbf{1}_N \approx -(1 + r_{f_t}) \frac{u''_t(C(t - \Delta))}{u'_{t-\Delta}(C(t - \Delta))} \mathbf{V}_{xy}(t - \Delta) C_Y(t - \Delta)$$

$$\left| \begin{array}{l} \text{where } \mathbf{V}_{xy}(t - \Delta) = \begin{bmatrix} \sigma_{r_1 Y_1} & \sigma_{r_1 Y_2} & \cdots & \sigma_{r_1 Y_M} \\ \sigma_{r_2 Y_1} & \sigma_{r_2 Y_2} & \cdots & \sigma_{r_2 Y_M} \\ \vdots & \ddots & & \vdots \\ \sigma_{r_N Y_1} & \sigma_{r_N Y_2} & \cdots & \sigma_{r_N Y_M} \end{bmatrix}_{N \times M} \end{array} \right.$$

| is the covariance matrix of  $\tilde{\mathbf{r}}_t$  and  $Y(t)$  conditional on  $Y(t - \Delta)$

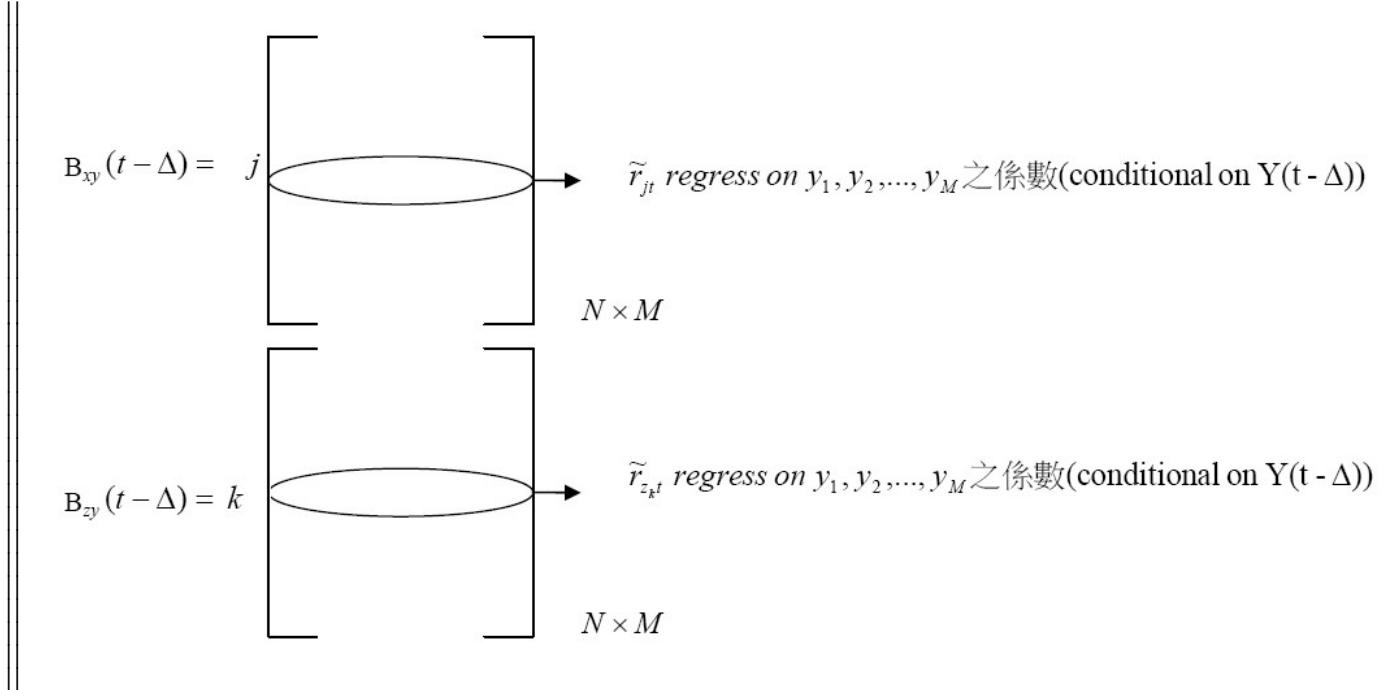
Define  $\tilde{r}_{z_k t}$  與  $y_k$  有最高之 correlation

$$E[\tilde{\mathbf{r}}_{zt} | Y(t - \Delta)] - r_{f_t} \mathbf{1}_M \approx -(1 + r_{f_t}) \frac{u''_t(C(t - \Delta))}{u'_{t-\Delta}(C(t - \Delta))} \mathbf{V}_{zy}(t - \Delta) C_Y(t - \Delta)$$

$$\left| \begin{array}{l} \text{where } \mathbf{V}_{zy}(t - \Delta) = \begin{bmatrix} \sigma_{z_1 Y_1} & \sigma_{z_1 Y_2} & \cdots & \sigma_{z_1 Y_M} \\ \sigma_{z_2 Y_1} & \sigma_{z_2 Y_2} & \cdots & \sigma_{z_2 Y_M} \\ \vdots & \ddots & & \vdots \\ \sigma_{z_N Y_1} & \sigma_{z_N Y_2} & \cdots & \sigma_{z_N Y_M} \end{bmatrix}_{M \times M} \end{array} \right.$$

| is the covariance matrix of  $\tilde{\mathbf{r}}_{zt}$  and  $Y(t)$  conditional on  $Y(t - \Delta)$

- 將  $-(1 + r_{f_t}) \frac{u''_t(C(t-\Delta))}{u'_t(C(t-\Delta))}$  與  $C_Y(t - \Delta)$  換掉  
 $\Rightarrow E[\mathbf{r}_t | Y(t - \Delta)] - r_{f_t} \mathbf{1}_N$   
 $\approx V_{xy}(t - \Delta) V_{zy}(t - \Delta)^{-1} (E[\mathbf{r}_{zt} | Y(t - \Delta)] - r_{f_t} \mathbf{1}_M)$   
 $\approx V_{xy}(t - \Delta) V_{yy}(t - \Delta)^{-1} (V_{zy}(t - \Delta) V_{yy}(t - \Delta)^{-1})^{-1} (E[\mathbf{r}_{zt} | Y(t - \Delta)] - r_{f_t} \mathbf{1}_M)$   
 $\equiv B_{xy}(t - \Delta) B_{zy}(t - \Delta) (E[\mathbf{r}_{zt} | Y(t - \Delta)] - r_{f_t} \mathbf{1}_M)$



- Aggregate wealth  $W(t) = W(Y(t), t) = \sum_{j=0}^N (S_j(Y(t), t) + x_j(Y(t), t))$   
 define  $\eta((Y(t), t)) \equiv \frac{C(Y(t), t)}{W(Y(t), t)}$  (在時間  $t$ , the aggregate consumption is a stochastic fraction of the aggregate wealth)

此時 (7.16.5) 變成

$$E[\tilde{r}_{jt} | \mathcal{F}_{t-1}] - r_{f_t} = -(1 + r_{f_t}) \text{Cov}(\tilde{r}_{jt}, \frac{u'_t(\eta(Y(t), t)W(t))}{u'_{t-1}(\eta(Y(t-1), t-1)W(t))})$$

(equity premium 終於跟 aggregate wealth, 即 market portfolio 扯上關係)

(此時  $\text{Cov}_{t-1}(\tilde{r}_{jt}, W(t))$  與  $\text{Cov}_{t-1}(\tilde{r}_{jt}, \eta(Y(t)))$  共同決定 risk premium)

- 如果  $\{Y(1), Y(2), \dots, Y(T)\}$  is a sequence of independent random variables, 則  $\text{Cov}_{t-1}(\tilde{r}_{jt}, W(t)) = \text{Cov}_{t-1}(\tilde{r}_{jt}, \tilde{M}_t)$  可單獨決定 risk premium
- $S_j(t) \equiv E\left[\sum_{s=t+1}^T \frac{u'_s(C(Y(s), s))}{u'_t(C(Y(t), t))} x_j(Y(s), s) | \mathcal{F}_t\right]$

$$\begin{aligned} & E[z | \mathcal{F}_t] = E[z | Y(t)] \\ & \text{但因 } z \text{ 只與 } Y(t+1), \dots, Y(T) \text{ 有關, 加上 } Y(t) \text{ 與 } Y(t+1), \dots, Y(T) \text{ 獨立} \\ & \Rightarrow E[z | Y(t)] = E[z] \end{aligned}$$

$$= \frac{E[\sum_{s=t+1}^T u'_s(C(Y(s), s)) x_j(Y(s), s) | \mathcal{F}_t]}{u'_t(C(Y(t), t))}$$

因  $0 < u'' < 0$ ,  $\Rightarrow C(t) \uparrow, u'(C(t)) \downarrow$  分母變小,  $S_j(t) \uparrow$ , 因一個  $C(t)$  不會對到 2 個  $S_j(t)$  之值, 所以可寫成  $S_j(C(t), t)$

- 此時 aggregate wealth 可寫成

$$\begin{aligned} W(t) &= \sum_{j=0}^N (S_j(Y(t), t) + x_j(Y(t), t)) \\ &= \sum_{j=0}^N S_j(C(t), t) + \sum_{j=0}^N x_j(Y(t), t) \\ &= \sum_{j=0}^N S_j(C(t), t) + C(Y(t), t)(C(t) \uparrow, S_j(C(t), t) \uparrow, W(t) \uparrow) \end{aligned}$$

( $W(t)$  is a strictly increasing function of  $C(t)$ , 因此存在  $g(W(t), t) = C(t)$  and  $g' > 0$ )

- Linear pricing relation for  $\tilde{r}_{jt}$

$$E[\tilde{r}_{jt} | Y(t-1)] - r_{f_t} = -(1 + r_{f_t}) \text{Cov}_{t-1}(\tilde{r}_{jt}, \frac{u'_t(g(W(t), t))}{u'_{t-1}(g(W(t-1), t-1))})$$

(如果  $\tilde{r}_{jt}$  與  $W(t)$  為正相關, 則  $\tilde{r}_{jt} \uparrow, W(t) \uparrow, g(W(t), t) \uparrow, u'_t(g(W(t), t)) \downarrow$ , 此時 risk premium 為正)

- $\{Y(1), Y(2), \dots, Y(T)\}$  假設為獨立, 相當於將 multiperiod problem 分割成 a sequence of disconnected single period problems in the sense that the realization of the state variables before time  $t$  conveys no information about the stochastic properties of  $y$  after  $t$
- 由  $C(t) = g(W(t), t)$ , 可得  $C(t) \approx C(t - \Delta) + g_W(W(t - \Delta), t - \Delta)\Delta W(t) + g_{t-\Delta}(W(t - \Delta), t - \Delta)\Delta t$ , 可按照 (7.21) 節, 導出 single-beta linear asset pricing relation with  $\tilde{r}_M$  as the pivotal variable