

Ch6. Valuation of Complex Securities and Options with Preference Restrictions

- 本章之目的是以 Ch5 之架構, 不對資產之分配或個人之效用做很特別之假設, 導出 complex securities 之價格. 例如: European call option

- $\begin{cases} \text{homogeneous belief } \pi_\omega \\ \text{time-additive and state-independent utility, } u_{i0} \text{ and } u_i \\ u_{i0} \text{ and } u_i \text{ are increasing, strictly concave, differentiable} \end{cases}$

$$(5.12.1) \lambda_i u'_{i0}(c_{i0}) = \phi_0, i = 1, 2, \dots, I$$

$$(5.12.2) \lambda_i \pi_\omega u'_i(c_{i\omega}) = \phi_\omega, \omega \in \Omega, i = 1, 2, \dots, I$$

若 \exists a representative agent with u_0 and u_1

$$\Rightarrow \phi_\omega = \frac{\pi_\omega u'_1(C_\omega)}{u'_0(C_0)}, \forall \omega \in \Omega \quad (5.23.1)$$

$$\text{for security } j, S_j = \sum_{\omega \in \Omega} \phi_\omega x_{j\omega}$$

(where $x_{j\omega}$ the is payoff of the asset j at state ω)

$$\Rightarrow S_j = E\left[\frac{u'_1(\tilde{C})}{u'_0(C_0)} \tilde{x}_j\right]$$

(where \tilde{x}_j : random payoff at time-1)

$$\parallel \text{riskless asset} \Rightarrow S_0 = E\left[\frac{u'_1(\tilde{C})}{u'_0(C_0)}\right] = \frac{1}{1+r_f} \quad (\text{or } r_f = \frac{1}{S_0} - 1)$$

$$\Rightarrow 1 = E\left[\frac{u'_1(\tilde{C}) \tilde{x}_j}{u'_0(C_0) S_j}\right] = \text{Cov}\left(\frac{u'_1(\tilde{C})}{u'_0(C_0)}, \frac{\tilde{x}_j}{S_j}\right) + E\left[\frac{u'_1(\tilde{C})}{u'_0(C_0)}\right] E\left[\frac{\tilde{x}_j}{S_j}\right]$$

$$\Rightarrow E\left[\frac{\tilde{x}_j}{S_j}\right] - (E\left[\frac{u'_1(\tilde{C})}{u'_0(C_0)}\right])^{-1} = -(E\left[\frac{u'_1(\tilde{C})}{u'_0(C_0)}\right])^{-1} \text{Cov}\left(\frac{u'_1(\tilde{C})}{u'_0(C_0)}, \frac{\tilde{x}_j}{S_j}\right)$$

$$\Rightarrow E\left[\frac{\tilde{x}_j}{S_j} - 1 - r_f\right] = -(1 + r_f) \text{Cov}\left(\frac{u'_1(\tilde{C})}{u'_0(C_0)}, \frac{\tilde{x}_j}{S_j}\right)$$

$$\Rightarrow E[\tilde{r}_j - r_f] = -(1 + r_f) \text{Cov}\left(\frac{u'_1(\tilde{C})}{u'_0(C_0)}, \tilde{r}_j\right)$$

$$\parallel \begin{aligned} & \text{兩期model} \\ & \Rightarrow \text{aggregate consumption } \tilde{C} = \text{aggregate wealth} \\ & = \text{market portfolio } \tilde{M} \end{aligned}$$

$$\Rightarrow E[\tilde{r}_j - r_f] = -(1 + r_f) \text{Cov}\left(\frac{u'_1(\tilde{M})}{u'_0(C_0)}, \tilde{r}_j\right)$$

- if u_1 is strictly concave

\Rightarrow 若 \tilde{r}_j 與 \tilde{C} (or \tilde{M}) 正相關 $\Rightarrow \tilde{r}_j$ 與 $u'_1(\tilde{C})$ (or $u'_1(\tilde{M})$) 負相關

$$\Rightarrow E[\tilde{r}_j - r_f] > 0$$

(if $\tilde{r}_j \uparrow, \tilde{C} \uparrow$, 表此 security 與 consumption 之波動一致, 故消費者持有此 security 無法 smooth consumption level, 故消費者不喜歡此 security, 此 security 需提供更高的 risk premium 來吸引人投資購買)

(或者是說, 好狀態時一單位的 consumption payoff 比壞狀態時一單位的 consumption payoff 來的差)

\Rightarrow if $\text{Cov}(\tilde{x}_j, \tilde{M}) > 0$ 時, $S_j \downarrow, \tilde{r}_j \uparrow$; if $\text{Cov}(\tilde{x}_j, \tilde{M}) < 0$ 時, $S_j \uparrow, \tilde{r}_j \downarrow$

$$\text{又 } E[\tilde{r}_m - r_f] = -(1 + r_f) \text{Cov}(\tilde{r}_m, \frac{u'_1(\tilde{M})}{u'_0(C_0)})$$

替換掉原式中此項

$$\Rightarrow E[\tilde{r}_j - r_f] = \frac{\text{Cov}(\tilde{r}_j, u'_1(\tilde{M}))}{\text{Cov}(\tilde{r}_m, u'_1(\tilde{M}))} E[\tilde{r}_m - r_f]$$

vs. p98 (4.13.1) CAPM

- 6.3節, 考慮 power utility (which exhibits preference towards skewness of time-1 random consumption), 不只 $\text{Cov}(\tilde{r}_j, \tilde{r}_m)$ 與 risk premium 有關, $\text{Cov}(\tilde{r}_j, \tilde{r}_m^2)$ (coskewness) 越高,risk premium 越低,security price 越高.

- Consumption-Based Asset Pricing Model (Lucas (1978))

$$U_t = u(C_t) + E_t[\rho u(C_{t+1})]$$

$$\max_{\theta} U_t$$

s.t. $C_t = e_t - p_t \theta$, $C_{t+1} = e_{t+1} + x_{t+1} \theta$

C_t : consumption level at t

ρ : discount factor

p_t : price of the security today

x_{t+1} : the payoff of the security at $t + 1$

θ : 購買多少單位之資產 at t

e_t : endowment at t

$$\text{FOC} \Rightarrow p_t u'(C_t) = E_t[\rho u'(C_{t+1})x_{t+1}]$$

or $p_t = E_t[\rho \frac{u'(C_{t+1})}{u'(C_t)} x_{t+1}]$

(i) if $x_{t+1} = 1 \Rightarrow \frac{1}{1+r_f} = E_t[\rho \frac{u'(C_{t+1})}{u'(C_t)}] = E_t[MRS_t^{t+1}]$

又 $1 = E_t[\rho \frac{u'(C_{t+1})}{u'(C_t)} \frac{x_{t+1}}{p_t}]$

$\Rightarrow 1 = E_t[MRS_t^{t+1}(1 + \tilde{r})]$

$$= \text{Cov}[MRS_t^{t+1}, \tilde{r}] + E_t[MRS_t^{t+1}]E_t[1 + \tilde{r}]$$

$\Rightarrow E_t[1 + \tilde{r}] - (1 + r_f) = -(1 + r_f)\text{Cov}[MRS_t^{t+1}, \tilde{r}]$

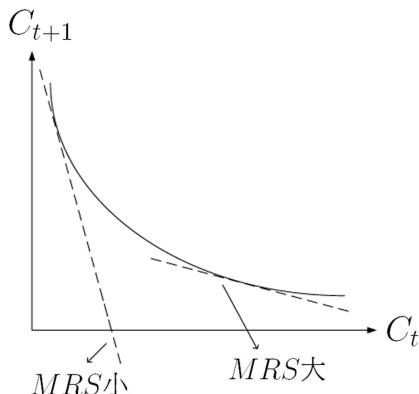
$\Rightarrow E_t[\tilde{r} - r_f] = -(1 + r_f)\text{Cov}[MRS_t^{t+1}, \tilde{r}]$

又 $E_t[\tilde{r}_m - r_f] = -(1 + r_f)\text{Cov}[MRS_t^{t+1}, \tilde{r}_m]$

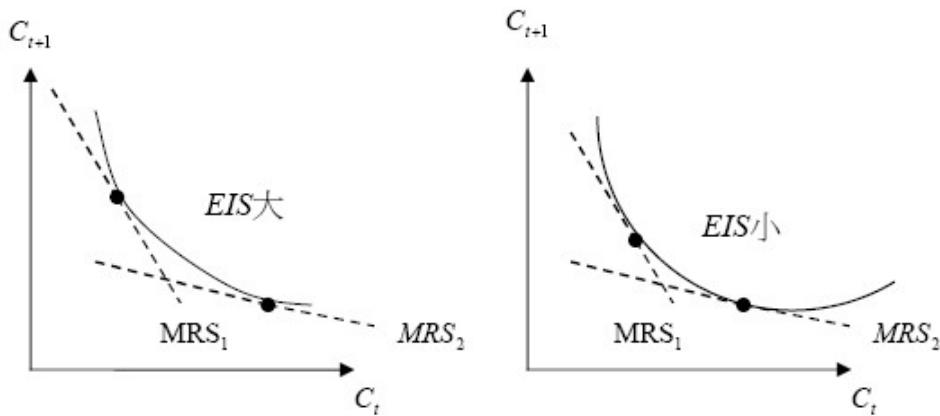
$\Rightarrow E_t[\tilde{r} - r_f] = \frac{\text{Cov}[MRS_t^{t+1}, \tilde{r}]}{\text{Cov}[MRS_t^{t+1}, \tilde{r}_m]}(E_t[\tilde{r}_m] - r_f)$

(與課本之結果很像，不過要注意的是 Consumption-based asset pricing model 是多期之模型，而課本的導証直到目前為止，都是單期之模型)

- Equity Premium Puzzle: The observed real risk free rate in America is 0.8% and the equity premium is about 6%. However, Mehra and Prescott (1985) found that given a proper value of the risk averse coefficient, one cannot generate a large enough risk premium to match the historical data through Lucas' model.
- Marginal rate of substitution (MRS) is the rate at which consumers are willing to give up units of one good in exchange for more units of another good, 例 $MRS_t^{t+1} \equiv \frac{\partial U/\partial C_{t+1}}{\partial U/\partial C_t}$ for power utility $= \rho(\frac{C_{t+1}}{C_t})^{-B}$, MRS 大, 表要犧牲很多之 C_t , 來換取一些 C_{t+1} (由圖中的斜率可看出), 且 MRS 大, 也可想成折現率小, 故 C_{t+1} 值較多之 C_t ; 反之, MRS 小, C_t 減少一點, C_{t+1} 增加很多 (由下圖可看出), 或想成折現率大, 故 C_{t+1} 值較少之 C_t (故 MRS 也可想成 C_{t+1} 之價格 in units of C_t)
- $MRS_t^{t+1} = \frac{\partial U/\partial C_{t+1}}{\partial U/\partial C_t} = -\frac{dC_t}{dC_{t+1}}|_{U \text{ is constant}} = -(\frac{1}{\text{斜率}})$



- Elasticity of intertemporal substitution (EIS) is the elasticity of the ratio of two inputs to a utility function with respect to the ratio of their marginal utilities, $EIS \equiv \frac{d \ln(C_{t+1}/C_t)}{d \ln(MRS_t^{t+1})}$ for power utility $= -\frac{1}{B}$ (一般取絕對值, 看成 $\frac{1}{B}$)
(measure how the ratio of factor inputs changes as the slope of the indifference curve changes, 即 MRS 變動 1% 時, consumption growth 會變動幾 %)



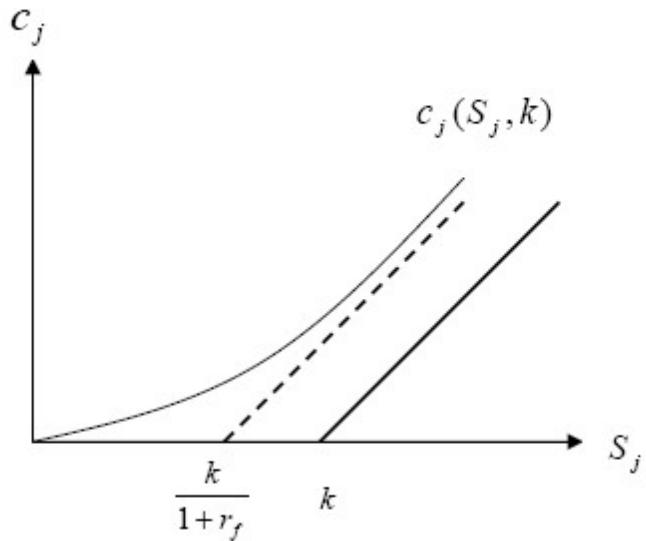
(indifferent curve 之曲度越大, EIS 小, 曲度小, EIS 大)

- 左圖, 當 MRS_1 上漲成 MRS_2 , 表 C_{t+1} 之價格相對較貴, 此時消費者可以選擇較多之 C_t 來替代, 反之當 MRS_2 下跌成 MRS_1 時, C_{t+1} 之價格相對便宜, 此時消費者可選擇較多之 C_{t+1} 來替代, 表示 C_t 與 C_{t+1} 之間的替代率高, 即消費者認為在何時消費沒差別
- 右圖, 當 MRS_1 上漲成 MRS_2 , 表 C_{t+1} 之價格相對較貴, 但消費者卻還是要選擇差不多比例之 C_t 與 C_{t+1} , 表 C_t 與 C_{t+1} 之間的替代率低, 即消費者會介意在 t 或是在 $t+1$ 消費

- EIS 大: 表消費者認為 C_{t+1} 之消費可取代 C_t 之消費, 所以今日有動機犧牲一點 C_t 去投資, 賺取報酬, 轉成 C_{t+1} 之增加 (因此 consumption process 由時間軸來看, 傾向較 volatile) (報酬上升, 容易吸引消費者延後消費)
 EIS 小: 表消費者認為 C_t 之消費不可由 C_{t+1} 之消費取代, 因此沒有動機減少今日的消費去投資 (報酬上升, 消費者也沒有意願延後消費)
(有論文用有錢人 EIS 大, 所以願意參與證券市場, 窮人 EIS 小, 所以不願參與證券市場, 來解釋美國的 participation rate 較低, 且股票大部分都在有錢人的手中, 其 consumption 較 volatile, 自然也會要求較高之風險貼水)
- 在 power utility 之架構下, risk averse coefficient 可同時控制風險趨避行為與 EIS 之大小, 當 $B \uparrow$, 消費者較風險趨避, 故希望 C_t 之出象越集中越好, 同時 EIS 小, C_t 與 C_{t+1} 間不可替代, 使得 consumption process 由時間軸來看也較 smooth, 意即 $B \uparrow$, 每一期之 C_t 的出象希望集中, 且跨期的消費變動也希望較 smooth
- Epstein and Zin (1989) 提出一種效用函數的架構, 將 B 與 EIS 分開:

$$V_t = \{(1 - \rho)C_t^{1-1/\psi} + \rho(E_t(V_{t+1}^{1-B}))^{\frac{1-1/\psi}{1-B}}\}^{\frac{1}{1-1/\psi}}$$
其中 ρ 為 discount factor, B 為風險趨避係數, ψ 為 EIS
- 以下大多用 arbitrage argument 來看 option prices, 主要目標是於 (6.10) 節:
在 (1) $t = 1$ 時, underlying asset 之 payoff 與之 aggregate consumption 是 jointly lognormal distribution
(2) the representative agent is with a CRRA utility
可導出類似 BS 的 option price formula
- Terminologies:
 S_j : underlying stock price at $t = 0$
 \tilde{x}_j : random payoff of one share of stock
 k : strike price
 $c_j(S_j, k)$: call option 今天之 price

- 想證明 $c_j(S_j, k) > \max[S_j - \frac{k}{1+r_f}, 0]$ (lower bound for European call options, $c \geq S_0 - ke^{-rT}$)



| | $\tilde{x}_j \geq k$ | $\tilde{x}_j < k$ |
|---------------------------|----------------------|-----------------------|
| short S_j | $-\tilde{x}_j$ | $-\tilde{x}_j$ |
| long $c_j(S_j, k)$ | $\tilde{x}_j - k$ | 0 |
| lending $\frac{k}{1+r_f}$ | k | k |
| total payoff at $t = 1$ | 0 | $k - \tilde{x}_j > 0$ |

$$\Rightarrow (i) c_j(S_j, k) - S_j + \frac{k}{1+r_f} > 0$$

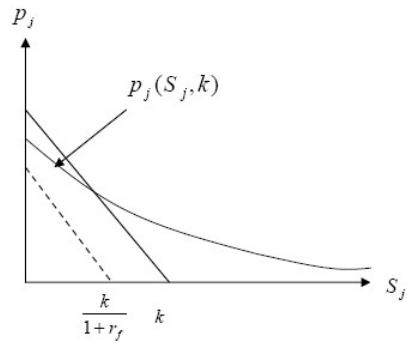
(若不成立, 則可能 something to be created from nothing)

(ii) 除此之外 $c_j(S_j, k)$ 為一權利, 而非義務 $\Rightarrow c_j(S_j, k) \geq 0$

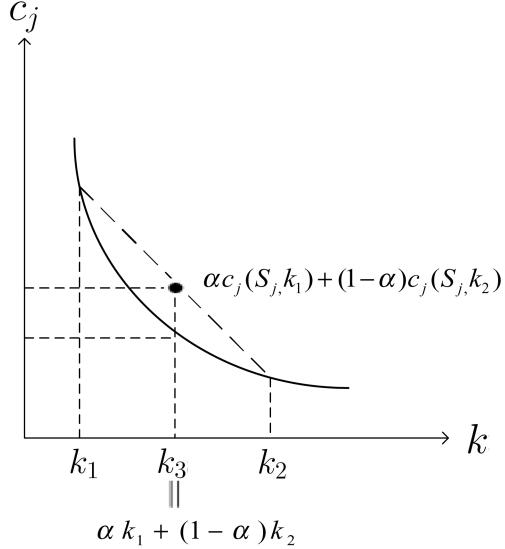
由 (i)(ii) 可知 $c_j(S_j, k) > \max[S_j - \frac{k}{1+r_f}, 0]$

(" \geq " \rightarrow " $>$ " if the probability that the option will be exercised is strictly between 0 and 1)

- For put, $p_j(S_j, k) > \max[\frac{k}{1+r_f} - S_j, 0]$ (lower bound for European put options, $p \geq ke^{-rT} - S_0$)



- 想證明 option price is a convex function of its exercise price
 $\Leftrightarrow \alpha c_j(S_j, k_1) + (1 - \alpha)c_j(S_j, k_2) \geq c_j(S_j, k_3)$
 (where $k_3 = \alpha k_1 + (1 - \alpha)k_2$)
 (圖)



- (i): long $\alpha c_j(S_j, k_1)$
- (ii): long $(1 - \alpha)c_j(S_j, k_2)$
- (iii): short $c_j(S_j, k_3)$
- (iv): total payoff

| | $\tilde{x}_j \leq k_1$ | $k_1 < \tilde{x}_j \leq k_3$ | $k_3 < \tilde{x}_j \leq k_2$ | $k_2 < \tilde{x}_j$ |
|-------|------------------------|---------------------------------|---|-----------------------------------|
| (i) | 0 | $\alpha(\tilde{x}_j - k_1)$ | $\alpha(\tilde{x}_j - k_1)$ | $\alpha(\tilde{x}_j - k_1)$ |
| (ii) | 0 | 0 | 0 | $(1 - \alpha)(\tilde{x}_j - k_2)$ |
| (iii) | 0 | 0 | $-(\tilde{x}_j - k_3)$ | $-(\tilde{x}_j - k_3)$ |
| (iv) | 0 | $\alpha(\tilde{x}_j - k_1) > 0$ | $\alpha(\tilde{x}_j - k_1) - (\tilde{x}_j - k_3)$ | 0 |

(其中 $\alpha(\tilde{x}_j - k_1) - (\tilde{x}_j - k_3) = (\alpha - 1)\tilde{x}_j - \alpha k_1 + k_3 = (\alpha - 1)\tilde{x}_j - \alpha k_1 + \alpha k_1 + (1 - \alpha)k_2 = (1 - \alpha)(k_2 - \tilde{x}_j) > 0$),

故得證

- $c_j(2S_j, 2k) = 2c_j(S_j, k)$

($c_j(S_j, k)$ is homogeneous of degree one in S_j and k)

(在發股票股利與股票分割時, strike price 與張數要同時調整)

- 上述是對 strike price k 作 weighted summation, 現在改對 \tilde{x}_j 作加總

$$\max\left[\sum_{j=1}^N \alpha_j \tilde{x}_j - k, 0\right] \leq \sum_{j=1}^N \alpha_j \max[\tilde{x}_j - k, 0]$$

(where \leq : 因 $\max(z, 0)$ 是 convex function and by Jensen's inequality)

$$\Rightarrow c^*(S^*, k) \leq \sum_{j=1}^N \alpha_j c_j(S_j, k)$$

意即大盤指數選擇權比個股選擇權比例加總便宜, 假設一個投資組合

$$S^* = \sum_{j=1}^N \alpha_j S_j, \text{ where } \sum_{j=1}^N \alpha_j = 1$$

$$\Rightarrow \text{其 } t = 1 \text{ 之 payoff 為 } \tilde{x}^* = \sum_{j=1}^N \alpha_j \tilde{x}_j$$

定義此投資組合之 call 的 payoff 為 $\max\left[\sum_{j=1}^N \alpha_j \tilde{x}_j - k, 0\right]$

其今日之 price 為 $c^*(S^*, k)$

- put-call parity $p_j(S_j, k) = \frac{k}{1+r_f} - S_j + c_j(S_j, k)$

| | $\tilde{x}_j < k$ | $\tilde{x}_j \geq k$ |
|---------------------------|-------------------|----------------------|
| short S_j | $-\tilde{x}_j$ | $-\tilde{x}_j$ |
| long $c_j(S_j, k)$ | 0 | $\tilde{x}_j - k$ |
| lending $\frac{k}{1+r_f}$ | k | k |
| total payoff | $k - \tilde{x}_j$ | 0 |

the payoff is the same as $p_j(S_j, k)$'s payoff, 故得證

- 希望證明 $c_j(S_j, k)$ is an increasing and convex function of S_j

(i) < increasing >

希望證明 if $S'_j > S_j \Rightarrow c_j(S'_j, k) > c_j(S_j, k)$

$$c_j(S'_j, k) = \frac{S'_j}{S_j} c_j(S_j, k \frac{S_j}{S'_j})$$

|| 等號是由 $c_j(2S_j, 2k) = 2c_j(S_j, k)$ 來的

$$\geq \frac{S'_j}{S_j} c_j(S_j, k) \text{ (因為 } \frac{S_j}{S'_j} < 1\text{)}$$

$$\geq c_j(S_j, k) \text{ (因為 } \frac{S'_j}{S_j} > 1\text{), 故得證}$$

(ii) < convex >

要證明 $\alpha c_j(S_j, k) + (1 - \alpha)c_j(S'_j, k) \geq c_j(\hat{S}_j, k)$, ($\hat{S}_j = \alpha S_j + (1 - \alpha)S'_j$)

先從 call 是 convex function of k 出發

$$rc_j(1, k_1) + (1 - r)c_j(1, k_2) \geq c_j(1, k_3), (k_3 = rk_1 + (1 - r)k_2)$$

$$\| \text{let } r = \alpha \frac{S_j}{\hat{S}_j}, k_1 = \frac{k}{S_j}, k_2 = \frac{k}{S'_j}$$

$$\Rightarrow \alpha \frac{S_j}{\hat{S}_j} c_j(1, \frac{k}{S_j}) + (1 - \alpha) \frac{S_j}{\hat{S}_j} c_j(1, \frac{k}{S'_j}) \geq c_j(1, k_3)$$

$$(\text{where } 1 - \alpha \frac{S_j}{\hat{S}_j} = \frac{\hat{S}_j - \alpha S_j}{\hat{S}_j} = \frac{(1-\alpha)S'_j}{\hat{S}_j})$$

$$\Rightarrow \alpha S_j c_j(1, \frac{k}{S_j}) + (1 - \alpha) S'_j c_j(1, \frac{k}{S'_j}) \geq \hat{S}_j c_j(1, k_3)$$

$$\Rightarrow \alpha c_j(S_j, k) + (1 - \alpha) c_j(S'_j, k) \geq c_j(\hat{S}_j, \hat{S}_j k_3)$$

(if $\hat{S}_j k_3 = k$, 則可得證)

$$\left\| \begin{aligned} \hat{S}_j k_3 &= \hat{S}_j [\alpha \frac{S_j}{\hat{S}_j} \frac{k}{S_j} + (1 - \alpha) \frac{S_j}{\hat{S}_j} \frac{k}{S'_j}] \\ &= \hat{S}_j [\alpha \frac{k}{S_j} + (1 - \alpha) \frac{k}{S'_j}] \\ &= \alpha k + (1 - \alpha) k = k \end{aligned} \right.$$

- 假設 (i) CRRA utility

$$(ii) \ln \left(\begin{array}{c} \tilde{C} \\ \tilde{x}_j \end{array} \right) \sim N(\quad)$$

由 MRS 之觀念出發，可推導出類似 BS formula 之公式（原本 BS formula 是在 continuous time model，且假設 stock price 是 geometric Brownian motion 和 instantaneous riskless interest rate 是 constant 得情況下導出）

$$(i) u_0(z_0) + u_1(z_1) = \frac{1}{1-B} z_0^{1-B} + \rho \frac{1}{1-B} z_1^{1-B}$$

(ρ : time preference parameter or subjective discount factor)

$$\text{由 } S_j = E[\frac{u'_1(\tilde{C})}{u'_0(C_0)} \tilde{x}_j] \text{ (6.2.3) (pricing kernal)}$$

$$\Rightarrow c_j(S_j, k) = E[\frac{u'_1(\tilde{C})}{u'_0(C_0)} \max(\tilde{x}_j - k, 0)]$$

$$(\text{where } \frac{u'_1(\tilde{C})}{u'_0(C_0)} = \rho (\frac{\tilde{C}}{C_0})^{-B})$$

$$\Rightarrow c_j(S_j, k) = \rho E[(\frac{\tilde{C}}{C_0})^{-B} \max(\tilde{x}_j - k, 0)]$$

(ii)

$$\left(\begin{array}{c} \ln \tilde{x}_j \\ \ln \tilde{C} \end{array} \right) \sim N(A, B)$$

$$A = \left(\begin{array}{c} \hat{\mu}_j \\ \hat{\mu}_c \end{array} \right)$$

$$B = \left(\begin{array}{cc} \sigma_j^2 & \kappa \sigma_j \hat{\sigma}_c \\ \kappa \sigma_j \hat{\sigma}_c & \hat{\sigma}_c^2 \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{c} \ln(\frac{\tilde{x}_j}{S_j}) \\ \ln \rho (\frac{\tilde{C}}{C_0})^{-B} \end{array} \right) \sim N(C, D)$$

$$C = \left(\begin{array}{c} \hat{\mu}_j - \ln S_j \\ -B \hat{\mu}_c + \ln \rho + B \ln C_0 \end{array} \right) = \left(\begin{array}{c} \mu_j \\ \mu_c \end{array} \right)$$

$$D = \left(\begin{array}{cc} \sigma_j^2 & -B \kappa \sigma_j \hat{\sigma}_c \\ -B \kappa \sigma_j \hat{\sigma}_c & B^2 \hat{\sigma}_c^2 \end{array} \right) = \left(\begin{array}{cc} \sigma_j^2 & \kappa \sigma_j \sigma_c \\ \kappa \sigma_j \sigma_c & \sigma_c^2 \end{array} \right),$$

where $\sigma_c = -B\hat{\sigma}_c$

$$\begin{aligned} \text{(i)+(ii)} \Rightarrow c_j(S_j, k) &= \rho E[(\frac{\tilde{C}}{C_0})^{-B} \max(\tilde{x}_j - k, 0)] \\ &= S_j \int_{-\infty}^{\infty} \int_{\ln(\frac{k}{S_j})}^{\infty} (e^z - \frac{k}{S_j}) e^y f(z, y) dz dy \end{aligned}$$

$$\begin{aligned} \left| \begin{array}{l} \text{其中 } z = \ln \frac{\tilde{x}_j}{S_j} \Rightarrow e^z = \frac{\tilde{x}_j}{S_j} \\ y = \ln \rho (\frac{\tilde{C}}{C_0})^{-B} \Rightarrow e^y = \rho (\frac{\tilde{C}}{C_0})^{-B} \end{array} \right. \\ = S_j \int_{-\infty}^{\infty} \int_{\ln(\frac{k}{S_j})}^{\infty} e^{z+y} f(z, y) dz dy \\ - k \int_{-\infty}^{\infty} \int_{\ln(\frac{k}{S_j})}^{\infty} e^y f(z, y) dz dy \end{aligned}$$

$$\begin{aligned} \left| \begin{array}{l} (1) = \int_{-\infty}^{\infty} \int_{\ln(\frac{k}{S_j})}^{\infty} e^y f(z, y) dz dy \\ (2) = \int_{-\infty}^{\infty} \int_{\ln(\frac{k}{S_j})}^{\infty} e^{y+z} f(z, y) dz dy \end{array} \right. \\ = S_j \cdot (2) - k(1) \end{aligned}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \int_a^{\infty} e^y f(z, y) dz dy = (e^{\mu_C + \frac{\sigma_C^2}{2}}) N(\frac{-a+\mu_j}{\sigma_j} + \kappa \sigma_C) \quad (6.10.4)$$

$$\begin{aligned} (2) \Rightarrow \int_{-\infty}^{\infty} \int_a^{\infty} e^{z+y} f(z, y) dz dy \\ = (e^{\mu_j + \mu_C + (\sigma_j^2 + 2\kappa \sigma_j \sigma_C + \sigma_C^2)/2}) N(\frac{-a+\mu_j}{\sigma_j} + \kappa \sigma_C + \sigma_j) \quad (6.10.5) \end{aligned}$$

if $a = -\infty \Rightarrow$

$$(1) = E[e^y] = E[\rho (\frac{\tilde{C}}{C_0})^{-B}] = e^{\mu_C + \frac{\sigma_C^2}{2}}$$

$$(2) = E[e^{z+y}] = E[\frac{\tilde{x}_j}{S_j} \rho (\frac{\tilde{C}}{C_0})^{-B}] = e^{\mu_j + \mu_C + (\sigma_j^2 + 2\kappa \sigma_j \sigma_C + \sigma_C^2)/2}$$

$$\text{又由 } S_j = E[\frac{u'_1(\tilde{C})}{u'_0(C_0)} \tilde{x}_j]$$

$$\text{for (1)} \Rightarrow \frac{1}{1+r_f} = E[\frac{u'_1(\tilde{C})}{u'_0(C_0)} 1] \Rightarrow e^{\mu_C + \frac{\sigma_C^2}{2}} = \frac{1}{1+r_f}$$

·由 $E[\rho (\frac{\tilde{C}}{C_0})^{-B}] = \frac{1}{1+r_f}$ 可看出 r_f 之變化來自於消費成長率之期望值

與 risk averse coefficient 之變化(因 $\sigma_c = -B\hat{\sigma}_c$)

·亦即 $\text{Cov}(\frac{\tilde{C}}{C_0}, r_f) > 0$ (r_f 與 $\frac{\tilde{C}}{C_0}$ 間互相影響)

(i) $r_f \uparrow$, 消費者願意犧牲 C_0 , 去投資 riskless bond, 以增加 \tilde{C} , 使得 $\frac{\tilde{C}}{C_0} \uparrow$

(ii) 若預期消費成長率高, 亦即今日之消費 C_0 , 相較於明日之消費 \tilde{C} , 是少很多的, 消費者會希望將部份之 \tilde{C} 移回今日消費, 此時就借錢, 使得 $r_f \uparrow$

$$\text{for (2)} \Rightarrow 1 = E[\frac{u'_1(\tilde{C})}{u'_0(C_0)} \frac{\tilde{x}_j}{S_j}] \Rightarrow e^{\mu_j + \mu_C + (\sigma_j^2 + 2\kappa \sigma_j \sigma_C + \sigma_C^2)/2} = 1$$

代入 $c_j(S_j, k) = S_j \cdot (2) - k \cdot (1)$

$$\Rightarrow c_j(S_j, k) = S_j N(Z_k + \sigma_j) - \frac{k}{1+r_f} N(Z_k)$$

其中 $Z_k = \frac{\ln(\frac{S_j}{k}) + (\mu_j + \kappa\sigma_j\sigma_C)}{\sigma_j}$

$$\left\| \begin{array}{l} \text{又由 } e^{\mu_C + \frac{\sigma_C^2}{2}} = \frac{1}{1+r_f} \text{ and } e^{\mu_j + \mu_C + (\sigma_j^2 + 2\kappa\sigma_j\sigma_C + \sigma_C^2)/2} = 1 \\ \Rightarrow \mu_j + \kappa\sigma_j\sigma_C = \ln(1+r_f) - \frac{\sigma_j^2}{2} \end{array} \right.$$

$$\Rightarrow Z_k = \frac{\ln(\frac{S_j}{k}) + \ln(1+r_f)}{\sigma_j} - \frac{\sigma_j^2}{2}$$

(risk averse coefficient 原本隱含在 σ_C 中, 但最後 σ_C 被取代了, 所以人的風險偏好不會影響 option 之價格)

- (1)(2) 之積分參考課本 p.166-p.168

- 只要 call option 之定價如上, 則 option 是 redundant asset, 而在均衡中因為每個人只要選擇 market portfolio 與 riskless asset 之組合, 就可以達到 PO, 因此沒有人會選 option (或是換句話說, 當均衡已形成, 則符合上述定價的 call option 放進均衡中, 也不會影響均衡 (does not upset equilibrium))

- (6.13節) Greek Letters

$$\frac{\partial C_j(S_j, k)}{\partial k}, \frac{\partial C_j(S_j, k)}{\partial S_j}, \frac{\partial C_j(S_j, k)}{\partial \sigma_f}, \frac{\partial C_j(S_j, k)}{\partial r_f}, \frac{\partial^2 C_j(S_j, k)}{\partial k^2}$$

- For firm j , $V_j = S_j + D_j$

$$S_j = c_j(V_j, k) = \rho E[\max[\tilde{x}_j - k, 0] (\frac{\tilde{C}}{C_0})^{-B}]$$

(Equity part 相當於擁有 call option) (6.14.2)

$$S_j = V_j N(Z_k + \sigma_j) - (1 + r_f)^{-1} k N(Z_k), \text{ where } Z_k = \frac{\ln(\frac{V_j}{k}) + \ln(1+r_f)}{\sigma_j} - \frac{1}{2} \sigma_j$$

$$D_j = \rho E[\min[\tilde{x}_j, k] (\frac{\tilde{C}}{C_0})^{-B}]$$

(k 為 D_j 之面額, \tilde{x}_j 為 firm j 下一期之 payoff) (6.14.3)

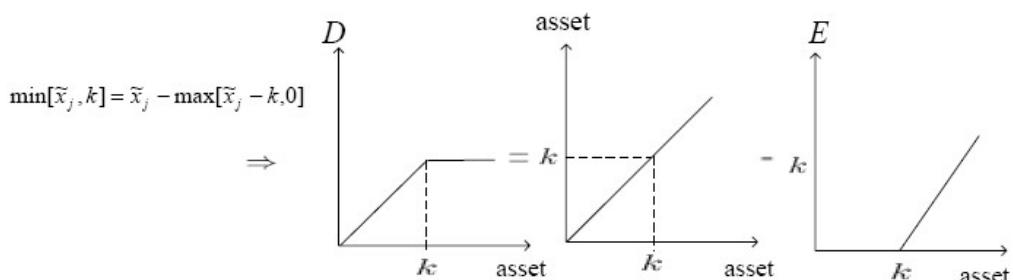
$$D_j = V_j - S_j = V_j(1 - N(Z_k + \sigma_j)) + (1 + r_f)^{-1} k N(Z_k)$$

- risky corporate debt 可由 option 之觀念來想

(1) 想成 bond holders 擁有公司

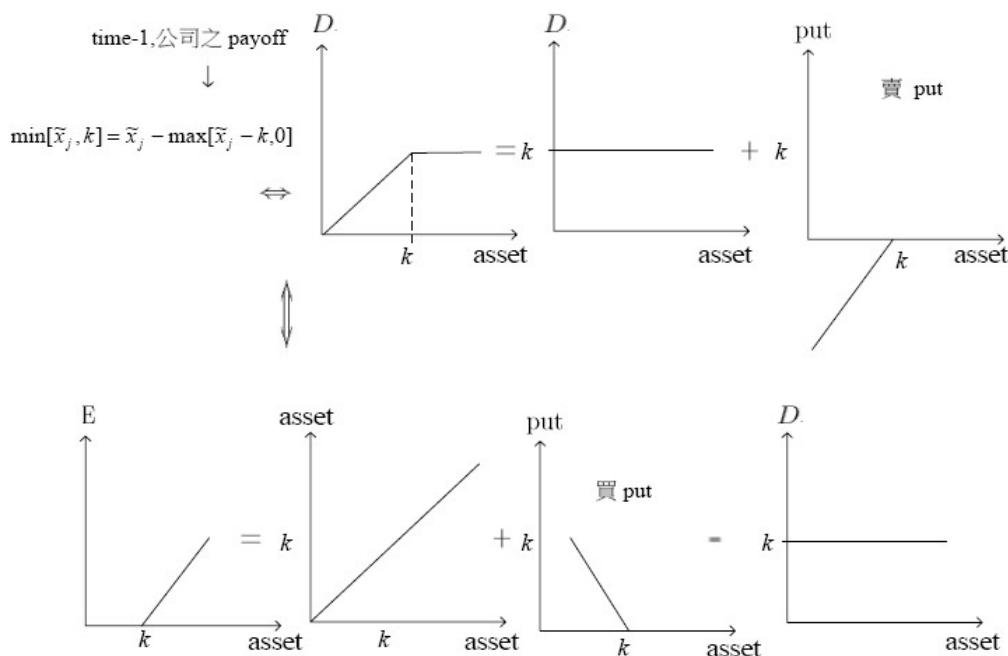
the bond holders can be viewed as holding the firm while selling a European call option on the value of firm with an exercise price equal to the face value of the debt to the equity holders ($\min[\tilde{x}_j, k] = \tilde{x}_j - \max[\tilde{x}_j - k, 0]$)

(圖)

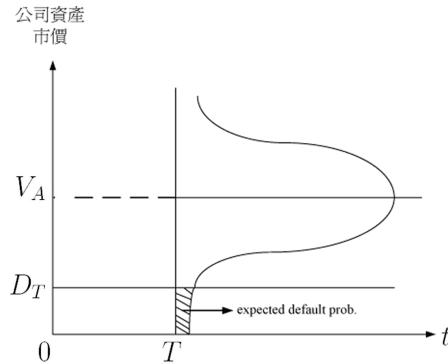


(2) 想成 equity holders 擁有公司

the bond holders are holding a riskless discount bond with a face value k , while at the same time selling a European put option on the value of the firm with an exercise price k to the equity holders. ($\min[\tilde{x}_j, k] = k - \max[k - \tilde{x}_j, 0]$)
 (圖)



- KMV credit rating model (Kealhofer, McQuown, and Vasicek 創立於1989年, 2002年2月, 與 Moody's 的風險管理系統部, 合併為 Moody's KMV 公司)



V_A : 公司資產今日市價

V_E : 公司權益今日價值 (as a call on V_A with a strike price D_T)

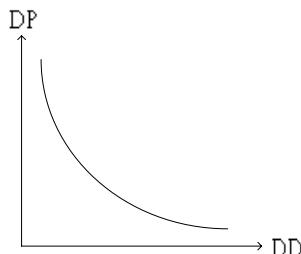
D_T : $t = T$ 時, 公司負債之面額

σ_E : 公司權益之波動度

σ_A : 公司市值之波動度

Default Distance (DD) = $\frac{E[V_A \text{ at } T] - D_T}{E[V_A \text{ at } T] \cdot \sigma_A}$ (DD \uparrow , default prob. \downarrow)

例如:



(DP 與 DD 之關係可用 exponential function 來 approximate $DP \approx \exp(a + b \cdot DD) + c$)

| DD | 0~1 | 1~2 | 2~3 | 3~4 | 4~5 | 5~6 |
|------|-----|------|------|-------|-------|-------|
| DP | 0.8 | 0.3 | 0.1 | 0.043 | 0.007 | 0.004 |
| 違約數目 | 720 | 450 | 200 | 150 | 28 | 17 |
| 公司數目 | 900 | 1500 | 2000 | 35000 | 40000 | 42000 |

- 由 Black-Scholes-Merton Formula

$$\Rightarrow V_E = V_A N(d_1) - D_T e^{-rT} N(d_2) \quad (1)$$

$$\text{where } d_1 = \frac{\ln(\frac{V_A}{D_T}) + (r + \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}}, d_2 = d_1 - \sigma_A \sqrt{T}$$

- 現實生活中 V_E 與 σ_E 已知, 但 V_A 與 σ_A 是未知

$$\text{assume } dV_E = V_E r dt + V_E \sigma_E dz$$

$$dV_A = V_A r dt + V_A \sigma_A dz$$

又因 $V_E(V_A, t)$ 為 V_A, t 之函數

$$\begin{aligned} \Rightarrow dV_E &= (\dots)dt + (\frac{\partial V_E}{\partial V_A} V_A \sigma_A) dZ \\ &= (\dots)dt + V_E \sigma_E dZ \\ \Rightarrow \sigma_E &= \frac{\frac{\partial V_E}{\partial V_A} V_A \sigma_A}{V_E} = \frac{N(d_1) V_A \sigma_A}{V_E} \end{aligned} \quad (2)$$

由 (1)(2) 聯立, 可求得 V_A 與 σ_A

- Expected default prob. = $P(V_A \text{ at } T \leq D_T) = P(\ln(V_A \text{ at } T) \leq \ln D_T)$

$$= P(\ln V_A + (r - \frac{\sigma_A^2}{2})T + \sigma_A \sqrt{T} \varepsilon_A \leq \ln D_T)$$

$$= P(\varepsilon_A \leq \frac{\ln(\frac{D_T}{V_A}) - (r - \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}})$$

$$= P(\varepsilon_A \leq -d_2) = N(-d_2)$$

- KMV 之好處: 用 V_E, σ_E, r, T, D_T 即可算出 expected default prob. 或是 default distance.

- KMV 之缺點 : (1) 假設 V_A 為 lognormal, 但違約考慮進來後, 並不是 lognormal

(2) expected default prob. 與歷史資料比偏低, 無法反應現狀

- 實務上: (1) KMV 建議 D_T (T 通常為 1 年) = 流動負債 + $\frac{1}{2}$ 長期負債 (2) 若是未公開上市公司, 則可用類似性質的公開上市公司資料, 求出 V_A 與 σ_A 和相關財報資訊之回歸式, 帶入非公開上市公司的這些財報資訊, 即可得其 V_A 與 σ_A
- 因 lognormal + lognormal \neq lognormal, 所以 6.10 之技巧, 對很多 portfolio 都不適用 (因 portfolio 之 payoff 與 aggregate consumption 已非 lognormal 分配), 但若我們可由 \tilde{C} 是 lognormal 分配, 求出 call on \tilde{C} 之 value, 再借助 5.19 小節中所提到的 pricing density (call on consumption 對 k 的二次微分), 則可應用來算其他 securities/portfolios 的價值.

- call option on \tilde{C} ($\ln \tilde{C} \sim N(\cdot, \cdot)$)

$$c_C(S_C, k) = S_C N(Z_k + \hat{\sigma}_C) - \frac{k}{1+r_f} N(Z_k)$$

$$\text{where } Z_k = \frac{\ln(S_C/k) + \ln(1+r_f)}{\hat{\sigma}_C} - \frac{1}{2}\hat{\sigma}_C^2$$

$$\hat{\sigma}_C^2 = \text{var}(\ln \tilde{C})$$

$S_C = \rho E[\tilde{C}(\frac{\tilde{C}}{C_0})^{-B}]$ ($t=1$ 之 aggregate \tilde{C} 今日的價格, 由式子 (6.2.3) 可看出)

- 若 $\Phi(k)$ 為 elementary claim price, 即當未來之 $\tilde{C} = k$ 時, 就得到一單位消費的 security, 今天之價格. 由 (5.19.2) 可知 the pricing density $\phi_c(k) = \frac{\Phi_c(k)}{dk} = \frac{\partial^2 c_C(k)}{\partial k^2} = \frac{n(Z_k)}{(1+r_f)\hat{\sigma}_C k}$ (6.15.5) 式

| | |
|--|---|
| | 6.15 節與 6.16 節 $\phi_c(k)$ 之 comparative statics |
| | $\frac{\partial \ln \phi_c(k)}{\partial \ln S_c}, \frac{\partial \ln \phi_c(k)}{\partial \ln \hat{\sigma}_c}, \frac{\partial \ln \phi_c(k)}{\partial \ln(1+r_f)^{-1}}, \frac{\partial \ln \phi_c(k)}{\partial \ln k}$ |

- (5.19.3) 式 $\Rightarrow S_x = \int \frac{\Phi_c(k)}{dk} E[\tilde{x} | \tilde{C} = k] dk = \int E[\tilde{x} | \tilde{C} = k] \phi_c(k) dk$
 $(6.2.3) \text{ 式 } \Rightarrow S_x = E\left[\frac{u'_1(\tilde{C})}{u'_0(C_0)} \cdot \tilde{x}\right] = E[E[MRS \cdot \tilde{x} | \tilde{C} = k] \cdot \text{prob.}(\tilde{C} = k)]$

$$\text{兩式相比 } \Rightarrow \frac{\phi_c(k)}{\text{prob}(\tilde{C}=k)} = MRS(\tilde{C} = k)$$

- if $\frac{\partial \ln MRS}{\partial \ln \tilde{C}} = \text{constant} \iff \text{CRRA utility}$

$$(\text{也可寫成 } \frac{\partial \ln(\phi_c(k)/\text{prob.}(\tilde{C}=k))}{\partial \ln k} = \frac{\partial \ln MRS(\tilde{C}=k)}{\partial \ln k} = \text{constant})$$

(例如 power utility 時, $MRS = (\frac{\tilde{C}}{C_0})^{-\beta}$, $\ln MRS = -\beta(\ln \tilde{C} - \ln C_0)$,

$$\frac{\partial \ln MRS}{\partial \ln \tilde{C}} = -\beta = \text{constant}$$

- (6.10) 節證明 lognormal+CRRA \Rightarrow BS, 而 (6.17) 節想證明在維持 lognormal 之假設下, BS \Rightarrow CRRA, 亦即用 BS 之 $c_C(S_C, k)$ 算出之 pricing density $\phi_C(k) = \frac{\partial^2 c_C(S_C, k)}{\partial k^2}$ 滿足 $\frac{\partial \ln(\phi_C(k)/\text{prob.}(\tilde{C}=k))}{\partial \ln k}$ 為 constant
- (6.17.1)~(6.17.3) 式想證明 $\frac{\partial \ln(\phi_C(k)/\text{prob.}(\tilde{C}=k))}{\partial \ln k}$ 為 constant, 但不知其中 u 為何?)

- (6.18 節) 由 (5.19.3) \Rightarrow 當一個 complex security 之 payoff 為 \tilde{x}_j , 其今日之 price
 $S_j = \int \phi_C(k) E[\tilde{x}_j | \tilde{C} = k] dk$
 $= \frac{1}{(1+r_f)\hat{\sigma}_C} \int_0^\infty \frac{n(Z_k)}{k} E[\tilde{x}_j | \tilde{C} = k] dk$

$$\begin{cases} \text{if } \begin{pmatrix} y \\ z \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_y \\ \mu_z \end{pmatrix}, \begin{pmatrix} \sigma_y^2 & \sigma_{yz} \\ \sigma_{yz} & \sigma_z^2 \end{pmatrix}\right) \\ \Rightarrow y|z \sim N(\mu_y + \beta_{yz}(z - \mu_z), \sigma_y^2 - \beta_{yz}\sigma_{yz})(\text{由 } f(y|z) = \frac{f(y,z)}{f(z)} \text{ 可導出}) \\ \text{where } \beta_{yz} = \frac{\sigma_{yz}}{\sigma_z^2} \end{cases}$$

$$\text{現在 } \begin{pmatrix} \ln \tilde{x}_j \\ \ln \tilde{C} \end{pmatrix} \sim N\left(\begin{pmatrix} \hat{\mu}_j \\ \hat{\mu}_C \end{pmatrix}, \begin{pmatrix} \sigma_j^2 & \sigma_{jC} \\ \sigma_{jC} & \hat{\sigma}_C^2 \end{pmatrix}\right)$$

$$\Rightarrow \ln \tilde{x}_j | \ln \tilde{C} \sim N(\hat{\mu}_j + \beta_{jC}(\ln \tilde{C} - \hat{\mu}_C), \sigma_j^2 - \beta_{jC}\sigma_{jC})$$

$$(\text{但6.18節}, \ln \tilde{x}_j | \ln \tilde{C} \sim N(\hat{\mu}_j - \frac{1}{2}\sigma_j^2 + \beta_{jc}(\ln \tilde{C} - \hat{\mu}_c + \frac{1}{2}\sigma_j^2),$$

$$\sigma_j^2 - \beta_{jC}\sigma_{jC}, \text{ 且 } \beta_{jC} = \frac{\sigma_{jC}}{\sigma_j \tilde{\sigma}_C})$$

$$\left\| \begin{array}{l} \ln(E[x]) = E[\ln x] + \frac{1}{2}\text{var}(\ln x) \\ E[e^y] = e^{\mu_y + \frac{1}{2}\sigma_y^2} (x = e^y \text{ or } \ln x = y) \end{array} \right.$$

$$E[\tilde{x}_j | \tilde{C} = k] = \exp\{\hat{\mu}_j + \beta_{jC}(\ln k - \hat{\mu}_C) - \frac{\sigma_j^2 - \beta_{jC}\sigma_{jC}}{2}\}$$

$$\Rightarrow S_j = \frac{1}{(1+r_f)\tilde{\sigma}_C} \int_0^\infty \frac{n(Z_k)}{k} \exp\{\hat{\mu}_j + \beta_{jC}(\ln k - \hat{\mu}_C) - \frac{\sigma_j^2 - \beta_{jC}\sigma_{jC}}{2}\} dk$$

(算不出積分)