

Ch5. Allocative Efficiency and the Valuation of State Contingent Securities

- “A state contingent consumption claim” is a security that pays one unit of the consumption good when one particular state of the world occurs and nothing otherwise.

(It is also called a Arrow-Debreu Security. Existing assets may be viewed as complex bundles of state contingent claims.)

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{|\Omega| \times 1}, \text{在今天之價格為 } \phi_w, \text{此亦稱為 A-D state price}$$

- If $\left\{ \begin{array}{l} \text{allocation of state contingent is efficient} \\ \text{individuals have time-additive state independent utility function} \end{array} \right.$
則均衡價的決定就好像由一個 representative agent 來決定，而此 agent 之 utility function 與經濟體中個人之財富分配有關
 \Rightarrow 均衡價與財富分配也有關 (5.21 - 5.26 節)
- 若 utility function 使得均衡價格與 initial wealth distribution 無關，則我們說這個 utility 有 aggregation property
- Two periods, one perishable consumption good,
 I individuals, $|\Omega|$ states, $w \in \Omega$
individual i 之 allocation of contingent claim is
 $\{(c_{i0}, c_{iw}, \omega \in \Omega) ; i = 1, 2, \dots, I\}$, which is feasible if

$$\sum_{i=1}^I c_{i0} = C_0, \sum_{i=1}^I c_{iw} = C_\omega$$

- 希望證明:

1. CE is PO under
 - (1) complete market
 - (2) perfect competition

(本書只討論此部份)

2. Any PO can be achieved as a CE

除了 (1)(2) 之外,

- (3) consumer's preference is convex (意即 u is concave)
- (4) firm's production sets are convex

- Pareto optimal or Pareto efficiency means that it is not possible to make someone better off without making others worse

此時, 根據 classical second welfare theorem, a “social planner” maximizes a linear combination of individuals' utility function using $\{\lambda_i\}$ as weights.

$$\max_{c_{i0}, c_{i\omega}} \sum_{i=1}^I \lambda_i \left[\sum_{\omega \in \Omega} \pi_{i\omega} u_{i\omega}(c_{i0}, c_{i\omega}) \right]$$

$$\text{s.t. } \sum_{i=1}^I c_{i\omega} = C_\omega, \forall \omega \in \Omega, \sum_{i=1}^I c_{i0} = C_0$$

(if $\pi_{i\omega} = \pi_\omega \Rightarrow$ homogeneous expectation)

$$\begin{aligned} \Rightarrow \max_{c_{i0}, c_{i\omega}} L &= \sum_{i=1}^I \lambda_i \left[\sum_{\omega \in \Omega} \pi_{i\omega} u_{i\omega}(c_{i0}, c_{i\omega}) \right] + \phi_0 [C_0 - \sum_{i=1}^I c_{i0}] \\ &\quad + \sum_{\omega \in \Omega} \phi_\omega [C_\omega - \sum_{i=1}^I c_{i\omega}] \end{aligned}$$

$$\frac{\partial}{\partial c_{i0}} \Rightarrow \lambda_i \sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}} = \phi_0, i = 1, \dots, I \quad (5.4.1)$$

$$\frac{\partial}{\partial c_{i\omega}} \Rightarrow \lambda_i \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}} = \phi_\omega, \omega \in \Omega, i = 1, \dots, I \quad (5.4.2)$$

\Rightarrow MRS (marginal rates of substitution)

$$\begin{aligned} &= \frac{\pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}}}{\sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}}} \\ &= \frac{\phi_\omega}{\phi_0}, \omega \in \Omega, i = 1, \dots, I \end{aligned}$$

\Rightarrow PO \Leftrightarrow MRS are equal across individuals for each state (MRS is independent of i)

- Competitive equilibrium \Rightarrow PO (if there exist a complete set of state contingent claims)

for any individual i

$$\max_{c_{i0}, c_{i\omega}} \sum_{\omega \in \Omega} \pi_{i\omega} u_{i\omega}(c_{i0}, c_{i\omega})$$

$$\text{s.t. } c_{i0} + \sum_{\omega \in \Omega} \phi_{\omega} c_{i\omega} = e_{i0} + \sum_{\omega \in \Omega} \phi_{\omega} e_{i\omega}$$

(今日之消費 + 購買未來之消費 = endowment)

(其中 ϕ_{ω} denotes the price at $t = 0$ of each state contingent claim, e_{i0} , $e_{i\omega}$ 指 endowments, 其中 e_{i0} 是以 consumption goods 型式存在, 而 $e_{i\omega}$ 是以持有 state contingent claims 之方式存在)

$$\max_{c_{i0}, c_{i\omega}} L =$$

$$\sum_{\omega \in \Omega} \pi_{i\omega} u_{i\omega}(c_{i0}, c_{i\omega}) + \theta_i [e_{i0} - c_{i0} + \sum_{\omega \in \Omega} \phi_{\omega} (e_{i\omega} - c_{i\omega})]$$

$$\frac{\partial}{\partial c_{i0}} \Rightarrow \sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}} = \theta_i$$

$$\frac{\partial}{\partial c_{i\omega}} \Rightarrow \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}} = \theta_i \phi_{\omega}, \omega \in \Omega$$

(其中 θ_i 為 shadow price. A shadow price is the change in the objective value of the optimal solution of an optimization obtained by relaxing the right hand side of the constraint by one unit.)

$$\text{MRS} = \frac{\pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}}}{\sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}}} = \phi_{\omega}, \omega \in \Omega$$

將 PO 中的 $\phi_0 = 1$, $\lambda_i = \theta_i^{-1}$, 則 CE \Rightarrow PO

(此時 CE being with a complete set of state contingent claims \Rightarrow market complete, 且 ϕ_{ω} 被稱為 state price)

(因 consumption good 為 numeraire, 所以 time-0 的 consumption good 之價值 ϕ_0 就為 1)

- 上述只考慮了 state contingent claims, 但實際上很可能 state contingent claims 並不存在, 而是存在 complex securities, which are bundles of state contingent claims.

- 下面考慮市場上存在個數等於 the number of states 的 linear independent complex securities 之情況 (we call this kind of economy as a securities markets economy)

S_j : price of complex security j at $t = 0$, $j = 1, \dots, N$,
which payoff at $t = 1$ is x_{jw} units of consumption goods in state w

$$x_j = \begin{bmatrix} x_{j1} \\ x_{j2} \\ \vdots \\ x_{j|\Omega|} \end{bmatrix}$$

individual i holds α_{ij} shares of searity j

- For individuals i

$$\max_{c_{i0}, \alpha_{ij}} \sum_{\omega \in \Omega} \pi_{i\omega} u_{i\omega}(c_{i0}, \sum_{j=1}^N \alpha_{ij} x_{j\omega}) \quad (\text{where } \sum_{j=1}^N \alpha_{ij} x_{j\omega} = c_{i\omega})$$

$$\text{s.t. } c_{i0} + \sum_j \alpha_{ij} S_j = e_{i0} + \sum_j \hat{\alpha}_{ij} S_j$$

(今日之消費 + 購買 complex securities = endowment)

(其中 individual i is endowed with $\hat{\alpha}_{ij}$ number of shares of security j , and e_{i0} of time-0 consumption)

$$\frac{\partial}{\partial \alpha_{ij}} \Rightarrow \sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}} x_{j\omega} = \theta_i S_j$$

$$\frac{\partial}{\partial c_{i0}} \Rightarrow \sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}} = \theta_i$$

$$\Rightarrow \sum_{\omega \in \Omega} \pi_{i\omega} \frac{\frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}}}{\sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}}} x_{j\omega} = S_j, j = 1, \dots, N$$

但之前 MRS 是指 between present consumption and “state contingent consumption” 之邊際替代率, 但這裡並不是, 所以還不能說此 CE 是 PO.

改寫成矩陣形態:

$$\Rightarrow \begin{pmatrix} x_{1\omega_1} & \dots & x_{1\omega_{|\Omega|}} \\ x_{2\omega_1} & \dots & x_{2\omega_{|\Omega|}} \\ \vdots & & \vdots \\ x_{N\omega_1} & \dots & x_{N\omega_{|\Omega|}} \end{pmatrix} \begin{pmatrix} \pi_{i\omega_1} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega}) / \partial c_{i\omega_1}}{\sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}}} \\ \pi_{i\omega_2} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega}) / \partial c_{i\omega_2}}{\sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}}} \\ \vdots \\ \pi_{i\omega_{|\Omega|}} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega}) / \partial c_{i\omega_{|\Omega|}}}{\sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}}} \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_N \end{pmatrix}$$

$\Rightarrow [x_{j\omega}] \times (\text{之前的 MRS}) = S$

if $N = |\Omega|$, 加上 linearly independent assumption

$\Rightarrow [x_{j\omega}]$ is invertible

$\Rightarrow \text{之前的 MRS} = [x_{j\omega}]^{-1}S$

$$\Rightarrow \pi_{i\omega} \frac{\frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i\omega}}}{\sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(c_{i0}, c_{i\omega})}{\partial c_{i0}}} = m_\omega, i = 1, \dots, I$$

(where m_ω is independent of i)

\Rightarrow 即使考慮 complex security, 只要 security 之數目夠多, 至少等於 $|\Omega|$, 則 CE \Rightarrow PO, 且 $m_\omega = \phi_\omega$

- 除此之外, state contingent claims can be created by forming portfolios of the complex securities

$$[x_{j\omega}] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{portfolio weight} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = [x_{j\omega}]^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

\Rightarrow optimal allocation of state contingent claims is achieved with individuals trading complex securities.

- Complete market economy 與 (complete) securities market economy 之共同特色為 the values of securities are additive
亦即當 $z_\omega = x_\omega + y_\omega$ (payoff 一樣) $S_z = S_x + S_y$
如果有 a complete set of state contingent claims

$$\Rightarrow X_\omega = \begin{bmatrix} 3 \\ 2 \\ 0 \\ \vdots \\ 4 \end{bmatrix} = 3 \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + 2 \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + 4 \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{則 } S_x &= \sum_{\omega \in \Omega} \phi_\omega x_\omega \\ \Rightarrow S_z &= \sum_{\omega \in \Omega} \phi_\omega z_\omega = \sum_{\omega \in \Omega} \phi_\omega (x_\omega + y_\omega) \\ &= \sum_{\omega \in \Omega} \phi_\omega x_\omega + \sum_{\omega \in \Omega} \phi_\omega y_\omega = S_x + S_y \end{aligned}$$

** 若 “=” 不成立

$\Rightarrow \exists$ arbitrage opportunity(買低賣高, 且將來之 payoff 可 offset)

如果是在一個 securities markets economy, 根據 (5.6.2)

$$\begin{aligned} S_z &= \sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(C_{i0}, C_{i\omega}) / \partial C_{i\omega}}{\sum_{\omega \in \Omega} \pi_{i\omega} \partial u_{i\omega}(C_{i0}, C_{i\omega}) / \partial C_{i0}} z_\omega \\ &= \sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(C_{i0}, C_{i\omega}) / \partial C_{i\omega}}{\sum_{\omega \in \Omega} \pi_{i\omega} \partial u_{i\omega}(C_{i0}, C_{i\omega}) / \partial C_{i0}} x_\omega + \sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i\omega}(C_{i0}, C_{i\omega}) / \partial C_{i\omega}}{\sum_{\omega \in \Omega} \pi_{i\omega} \partial u_{i\omega}(C_{i0}, C_{i\omega}) / \partial C_{i0}} y_\omega \\ &= S_x + S_y \end{aligned}$$

- Modigliani and Miller Thorem

The value of firms in the same risk class are determined independently of their capital structure in a frictionless market

因為公司 security 之價值只與其下一期 payoff 有關,

假設 2 firms

- (1) 下一期之 payoff 是成比例的 $\alpha x_{1\omega} = x_{2\omega}$,
- (2) firm 1 is 100% equity, firm 2 is partly debt

$$\Rightarrow \alpha x_{1\omega} = x_{2\omega} = E_{2\omega} + D_{2\omega}$$

$$\Rightarrow \alpha S_1 = S_2 = S_{2E} + S_{2D}$$

但並不表示, 當公司改變其資本結構時 (recapitalization), 公司之價值不變:

(i) 若在 incomplete market 中, 公司 recapitalization, 則有新 security 出現, equilibrium 重新形成, 各個 ϕ_w 也不同了, 使得公司之 value 有些變化.

(ii) 若 recapitalization 並未形成新的 security, 則公司依然需買回或賣出部分之 security, 則人們也須調整其投資組合, 除非存在唯一的均衡, 不然均衡改變了, 公司價值也變了

- 但是在現實中，通常 complex securities 之數目，小於 state of nature 之數目，但可由 option market 中的 put or call 淉出 complete market assume $|\Omega|=5$ (想成股價有5種可能)

定義 state index portfolio (由 portfolio of corporate securities 所構成)：每種 state 有不同之 payoff, 例

$$\begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{bmatrix}$$

Considering writing European call option with different strike price $k = 2, 4, 6, 8$

可以 complete market

$$k = 2 \Rightarrow \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

$$k = 4 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 6 \end{bmatrix}$$

$$k = 6 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

$$k = 8 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

\Rightarrow

$$[x_{j\omega}] = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 \\ 6 & 4 & 2 & 0 & 0 \\ 8 & 6 & 4 & 2 & 0 \\ 10 & 8 & 6 & 4 & 2 \end{bmatrix}^T$$

invertible, 可以 complete market

- 但上述理論假設了存在一個 state index portfolio, 因為 the realized payoff of the state index portfolio reveal the true state of nature, 所以其他 complex securities 之 payoff 都是此 state index portfolio payoff 之函數. 但在現實中未必存在此種 state index portfolio.
- 若 (i) homogeneous beliefs π_ω
 (ii) state-independent time-additive utility

$$u_{i\omega}(c_{i0}, c_{i\omega}) = u_{i0}(c_{i0}) + u_i(c_{i\omega})$$
 則未必要如同 5.6 - 5.8 節中所說的, 在 securities markets economy 中, 要 $N = |\Omega|$ 之 complete market, 或用 state index portfolio 與相對應的選擇權, 作出 complete market, 才能使 CE \Rightarrow PO 成立
- 原本要 $N = |\Omega|$ 之原因是希望 $[x_{j\omega}]$ is invertible, 使得 ϕ_ω 可算出, 而在 (i)(ii) 之假設下, 因為有 PO sharing rule (5.12-5.13 節), 只需知道 $\phi(k) = \phi(c_\omega = k)$ 即可求出 complex security 之價值, 亦即並不需從 complex security payoff 與其今日的 value 反推 ϕ_ω
- 不用 state index portfolio 之原因是因 (i)(ii) 可得 PO sharing rule 存在, 因此可用 aggregate consumption 取代此 state index portfolio, 而所有選擇權變成 options on aggregate consumption, 而 $\phi(k)$ 可由這些 options 算出

- homogeneous belief PO

$$\max_{c_{i0}, c_{i\omega}} L = \sum_{i=1}^I \lambda_i \left[\sum_{\omega \in \Omega} \pi_\omega (u_{i0}(c_{i0}) + u_i(c_{i\omega})) \right] \\ + \phi_0 [C_0 - \sum_{i0}^I c_{i0}] + \sum_{\omega \in \Omega} \phi_\omega [C_\omega - \sum_{i=1}^I c_{i\omega}]$$

$$(5.4.1) \text{ vs. } \frac{\partial}{\partial c_{i0}} \Rightarrow \lambda_i \sum_{\omega \in \Omega} \pi_{i\omega} \frac{\partial u_{i0}(c_{i0})}{\partial c_{i0}} = \lambda_i u'_{i0}(c_{i0}) \sum_{\omega \in \Omega} \pi_\omega \\ = \lambda_i u'_{i0}(c_{i0}) = \phi_0$$

$$(5.4.2) \text{ vs. } \frac{\partial}{\partial c_{i\omega}} \Rightarrow \lambda_i \pi_\omega u'_i(c_{i\omega}) = \phi_\omega \\ \Rightarrow \frac{\pi_\omega u'_i(c_{i\omega})}{u'_{i0}(c_{i0})} = \frac{\phi_\omega}{\phi_0}, \forall \omega (PO)$$

又由上兩式

$$\Rightarrow \begin{cases} \lambda_i u'_{i0}(c_{i0}) = \lambda_k u'_{k0}(c_{k0}) = \phi_0, \forall i, k, \\ \pi_\omega \lambda_i u'_i(c_{i\omega}) = \pi_\omega \lambda_k u'_k(c_{k\omega}) = \phi_\omega, \text{ 即 } \lambda_i u'_i(c_{i\omega}) = \lambda_k u'_k(c_{k\omega}), \forall i, k \end{cases}$$

- 此時 for any ω, ω' , if $C_\omega > C_{\omega'}$, then $c_{i\omega} > c_{i\omega'} \forall i$
(經濟好, 大家好; 經濟差, 大家差, 沒有例外, 像是彼此分享一樣)

if $C_\omega > C_{\omega'}$, assume $\exists c_{k\omega} < c_{k\omega'}$

$$\text{由 strictly concavity of } u_k \Rightarrow \lambda_i u'_i(c_{i\omega}) = \lambda_k u'_k(c_{k\omega}) > \lambda_k u'_k(c_{k\omega'}) \\ = \lambda_i u'_i(c_{i\omega'})$$

由 strictly concavity of $u_i \Rightarrow c_{i\omega} < c_{i\omega'}$

\Rightarrow 當 $C_\omega > C_{\omega'}$ 時, 只要有人之 $c_{k\omega} < c_{k\omega'}$,

則全部的人 $c_{i\omega} < c_{i\omega'}, \forall i \Rightarrow C_\omega < C_{\omega'} \Rightarrow$ 產生矛盾

(而且因為 $C_\omega > C_{\omega'}$, 一定會有 $\exists k$, 使得 $c_{k\omega} > c_{k\omega'}$, 不然不能會使 market clear, 當對 k 而言 $c_{k\omega} > c_{k\omega'}$, 表示對所有 i 而言, $c_{i\omega} > c_{i\omega'}$)

- 此時個人消費 $\tilde{c}_i(c_{i0})$ 與 aggregate consumption $\tilde{C}(C_0)$ 一定存在某種關係 $c_i = f_i(\tilde{C})$ ($c_{i0} = f_{i0}(C_0)$, 其中 f_{i0}, f_i 是 strictly increasing function, 意即 given \tilde{C} , 就可得到個人之 consumption \tilde{c}_i . f_{i0} 's and f_i 's describe the Pareto optimal allocation of time-0 and time-1 aggregate to different individuals, and they are called the Pareto optimal sharing rules.
- f_i, f_{i0} 只要是 strictly increasing 即可, 未必要是 linear, 但如果 f_i, f_{i0} 是 linear 且經濟體中有 riskless asset, 則不需要 complete 之假設, a CE in a securities markets economy is PO

- Suppose the PO sharing rule is linear:

$$\tilde{c}_i = f_i(\tilde{C}) = a_i + b_i \tilde{C}$$

要達成上述之消費計畫, 個人需買 a_i shares 之 riskless bond 與 b_i 單位之 market portfolio (因 market portfolio 之 payoff 為 \tilde{C}), 剩下就在 $t=0$ 消費 (之後會再加以說明)

- given $\lambda = [\lambda_i]_{I \times 1}$,

假設 PO linear sharing rules: $\tilde{c}_i(\lambda) = a_i(\lambda) + b_i(\lambda)\tilde{C}, \forall i$

由 PO + homogeneous belief $\Rightarrow \lambda_i u'_i(c_{i\omega}) = \lambda_k u'_k(c_{k\omega})$

Considering the linear sharing rule

$$\Rightarrow \lambda_i u'_i(a_i(\lambda) + b_i(\lambda)\tilde{C}) = \lambda_k u'_k(a_k(\lambda) + b_k(\lambda)\tilde{C}), \forall i, k$$

$$\frac{\partial}{\partial \tilde{C}} \Rightarrow \lambda_i u''_i(\tilde{c}_i(\lambda))b_i(\lambda) = \lambda_k u''_k(\tilde{c}_k(\lambda))b_k(\lambda), \forall i, k$$

$$\Rightarrow \lambda_k u''_k(\tilde{c}_k(\lambda)) = \frac{b_i(\lambda)}{b_k(\lambda)} \lambda_i u''_i(\tilde{c}_i(\lambda))$$

$$\frac{\partial}{\partial \lambda_i} \Rightarrow u'_i(\tilde{c}_i(\lambda)) + \lambda_i u''_i(\tilde{c}_i(\lambda))(a_{ii}(\lambda) + b_{ii}(\lambda)\tilde{C})$$

$$= \lambda_k u''_k(\tilde{c}_k(\lambda))(a_{ki}(\lambda) + b_{ki}(\lambda)\tilde{C}), \forall i, k$$

$$\Rightarrow u'_i(\tilde{c}_i(\lambda)) + \lambda_i u''_i(\tilde{c}_i(\lambda))(a_{ii}(\lambda) + b_{ii}(\lambda)\tilde{C})$$

$$= \frac{b_i(\lambda)}{b_k(\lambda)} \lambda_i u''_i(\tilde{c}_i(\lambda))(a_{ki}(\lambda) + b_{ki}(\lambda)\tilde{C}), \forall i, k \text{ (replace } \lambda_k u''_k(\tilde{c}_k(\lambda)))$$

等式兩邊取 $\sum_{k=1}^I$

$$\Rightarrow I[u'_i(\tilde{c}_i(\lambda)) + \lambda_i u''_i(\tilde{c}_i(\lambda))(a_{ii}(\lambda) + b_{ii}(\lambda)\tilde{C})]$$

$$= b_i(\lambda) \lambda_i u''_i(\tilde{c}_i(\lambda)) \sum_{k=1}^I \frac{a_{ki}(\lambda) + b_{ki}(\lambda)\tilde{C}}{b_k(\lambda)}, \forall i$$

$$\Rightarrow u'_i(\tilde{c}_i(\lambda)) = -\lambda_i u''_i(\tilde{c}_i(\lambda))(a_{ii}(\lambda) + b_{ii}(\lambda)\tilde{C})$$

$$+ \frac{1}{I} b_i(\lambda) \lambda_i u''_i(\tilde{c}_i(\lambda)) \sum_{k=1}^I \frac{a_{ki}(\lambda) + b_{ki}(\lambda)\tilde{C}}{b_k(\lambda)}, \forall i$$

$$\Rightarrow -\frac{u'_i(\tilde{c}_i(\lambda))}{u''_i(\tilde{c}_i(\lambda))} = \lambda_i(a_{ii}(\lambda) + b_{ii}(\lambda)\tilde{C})$$

$$- \frac{\lambda_i}{I} b_i(\lambda) \left[\sum_{k=1}^I \frac{a_{ki}(\lambda)}{b_k(\lambda)} + \sum_{k=1}^I \frac{b_{ki}(\lambda)\tilde{C}}{b_k(\lambda)} \right], \forall i$$

$$\therefore \text{linear sharing rule} \Rightarrow \tilde{C} = \frac{\tilde{c}_i(\lambda) - a_i(\lambda)}{b_i(\lambda)}$$

$$\begin{aligned}
\therefore -\frac{u'_i(\tilde{c}_i(\lambda))}{u''_i(\tilde{c}_i(\lambda))} &= \lambda_i a_{ii}(\lambda) + \frac{\tilde{c}_i(\lambda) - a_i(\lambda)}{b_i(\lambda)} \lambda_i b_{ii}(\lambda) - \frac{\lambda_i}{I} b_i(\lambda) \sum_{k=1}^I \frac{a_{ki}(\lambda)}{b_k(\lambda)} \\
&\quad - \frac{\lambda_i}{I} (\tilde{c}_i(\lambda) - a_i(\lambda)) \sum_{k=1}^I \frac{b_{ki}(\lambda)}{b_k(\lambda)}, \forall i \\
&= [\lambda_i a_{ii}(\lambda) - \frac{\lambda_i}{I} b_i(\lambda) \sum_{k=1}^I \frac{a_{ki}(\lambda)}{b_k(\lambda)} - \frac{a_i(\lambda)}{b_i(\lambda)} \lambda_i b_{ii}(\lambda) + \frac{\lambda_i}{I} a_i(\lambda) \sum_{k=1}^I \frac{b_{ki}(\lambda)}{b_k(\lambda)}] \\
&\quad + [\lambda_i \frac{b_{ii}(\lambda)}{b_i(\lambda)} - \frac{\lambda_i}{I} \sum_{k=1}^I \frac{b_{ki}(\lambda)}{b_k(\lambda)}] \tilde{c}_i(\lambda) \\
&= A_i + B_i \tilde{c}_i(\lambda), \forall i
\end{aligned}$$

(因上式對於任意 i 都成立, exercise 5.6 證明其實 $B_i = B$)

- Necessary and sufficient condition for Pareto optimal sharing rule to be linear: $-\frac{u'_i(z)}{u''_i(z)} = A_i + Bz$, for instance, u_i can be

$$u'_i(z) = \rho_i (A_i + Bz)^{-\frac{1}{B}}, \text{ when } B \neq 0$$

or

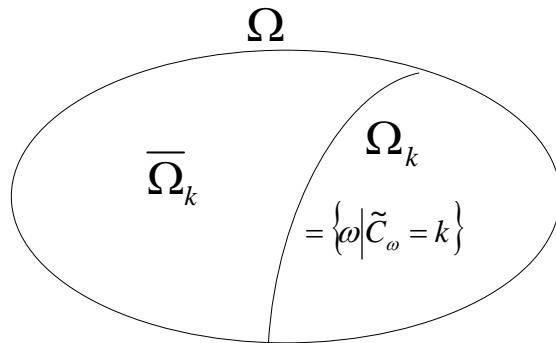
$$u'_i(z) = \rho_i \exp\left\{-\frac{z}{A_i}\right\}, \text{ when } B = 0$$

其中 ρ_i 可想成 subjective time preference parameter

- Necessary and sufficient condition 之左邊為 $\frac{1}{R_A}$, 稱 risk tolerance, $T_i(z)$, 比值越大, 代表個人越能容忍風險. $B_i = \frac{\partial T_i(z)}{\partial z} = \sigma_i(z)$, 稱 cautiousness, 若比值為 +, 則代表當消費水準越高, 越能容忍風險 (越不風險趨避)
- (5.14.7), (5.14.8) 與 (1.27.1), (1.27.2) 很像, 意即滿足 PO sharing rule 是 linear 時, two fund monetary separation 會成立, 意即對個人而言, 不同的 initial wealth (在此等同於不同 λ), 個人只會在 riskless asset 與 risky portfolio 中做不同比例投資, 若 $B_i = B$, 表示所有人的風險趨避係數一樣 則所有人的 risky portfolio 組成都會一樣, 當 market clear 時, 表示個人持有之 risky portfolio 必為 market portfolio (也呼應 (5.13.1) 式的含意, 參考講義上一頁最上方)

- 在 PO sharing rule 下, 當 \tilde{C} 一樣, PO 之均衡一樣, 即使不知所有之 state price ϕ_ω , 仍可得出 complex security 之 value
 - * define $\phi(k)$ be the price of an “elementary claim” on $t = 1$ consumption that pay one unit of consumption iff $\tilde{C} = k$, and let $\pi(k)$ be the prob ability that $\tilde{C} = k$, i.e.

$$\text{at } t = 1, \begin{cases} \omega \in \Omega_k \Rightarrow \text{payoff} = 1 \\ \omega \in \bar{\Omega}_k \Rightarrow \text{payoff} = 0 \end{cases}$$



此種 elementary claim 之 price at $t = 0$ 為 $\phi(k)$

由 (5.12.1) 與 (5.12.2) 之 PO 的 MRS $\Rightarrow \phi_\omega = \frac{\pi_\omega u'_i(c_{i\omega})}{u'_{i0}(c_{i0})}$

$$\begin{aligned} \phi(k) &= \sum_{\omega \in \Omega_k} \phi_\omega \\ &= \frac{u'_i(f_i(k))}{u'_{i0}(c_{i0})} \sum_{\omega \in \Omega_k} \pi_\omega \end{aligned}$$

(when $\omega \in \Omega_k, \tilde{C} = k, c_{i\omega} = f_i(\tilde{C}) = f_i(k)$)

$$= \frac{u'_i(f_i(k))}{u'_{i0}(c_{i0})} \pi(k)$$

- \forall complex security, 只要知道

(1) expected payoffs conditional on \tilde{C}

(2) the prices of the elementary claims on \tilde{C}

由 (1)(2) 即可知道這個 complex security 之價值

$$\begin{aligned} S_x &= \sum_{\omega \in \Omega} \phi_\omega x_\omega = \sum_k \sum_{\omega \in \Omega_k} \phi_\omega x_\omega \\ &= \sum_k \frac{u'_i(f_i(k))}{u'_{i0}(c_{i0})} \sum_{\omega \in \Omega_k} \pi_\omega x_\omega \\ &= \sum_k \phi(k) \sum_{\omega \in \Omega_k} \frac{\pi_\omega}{\pi(k)} x_\omega \\ &= \sum_k \phi(k) E[\tilde{x} | \tilde{C} = k] \end{aligned}$$

- Prices for the elementary claim $\phi(k)$ 可由 butterfly spread on the time-1 aggregate consumption 做出來

$(c(k))$: call option with exercise price of k units aggregate consumption)

$c(\tilde{C} = k)$	$c(0)$	$c(1)$	$c(2)$	$c(3)$	$c(4)$
$\tilde{C} = 1$	1	0	0	0	0
$\tilde{C} = 2$	2	1	0	0	0
$\tilde{C} = 3$	3	2	1	0	0

\Rightarrow

$$c(0) - c(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$c(1) - c(2) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$(c(0) - c(1)) - (c(1) - c(2)) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \phi(1)$$

同理可求出 $\phi(2)$ 與 $\phi(3)$

ex. $c(0) = 1.7, c(1) = 0.8, c(2) = 0.1, c(3) = 0, c(4) = 0$

$$\Rightarrow \phi(1) = 0.2, \phi(2) = 0.6, \phi(3) = 0.1$$

$\Rightarrow \phi(1) + \phi(2) + \phi(3) = 0.9 \leftarrow$ riskless asset 今日之價格

$$\Rightarrow r_f = \frac{1}{0.9} - 1 = 0.111$$

- 若 strike price 之間隔不是 1, 而是 Δ

$$\text{則 } \phi(k) = \frac{1}{\Delta}[(c(k - \Delta) - c(k)) - (c(k) - c(k + \Delta))]$$

$$\Rightarrow \frac{\phi(k)}{\Delta} = \frac{1}{\Delta^2}[(c(k + \Delta) - c(k)) - (c(k) - c(k - \Delta))]$$

$$\Rightarrow \frac{\phi(k)}{dk} = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta^2}[(c(k + \Delta) - c(k)) - (c(k) - c(k - \Delta))]$$

$$= \frac{\partial^2 c(k)}{\partial k^2}$$

$$\Rightarrow \phi(k) = \frac{\partial^2 c(k)}{\partial k^2} dk$$

此時 $\frac{\phi(k)}{dk} = \frac{\partial^2 c(k)}{\partial k^2}$ is interpreted as the “pricing density” for elementary claims on aggregate consumption. 若考慮一個 security, 其 payoff is one unit of consumption iff $\tilde{C} \in A$, 則此 security 之價格爲

$$\Rightarrow \int_A \frac{\phi(k)}{dk} dk = \int_A \frac{\partial^2 c(k)}{\partial k^2} dk$$

$$\Rightarrow \text{此時 (5.16.3) 變成 } S_x = \int \frac{\phi(k)}{dk} E[\tilde{x} | \tilde{C} = k] dk$$

$$= \int \frac{\partial^2 c(k)}{\partial k^2} E[\tilde{x} | \tilde{C} = k] dk$$

- 選擇權對 strike price 之二次微分可當作 underlying asset 之 (risk neutral) prob.density function 加上折現 (而此項其實就是所謂 pricing kernel, 在算任何 payoff 今日之價值時, 都要代在其中算)

$$C(S_0, K, T) = \int_0^\infty e^{-rT} \max\{S_T - K, 0\} q(S_T) dS_T$$

$$\frac{\partial C(S_0, K, T)}{\partial K} = \int_K^\infty -e^{-rT} q(S_T) dS_T$$

$$\frac{\partial^2 C(S_0, K, T)}{\partial K^2} = e^{-rT} q(S_T) \Big|_{S_T=K}$$

$$\Rightarrow q(S_T)|_{S_T=K} = \frac{\partial^2 C(S_0, K, T)}{\partial K^2} e^{rT}$$

$$\left\| \begin{array}{l} \frac{\partial C(S_0, K, T)}{\partial K} = -e^{-rT} N(d_2) \\ \frac{\partial^2 C(S_0, K, T)}{\partial K^2} = \frac{e^{-rT}}{K\sigma\sqrt{T}} n(d_2) \end{array} \right.$$

$= \frac{1}{K\sigma\sqrt{T}} n(d_2) = \frac{1}{\sqrt{2\pi}K\sigma\sqrt{T}} e^{-\frac{d_2^2}{2}}$ (可看成 $S_T = K$ 之 log-normal pdf), where $d_2 = \frac{\ln(S_0/K) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$ (剛好等於 $S_T = K$ 時之 $\varepsilon \sim N(0, 1)$ 之值)

因此任意 payoff 今日之價格 = $\int_0^\infty e^{-rT} \cdot \text{payoff} \cdot q(S_T) \cdot dS_T = \int_0^\infty \text{payoff} \cdot \frac{\partial^2 C}{\partial K^2} \cdot dS_T$

- 在兩期模型中, individuals will consume all their wealth at $t = 1 \Rightarrow t = 1$

時, aggregate wealth = aggregate consumption = aggregate endowment
所以可將 aggregate consumption 想成 market portfolio

$$\Rightarrow S_x = \sum_k \phi(k) E[\tilde{x} | \tilde{M} = k]$$

(只有在兩期模型且經濟體中只有一個 perishable consumption good 才成立)

- In a competitive economy, 若

$$\left\{ \begin{array}{l} \text{complete market} \\ \text{individuals have homogeneous beliefs} \\ \text{time-additive, state independent utility} \\ u' > 0, u'' < 0, \text{ 且可微} \end{array} \right.$$

\Rightarrow 考慮多個個人之 CE 均衡, 會與經濟體中只存在一個 representative agent 之均衡一樣, 亦即同樣之均衡價格 ϕ_w , 可由 representative agent model 求得 (證明如下)

- 個人 $\{c_{i0}, c_{i\omega}; \omega \in \Omega, i = 1, 2, \dots, I\}, \{u_{i0}, u_i\}$
 representative agent $\{C_{i0}, C_{i\omega}; \omega \in \Omega\}, \{u_0, u_1\}$
 定義 representative agent 之 utility function

$$u_0(z) = \max_{z_i} \sum_{i=1}^I \lambda_i u_{i0}(z_i)$$

$$\text{s.t. } \sum_{i=1}^I z_i = z = C_0$$

$$u_1(z) = \max_{z_i} \sum_{i=1}^I \lambda_i u_i(z_i)$$

$$\text{s.t. } \sum_{i=1}^I z_i = z = C_\omega$$

(C_w 在各 individuals 中分配, 使得以 λ_i 為 weighted 之 representative agent 之 utility 最大, 也就是整個社會 utility 最大)

根據上面 u_0 與 u_1 之定義

$$u'_0(C_0) = \sum_{i=1}^I \lambda_i u'_{i0}(c_{i0}^*) \frac{dc_{i0}^*}{dC_0} = \sum_{i=1}^I \frac{dc_{i0}^*}{dC_0} = 1$$

(where $\lambda_i u'_{i0}(c_{i0}^*) = \phi_0 = 1$, 參考 (5.12.1))

$$u'_1(C_\omega) = \sum_{i=1}^I \lambda_i u'_i(c_{i\omega}^*) \frac{dc_{i\omega}^*}{dC_\omega} = \frac{\phi_\omega}{\pi_\omega} \sum_{i=1}^I \frac{dc_{i\omega}^*}{dC_\omega} = \frac{\phi_\omega}{\pi_\omega}$$

(where $\lambda_i u'_i(c_{i\omega}^*) = \frac{\phi_\omega}{\pi_\omega}$, 參考 (5.12.2))

在之前已知當 individuals 之消費為 $c_{i0}^*, c_{i\omega}^*$ 時, 社會是 PO,
 即整個社會 utility 最大 \Rightarrow 在此情況下, 依照 $z_i = c_{i0}^*, z_i = c_{i\omega}^*$
 之分配, 會最大化 representative agent 之 utility

- For the representative agent

$$u = u_0(C_0) + \sum_{\omega} \pi_{\omega} u_1(C_{\omega})$$

其 MRS = $\frac{\frac{\partial u}{\partial C_{\omega}}}{\frac{\partial u}{\partial C_0}} = \frac{\pi_{\omega} u'_1(C_{\omega})}{u'_0(C_0)} = \frac{\pi_{\omega} \phi_{\omega}}{1} = \phi_{\omega}$, which is the same as those in

the previous CE Model. 不過因為 the utility of the representative agent

與 initial endowment distribution 有關. 故 ϕ_{ω} 與 λ_i 有關 (individual utility)

以 λ_i 為 weight 加總形成 representative agent 之 utility) 表示 ϕ_{ω} 還是與 initial endowment distribution 有關

- In a heterogeneous-agent economy, if the equilibrium prices are determined independently of the distribution of initial endowments

\Rightarrow aggregation property

- linear risk tolerance + identical cautiousness + identical time preference

\Rightarrow aggregation property

根據 (5.14.6), 要滿足 linear tolerance + identical cautiousness 有 $u_i(z) = \frac{1}{B-1}(A_i + Bz)^{1-\frac{1}{B}}$ 與 $u_i(z) = -A_i \exp\left\{-\frac{z}{A_i}\right\}$
(與 (5.14.7), (5.14.8) 或 (1.27.1), (1.27.2) 很像)

- (5.25節) 用 $u_i(z) = \frac{1}{B-1}(A_i + Bz)^{1-\frac{1}{B}}$, 證明 ϕ_ω 與一開始財富分配無關 (5.25.6)

$$\text{if } u_i(z_i) = \frac{1}{B-1}(A_i + Bz_i)^{1-\frac{1}{B}}$$

$$\text{For } \omega \in \Omega, u_1(z_\omega) = \max_{z_{i\omega}} \sum_{i=1}^I \lambda_i \rho_i u_i(z_{i\omega}), \text{s.t. } \sum_{i=1}^I z_{i\omega} = z_\omega$$

$$u_1(z_\omega) = ?(\text{求完 } u_0(z_0) \text{ 與 } u_1(z_\omega), \text{ 才能看 } \phi_\omega = \frac{\pi_\omega u'_1(l_\omega)}{u'_0(l_0)})$$

$$\max_{z_{i\omega}} L = \sum_{i=1}^I \lambda_i \rho_i u_i(z_{i\omega}) + \theta [z_\omega - \sum_{i=1}^I z_{i\omega}]$$

$$\frac{\partial L}{\partial z_{i\omega}} \Rightarrow \rho_i (A_i + Bz_{i\omega}^*)^{-\frac{1}{B}} = \frac{\theta}{\lambda_i}, i = 1, 2, \dots, I \quad (1)$$

$$\frac{\partial L}{\partial \theta} \Rightarrow \sum_{i=1}^I z_{i\omega}^* = z_\omega$$

$$\text{將 (1) 式的 } -B \text{ 次方, } \sum_{i=1}^I \text{ 起來} \Rightarrow \sum_{i=1}^I \rho_i^{-B} (A_i + Bz_{i\omega}^*) = \theta^{-B} \sum_{i=1}^I \lambda_i^B$$

$$\stackrel{\rho_i = \rho}{=} \rho^{-B} (\sum_{i=1}^I A_i + Bz_\omega) = \theta^{-B} \sum_{i=1}^I \lambda_i^B$$

$$\Rightarrow \theta = \rho (\sum_{i=1}^I A_i + Bz_\omega)^{-\frac{1}{B}} (\sum_{i=1}^I \lambda_i^B)^{\frac{1}{B}}$$

$$\text{由 (1)} \Rightarrow \lambda_i \rho u_i(z_{i\omega}^*) = \frac{\lambda_i \rho}{B-1} (A_i + Bz_{i\omega}^*)^{1-\frac{1}{B}} = \frac{\theta}{B-1} (A_i + Bz_{i\omega}^*)$$

$$\stackrel{\theta \text{ 代入}}{=} \frac{1}{B-1} \rho (\sum_{i=1}^I A_i + Bz_\omega)^{-\frac{1}{B}} (\sum_{i=1}^I \lambda_i^B)^{\frac{1}{B}} (A_i + Bz_{i\omega}^*)$$

$$\Rightarrow u_1(z_\omega) = \sum_{i=1}^I \lambda_i \rho_i u_i(z_{i\omega}^*)$$

$$= \sum_{i=1}^I \frac{1}{B-1} \rho (\sum_{i=1}^I A_i + Bz_\omega)^{-\frac{1}{B}} (\sum_{i=1}^I \lambda_i^B)^{\frac{1}{B}} (A_i + Bz_{i\omega}^*)$$

$$= \frac{1}{B-1} \rho (\sum_{i=1}^I A_i + Bz_\omega)^{1-\frac{1}{B}} (\sum_{i=1}^I \lambda_i^B)^{\frac{1}{B}}$$

$$\Rightarrow \phi_\omega = \frac{\pi_\omega u'_1(C_\omega)}{u'_0(C_0)} = \frac{\rho \pi_\omega (\sum_{i=1}^I A_i + BC_\omega)^{-\frac{1}{B}}}{(\sum_{i=1}^I A_i + BC_0)^{-\frac{1}{B}}}, \forall \omega \in \Omega$$

- (5.26節) 用 $u_i(z) = -A_i \exp\left\{-\frac{z}{A_i}\right\}$, 允許不同之 $\pi_{i\omega}$ (heterogeneous belief) 與 ρ_i (different time preference), 還是可得出 ϕ_ω 與一開始財富分配無關, 結果為 (5.26.10) 式

$$u_i^i(z_i) = -A_i \exp\left(-\frac{z_i}{A_i}\right)$$

For $\omega \in \Omega$, $u_1(z_\omega) = \max_{z_{i\omega}} \sum_{i=1}^I \lambda_i \rho_i u_i(z_{i\omega})$, s.t. $\sum_{i=1}^I z_{i\omega} = z_\omega$

$$u_1(z_\omega) = ?$$

$$\max_{z_{i\omega}} L = \sum_{i=1}^I \lambda_i \rho_i u_i(z_{i\omega}) + \theta[z_\omega - \sum_{i=1}^I z_{i\omega}]$$

$$\frac{\partial L}{\partial z_{i\omega}} \Rightarrow \lambda_i \rho_i \exp\left(-\frac{z_{i\omega}^*}{A_i}\right) = \theta, i = 1, 2, \dots, I \quad (1)$$

$$\frac{\partial}{\partial \theta} \Rightarrow \sum_{i=1}^I z_{i\omega}^* = z_\omega$$

$$(1) \text{ 式取 } A_i \text{ 次方} \Rightarrow (\lambda_i \rho_i)^{A_i} \exp(-z_{i\omega}^*) = \theta^{A_i}$$

$$\Rightarrow \prod_{i=1}^I (\lambda_i \rho_i)^{A_i} \exp \Rightarrow \theta = \prod_{i=1}^I (\lambda_i \rho_i)^{\frac{A_i}{\sum_{i=1}^I A_i}} \exp\left(-\frac{z_\omega}{\sum_{i=1}^I A_i}\right)$$

$$u_1(z_\omega) = \sum_{i=1}^I \lambda_i \rho_i u_i(z_{i\omega}^*)$$

$$= \sum_{i=1}^I \lambda_i \rho_i (-A_i) \exp\left(-\frac{z_{i\omega}^*}{A_i}\right)$$

$$= \sum_{i=1}^I (-A_i) \theta$$

$$= \sum_{i=1}^I (-A_i) \prod_{i=1}^I (\lambda_i \rho_i)^{\frac{A_i}{\sum_{i=1}^I A_i}} \exp\left(-\frac{z_\omega}{\sum_{i=1}^I A_i}\right)$$

$$\Rightarrow \phi_\omega = \frac{\pi_\omega u'_1(C_\omega)}{u'_0(C_0)} = \frac{\pi_\omega \sum_{i=1}^I (-A_i) \prod_{i=1}^I (\lambda_i \rho_i)^{\frac{A_i}{\sum_{i=1}^I A_i}} \exp\left(-\frac{C_\omega}{\sum_{i=1}^I A_i}\right) \cdot \left(-\frac{1}{\sum_{i=1}^I A_i}\right)}{\sum_{i=1}^I (-A_i) \prod_{i=1}^I (\lambda_i)^{\frac{A_i}{\sum_{i=1}^I A_i}} \exp\left(-\frac{C_0}{\sum_{i=1}^I A_i}\right) \left(-\frac{1}{\sum_{i=1}^I A_i}\right)}$$

$$= \frac{\pi_\omega \prod_{i=1}^I (\rho_i)^{\frac{A_i}{\sum_{i=1}^I A_i}} \exp\left(-\frac{C_\omega}{\sum_{i=1}^I A_i}\right)}{\exp\left(-\frac{C_0}{\sum_{i=1}^I A_i}\right)}$$

||課本中考慮 $\pi_{i\omega}$ 之情況, 而這邊是考慮 π_ω 之情況