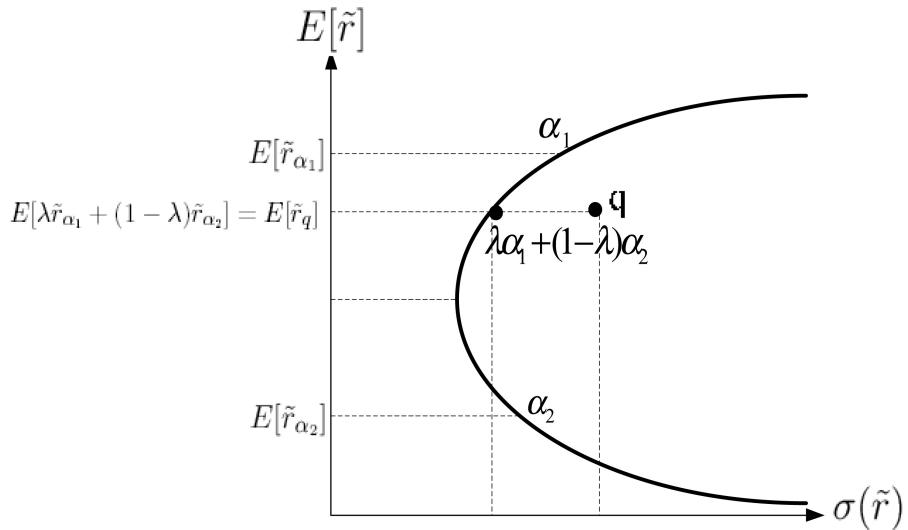


Ch4. Two Fund Separation and CAPM

- Two Fund Separation 與 portfolio frontier
 - (i) Any portfolio on the portfolio frontier can be generated by a linear combination of two frontier portfolios or mutual funds.
 - (ii) Two (mutual) fund separation:
given any feasible portfolio, there exists a portfolio of two mutual funds such that individuals prefer at least as much as the original portfolio.
由 (i), (ii) 加上 u is concave
 1. 由 SSD 可以推得, 當有 two fund separation, 此 two funds 都會在 portfolio frontier 上.
 2. 在均衡時, market should be clear, 加上 market protfolio 為個人的 optimal portfolio 之 convex combination, 因此 market portfolio 也會在 portfolio frontier, 此時 r_m 可當成 1 個 fund, 若有 riskless asset, r_f 則當另一個 (monetary) fund. \Rightarrow CAPM

- A vector of risky asset returns $\tilde{r} = (\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_N)$ exhibits “Two Fund separation” if there exists two mutual funds (or portfolio α_1, α_2), then for \forall portfolio q , there exists a scalar λ , s.t
$$E[u(\lambda\tilde{r}_{\alpha_1} + (1 - \lambda)\tilde{r}_{\alpha_2})] \geq E[u(\tilde{r}_q)], \text{ where } u \text{ is concave}$$
$$\Leftrightarrow \lambda\alpha_1 + (1 - \lambda)\alpha_2 \stackrel{\text{SSD}}{\geq} q$$
$$\Rightarrow E[\lambda\tilde{r}_{\alpha_1} + (1 - \lambda)\tilde{r}_{\alpha_2}] = E[\tilde{r}_q]$$
$$\text{and } \text{Var}(\lambda\tilde{r}_{\alpha_1} + (1 - \lambda)\tilde{r}_{\alpha_2}) \leq \text{Var}(\tilde{r}_q)$$
$$\Rightarrow \alpha_1, \alpha_2 \text{ 一定要在 portfolio frontier 上.}$$
(參考下圖, 圖中的 portfolio frontier 為 N risky assets 的 portfolio frontier)



- 取 $\alpha_1 = p$ ($\neq mvp$), $\alpha_2 = zc(p)$, $\beta_{qp} = \frac{\text{Cov}(\tilde{r}_q, \tilde{r}_p)}{\sigma^2(\tilde{r}_p)}$ 來作 λ ,

根據 (3.17) 節

$$\begin{aligned}\Rightarrow \tilde{r}_q &= (1 - \beta_{qp})\tilde{r}_{zc(p)} + \beta_{qp}\tilde{r}_p + \tilde{\varepsilon}_{qp}, \text{ where } E[\tilde{\varepsilon}_{qp}] = 0 \\ &= \tilde{Q}(\beta_{qp}) + \tilde{\varepsilon}_{qp}\end{aligned}$$

其中 $\tilde{Q}(\beta_{qp})$ is the dominating portfolio

$$\text{Two Fund Separation} \Leftrightarrow E[u(\tilde{Q}(\beta_{qp}))] \geq E[u(\tilde{r}_q)]$$

- 要證明 Two fund separation $\iff E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] = 0, \forall q$

(其實就是要證 $SSD \geq$)

(\Leftarrow)

$$\begin{aligned} E[u(\tilde{r}_q)] &= E[u(\tilde{Q}(\beta_{qp}) + \tilde{\varepsilon}_{qp})] \\ &= E[E[u(\tilde{Q}(\beta_{qp}) + \tilde{\varepsilon}_{qp})|\tilde{Q}(\beta_{qp})]] \end{aligned}$$

$$\begin{cases} \text{if } u \text{ is concave} \\ \text{根據 Jensen's inequality} \\ \Rightarrow E[u(\tilde{z})] \leq u(E[\tilde{z}]) \end{cases}$$

$$\leq E[u(E[\tilde{Q}(\beta_{qp}) + \tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})])](\text{if } E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] = 0)$$

$$= E[u(\tilde{Q}(\beta_{qp}))]$$

(\Rightarrow)

$$\text{若 two fund separation 成立} \Rightarrow E[u(\tilde{Q}(\beta_{qp}))] \geq E[u(\tilde{r}_q)]$$

$$\Rightarrow \max_a E[u(a\tilde{r}_q + (1-a)\tilde{Q}(\beta_{qp}))] \text{ 之解應為 } a = 0$$

亦即 FOC = 0 when $a = 0$

$$\Rightarrow E[u'(a\tilde{r}_q + (1-a)\tilde{Q}(\beta_{qp}))(\tilde{r}_q - \tilde{Q}(\beta_{qp}))]$$

$$\stackrel{a=0}{=} E[u'(\tilde{Q}(\beta_{qp}))\tilde{\varepsilon}_{qp}] = 0$$

\Downarrow 想證明

$$\forall q, E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] = 0$$

(利用反證法, 假設 $\exists q, E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] \neq 0$, 若如此會造成 $E[u'(\tilde{Q}(\beta_{qp}))\tilde{\varepsilon}_{qp}] \neq 0$, 即得証)

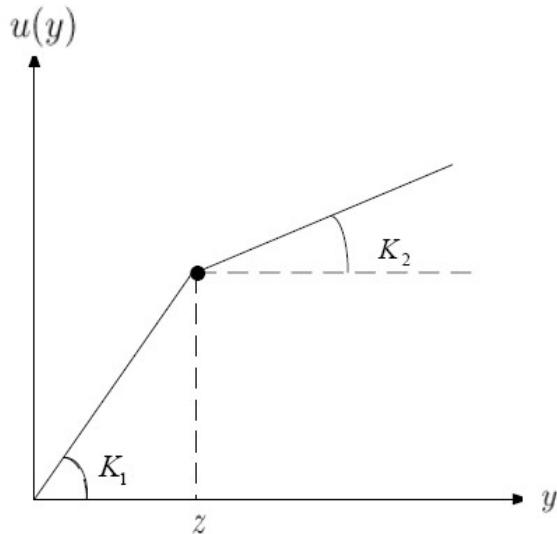
- 由 $E[\tilde{\varepsilon}_{qp}] = 0$ 與 $\exists q, \text{s.t } E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] \neq 0$

\Rightarrow For $\forall z$,

$$\begin{aligned} E[\tilde{\varepsilon}_{qp}] &= E[E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})]] \\ &= \int_{-\infty}^z E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] dF(\tilde{Q}) + \int_z^\infty E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] dF(\tilde{Q}) = 0 \\ \Rightarrow \int_{-\infty}^z E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] dF(\tilde{Q}) &= - \int_z^\infty E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] dF(\tilde{Q}) \neq 0 \\ (\text{因 } E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] &\neq 0) \end{aligned}$$

assume $u(y) = \begin{cases} K_1 y & \text{if } y < z \\ K_1 z + K_2(y - z) & \text{if } y \geq z \end{cases}$ (參考下圖)

(由圖中可看出 u 滿足 concave 的條件)



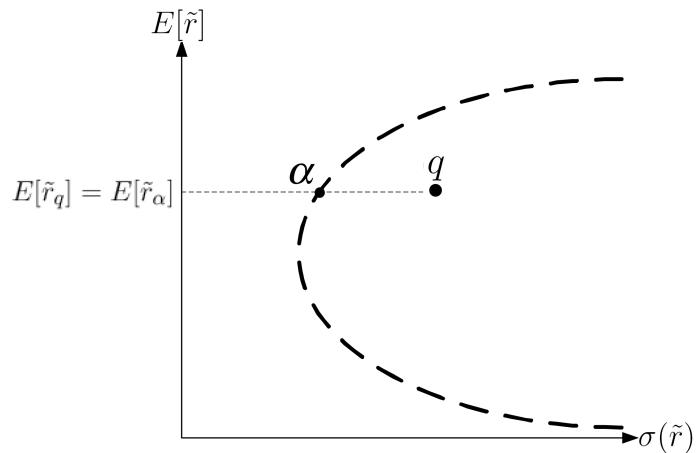
$$\begin{aligned} E[u'(\tilde{Q}(\beta_{qp}))\tilde{\varepsilon}_{qp}] &= E[E[u'(\tilde{Q}(\beta_{qp}))\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})]] \\ &= E[u'(\tilde{Q}(\beta_{qp}))E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})]] \\ &= K_1 \int_{-\infty}^z E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] dF(\tilde{Q}) \\ &\quad + K_2 \int_z^\infty E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] dF(\tilde{Q}) \\ &= (K_1 - K_2) \int_{-\infty}^z E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] dF(q) \neq 0 \end{aligned}$$

因 $E[\tilde{\varepsilon}_{qp}|\tilde{Q}(\beta_{qp})] \neq 0$, 造成 $\longrightarrow \longleftarrow$, 故得證

- One fund separation, 亦即存在 portfolio α 使得 $E[u(\tilde{r}_\alpha)] \geq E[u(\tilde{r}_q)]$ for all q , where u is concave.

\Updownarrow

For any q , $\tilde{r}_q = \tilde{r}_\alpha + \tilde{\varepsilon}_q$, 且 $E[\tilde{\varepsilon}_q | \tilde{r}_\alpha] = 0$ (亦即 $E[\tilde{r}_\alpha] = E[\tilde{r}_q]$, $\text{Var}(\tilde{r}_\alpha) \leq \text{Var}(\tilde{r}_q)$) 所有 return 都一樣, 且 α 為 mvp (且 portfolio frontier degenerates to this point), 才有可能有 one fund separation.



- If $r_i \sim N((), [])$, and nonidentical expectations

\Rightarrow two fund separation 成立

$$\tilde{r}_q = (1 - \beta_{qp})\tilde{r}_{zc(p)} + \beta_{qp}\tilde{r}_p + \tilde{\varepsilon}_{qp},$$

where $\tilde{r}_{zc(p)}$, \tilde{r}_p , and $\tilde{\varepsilon}_{qp}$ are uncorrelated.

但因 multinormal 之假設

\Rightarrow 變成 independent

$$\Rightarrow E[\tilde{\varepsilon}_{qp} | \tilde{Q}(\beta_{qp})] = E[\tilde{\varepsilon}_{qp}] = 0, \text{ 故得證}$$

- If $r_i \sim N((\), [\])$ and identical expectations,
 \Rightarrow one fund separation

$$\tilde{r}_q = \tilde{r}_{mvp} + \tilde{\varepsilon}_q$$

where \tilde{r}_{mvp} and $\tilde{\varepsilon}_q$ are uncorrelated.

$$\begin{aligned} & \left| \begin{aligned} & \text{Cov}(\tilde{r}_q, \tilde{r}_{mvp}) \\ & = \text{Cov}(\tilde{r}_{mvp}, \tilde{r}_{mvp}) + \text{Cov}(\tilde{\varepsilon}_q, \tilde{r}_{mvp}) \\ & \text{由3.12} \\ & = \text{Var}(\tilde{r}_{mvp}) \\ & \Rightarrow \text{Cov}(\tilde{\varepsilon}_q, \tilde{r}_{mvp}) = 0 \end{aligned} \right| \end{aligned}$$

因 multinormal 之假設

\Rightarrow 变成 independent

$$\Rightarrow E[\tilde{\varepsilon}_q | \tilde{r}_{mvp}] = E[\tilde{\varepsilon}_q] = 0$$

\Rightarrow one fund separation.

- (4.8 和 4.9 節) Two fund separation + market clearing

\Rightarrow market portfolio is on the portfolio frontier

證明如下：

(1) $i = 1, 2, \dots, I$ 個 individuals

(i) 每人有 W_0^i wealth (由 N risky assets 組成每人之 wealth)

$$\sum_{i=1}^I W_0^i = W_{m0} = \text{the total wealth in the economy}$$

= the value of all securities market portfolio

(ii) w_{ij} : i 投資其 wealth 在 security j 之比重

for any security j in the market

$$\sum_{i=1}^I w_{ij} W_0^i = w_{mj} W_{m0}$$

$$\Rightarrow \sum_{i=1}^I w_{ij} \frac{W_0^i}{W_{m0}} = w_{mj}$$

(where w_{mj} : j asset 佔市場之 weight)

\Rightarrow the market portfolio weights are a convex combination of the portfolio weights of individuals.

(2) 當 two fund separation 成立, 因此 two funds 必在 portfolio frontier 上, 個人所 prefer 的 portfolio 也是 two funds 之組合. (需要 two fund separation 成立, 應是因為要說明對任何 u 為 concave 之個人, 其 prefer 或者是說持有的 portfolio, 一定在 portfolio frontier 上面)

(3) 再加上 frontier portfolio 之組合, 亦在 portfolio frontier 上, 亦即 two fund 之組合會在 portfolio frontier 上, 因此可以推得個人 prefer 之 portfolio 也在 portfolio frontier 上.

(4) 而 market portfolio 是個人 portfolio 之組合 ((1) 中所證)

\Rightarrow market portfolio 在 portfolio frontier 上

- 因 \tilde{r}_m 在 portfolio frontier 上 \Rightarrow 用 \tilde{r}_m 與 $\tilde{r}_{zc(m)}$ 組合出所有之 \tilde{r}_q , 其中 q 為任意 portfolio.

$$E[\tilde{r}_q] = (1 - \beta_{qm})E[\tilde{r}_{zc(m)}] + \beta_{qm}E[\tilde{r}_m]$$

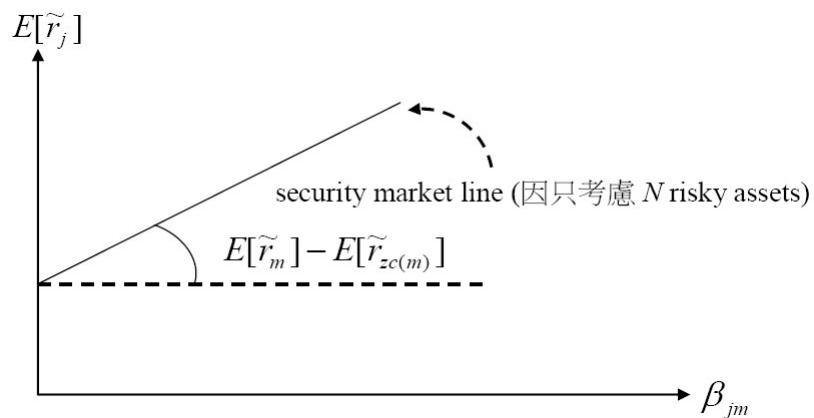
for any security j (個股或是各資產)

$$E[\tilde{r}_j] = (1 - \beta_{jm})E[\tilde{r}_{zc(m)}] + \beta_{jm}E[\tilde{r}_m]$$

(q and j 未必在 frontier 上)

上述式子表示, in equilibrium, there is a linear restriction on the expected rate of return of any risky asset

亦可寫成 $E[\tilde{r}_j] = E[\tilde{r}_{zc(m)}] + \beta_{jm}(E[\tilde{r}_m] - E[\tilde{r}_{zc(m)}])$ (圖)



在 $E[\tilde{r}_j]$ 與 β_{jm} 平面上形成 security market line (因只考慮 N risky assets)

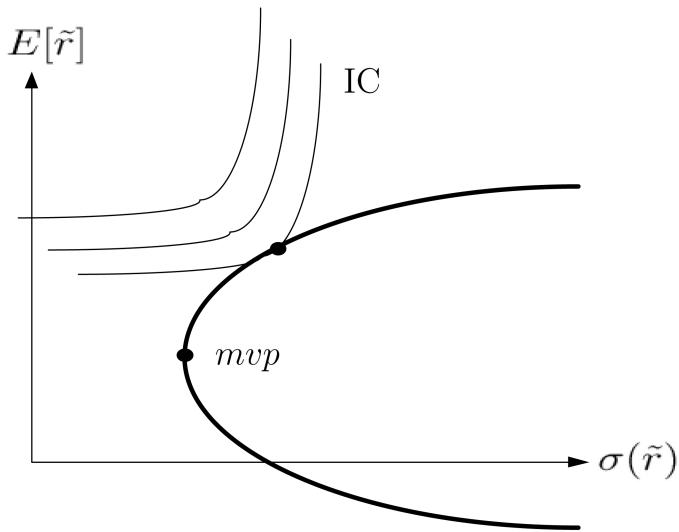
若 $E[\tilde{r}_m] < E[\tilde{r}_{zc(m)}]$ (此時 market portfolio 非 efficient portfolio)

$\Rightarrow E[\tilde{r}_j] = E[\tilde{r}_m] + \beta_{jm}(E[\tilde{r}_{zc(m)}] - E[\tilde{r}_m])$

- (4.11節) 要證明 market portfolio is an efficient portfolio under a special case
assume (i) $r_i \sim N(\mu, \sigma^2)$ (使成 mean-variance model)
(ii) u is increasing and strictly concave

$\left\{ \begin{array}{l} \text{individual prefers higher expected rate of return} \\ \text{individual prefers a portfolio with lower s.d.} \end{array} \right.$

\Rightarrow IC 在 $E[\tilde{r}] - \sigma(\tilde{r})$ 平面上為正斜率 (p.94~95 為證明 IC 斜率為正)(圖)



$$\because \tilde{r}_j \sim N(\mu_j, \sigma_j^2) \Rightarrow \tilde{r}_p \sim N(E[\tilde{r}_p], \sigma^2(\tilde{r}_p))$$

$$\therefore E[u_i(W_0^i(1 + \tilde{r}_p))] = E[u_i(W_0^i(1 + E[\tilde{r}_p] + \sigma(\tilde{r}_p)\tilde{z}))],$$

where $\tilde{z} \sim N(0, 1)$

$$\text{Define } V_i(\tilde{r}_p, \sigma(\tilde{r}_p)) = E[u_i(W_0^i(1 + E[\tilde{r}_p] + \sigma(\tilde{r}_p)\tilde{z}))]$$

想求 indifference curve 之斜率

$$\therefore \frac{\partial V_i}{\partial E[\tilde{r}_p]} dE[\tilde{r}_p] + \frac{\partial V_i}{\partial \sigma(\tilde{r}_p)} d\sigma(\tilde{r}_p) = 0$$

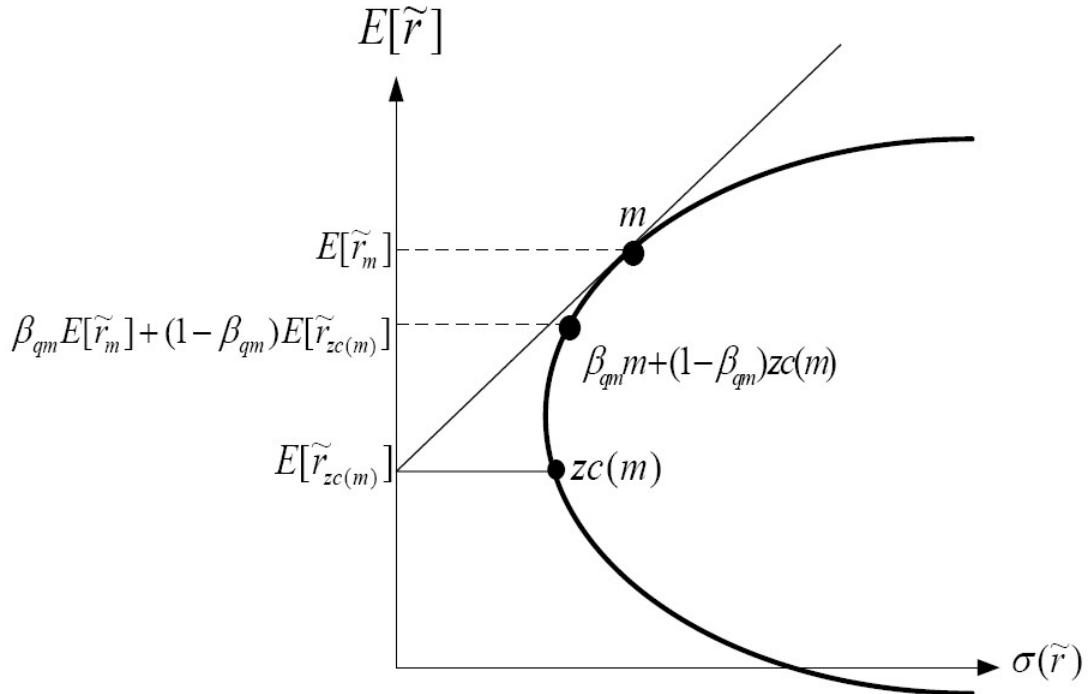
$$\therefore \text{IC 之斜率} = \frac{dE[\tilde{r}_p]}{d\sigma(\tilde{r}_p)} = -\frac{\frac{\partial V_i}{\partial \sigma(\tilde{r}_p)}}{\frac{\partial V_i}{\partial E[\tilde{r}_p]}} > 0$$

$$\left\| \begin{array}{l} \frac{\partial V_i}{\partial \sigma(\tilde{r}_p)} = E[u'_i(\tilde{W}^i)\tilde{z}W_0^i] = W_0^i \operatorname{Cov}(u'_i(\tilde{W}^i), \tilde{z}) < 0 \\ \frac{\partial V_i}{\partial E[\tilde{r}_p]} = E[u'_i(\tilde{W}^i)W_0^i] > 0 \\ \text{其中 } \tilde{W}^i = W_0^i(1 + E[\tilde{r}_p] + \sigma(\tilde{r}_p)\tilde{z}) \end{array} \right.$$

由 IC 之正斜率 \Rightarrow 所有人都選 efficient portfolio, 而 market portfolio 又是個人 portfolio 之 convex 組合 \Rightarrow market portfolio 是 efficient portfolio

- Zero-Beta Capital Asset Pricing Model (N risky assets)

$$E[\tilde{r}_q] = E[\tilde{r}_{zc(m)}] + \beta_{qm}(E[\tilde{r}_m] - E[\tilde{r}_{zc(m)}])$$
 (where $(E[\tilde{r}_m] - E[\tilde{r}_{zc(m)}]) > 0$)



- 若 \exists a riskless asset (以下均考慮此狀況)

if $r_f < \frac{A}{C}$, 取切點 portfolio e 與 $zc(e) = r_f$ 當 α_1, α_2

(此時, e 與 r_f 均在 portfolio frontier 上) (注意此時並未說 e 是 market portfolio)

$$\tilde{r}_q = (1 - \beta_{qe})r_f + \beta_{qe}\tilde{r}_e + \tilde{\varepsilon}_{qe} \quad (3.19.2\text{式})$$

$$(\text{where } (1 - \beta_{qe})r_f + \beta_{qe}\tilde{r}_e = \tilde{Q}(\beta_{qe}))$$

the necessary and sufficient condition for two fund separation is:

$$E[\tilde{\varepsilon}_{qe}|\tilde{Q}(\beta_{qe})] \stackrel{\text{因 } r_f \text{ 非隨機}}{=} E[\tilde{\varepsilon}_{qe}|\tilde{r}_e] = 0.$$

(亦即 $E[\tilde{\varepsilon}_{qe}|\tilde{r}_e] = 0$ 為 $\{(\tilde{r}_j)_{j=1}^N, r_f\}$ exhibit two fund separation 之 necessary and sufficient condition)

$If r_f > \frac{A}{C}$, 取切點 portfolio e' 與 $zc(e') = r_f$ 當 α_1, α_2 $\tilde{r}_q = (1 - \beta_{qe'})r_f + \beta_{qe'}\tilde{r}_{e'} + \tilde{\varepsilon}_{qe'}$ (where $(1 - \beta_{qe'})r_f + \beta_{qe'}\tilde{r}_{e'} = \tilde{Q}(\beta_{qe'})$) necessary and sufficient condition for two fund separation $E[\tilde{\varepsilon}_{qe'} \tilde{Q}(\beta_{qe'})] = E[\tilde{\varepsilon}_{qe'} \tilde{r}_{e'}] = 0$
--

證明之方式與之前類似, 請參考 (4.4) 節

- $r_f + \text{two fund separation} = \text{two fund monetary separation}$
- Two fund monetary separation + $(r_f < \frac{A}{C})$ + risky assets are in strictly positive supply (risky assets 之 supply 為正, 表若要 market clearing, 其 demand 亦要為正, 表示大家都會投資 risky asset, 亦即不可全部去投資 r_f) + 所有人之 efficient portfolio 之 weighted sum, 也就是 market portfolio, 也會在 N risky assets and 1 riskless asset 之 efficient portfolio 上 \Rightarrow 切點 e , 剛好是 market portfolio

$$\Rightarrow E[\tilde{r}_q] - r_f = \beta_{qm}(E[\tilde{r}_m] - r_f)$$

(Capital Asset Pricing Model: CAPM)(independently derived by Lintner (1965), Mossin (1965), and Sharpe (1964))

說明: 在 1 riskless asset 與 N risky assets 之情況下, 所有人都投資部分 r_f , 部分 e , 且所有之投資組合加總, 也在 1 riskless asset 與 N risky assets 之 portfolio frontier 上, 又因 market clear, 所有人之 e 之加總, 恰為 market portfolio (market

portfolio 是市場上所有 securities 所形成之 portfolio, 所以一定也在只有 N risky assets 所形成之 portfolio frontier 上, e 點恰為此時, 考慮 1 riskless asset 與 N risky assets 的 portfolio frontier 與 all risky assets 之 portfolio frontier 之唯一交點, 所以 e 點必為 market portfolio)

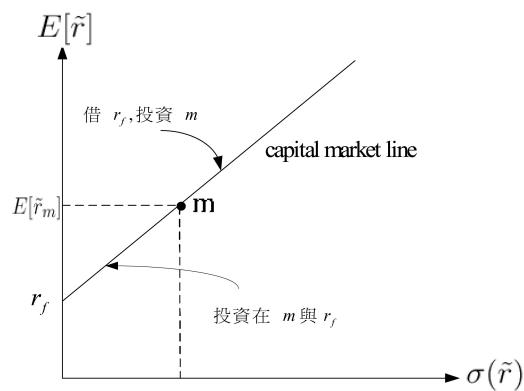
- 當 $r_f = \frac{A}{C}$, 由 section 3.18 可知 an individual puts all his wealth into the riskless asset and holds a zero-weighted-sum portfolio \Rightarrow riskless asset be in strictly positive supply, risky assets be in zero net supply.

但其實在 market clearing 之情況下, risky assets 之 demand 與 supply 需均為正 (表大家都會有部分資產投資 risky assets)

- 若 $r_f > \frac{A}{C}$, 則風險趨避投資人更是不可能投資期望報酬率小於 r_f 的資產
if $E[\tilde{r}] < r_f$
 $\Rightarrow E[u(W_0(1 + \tilde{r}))] \leq u(E[W_0(1 + \tilde{r})]) < u(W_0(1 + r_f))$
所有人投資 r_f , 使得 market 不能 clear
 \Rightarrow 不是均衡

- 由上述可知, 唯有 $r_f < \frac{A}{C}$, 才是 equilibrium, 且此時 risk premium of the market portfolio is positive (亦即 $E[\tilde{r}_m] > r_f$)

- Capital Market Line



- If $\tilde{r}_j \sim N((), [])$, 由另一角度, 看 CAPM

$$E[u'_i(\tilde{W}_i)(\tilde{r}_j - r_f)] = 0, \forall i, j$$

其中 $\tilde{W}_i = W_0^i(1 + r_f + \sum_{j=1}^N w_{ij}(\tilde{r}_j - r_f))$

$$\Rightarrow E[u'_i(\tilde{W}_i)]E[(\tilde{r}_j - r_f)] = -\text{Cov}(u'_i(\tilde{W}_i), \tilde{r}_j)$$

|| Stein's lemma
 || if \tilde{X} and \tilde{Y} are bivariate normally distributed
 || $\text{Cov}(g(\tilde{X}), \tilde{Y}) = E[g'(\tilde{X})]\text{Cov}(\tilde{X}, \tilde{Y})$ (很像計量的 delta method)

$$\Rightarrow E[u'_i(\tilde{W}_i)]E[(\tilde{r}_j - r_f)] = -E[u''_i(\tilde{W}_i)]\text{Cov}(\tilde{W}_i, \tilde{r}_j)$$

|| define i 之global ARA
 || $\theta_i \equiv -\frac{E[u''_i(\tilde{W}_i)]}{E[u'_i(\tilde{W}_i)]}$ (一般之ARA 沒加 $E[]$)

$$\Rightarrow \frac{1}{\theta_i} E[\tilde{r}_j - r_f] = \text{Cov}(\tilde{W}_i, \tilde{r}_j)$$

$$\Rightarrow (\sum_i \theta_i^{-1}) E[\tilde{r}_j - r_f] = \sum_i \text{Cov}(\tilde{W}_i, \tilde{r}_j) = \text{Cov}(\sum_i \tilde{W}_i, \tilde{r}_j) = \text{Cov}(\tilde{M}, \tilde{r}_j)$$

$$(\text{where } \tilde{M} = \sum_i \tilde{W}_i = W_{m0}(1 + \tilde{r}_m))$$

$$\Rightarrow E[\tilde{r}_j - r_f] = W_{m0}(\sum_i \theta_i^{-1})^{-1} \text{Cov}(\tilde{r}_m, \tilde{r}_j)$$

$$(\text{where } W_{m0}(\sum_i \theta_i^{-1})^{-1}: \text{aggregate RRA})$$

If $r_j = r_m$

$$\Rightarrow E[\tilde{r}_m - r_f] = W_{m0}(\sum_i \theta_i^{-1})^{-1} \text{Var}(\tilde{r}_m)$$

$$W_{m0}(\sum_i \theta_i^{-1})^{-1} = \frac{E[\tilde{r}_m - r_f]}{\text{Var}(\tilde{r}_m)} \text{ 代回原式}$$

$$\Rightarrow E[\tilde{r}_j - r_f] = \frac{E[\tilde{r}_m - r_f]}{\text{Var}(\tilde{r}_m)} \text{Cov}(\tilde{r}_m, \tilde{r}_j) = \beta_{jm} E[\tilde{r}_m - r_f]$$

(CAPM)

- Considering the quadratic utility, 用另一個角度看 CAPM. (4.16 節)

$$u_i(z) = a_i z - \frac{b_i}{2} z^2$$

$$\Rightarrow u'_i(z) = a_i - b_i z$$

$$\text{代入 } E[u'_i(\tilde{W}_i)] E[\tilde{r}_j - r_f] = -\text{Cov}(u'_i(\tilde{W}_i), \tilde{r}_j)$$

$$\Rightarrow E[a_i - b_i \tilde{W}_i] E[\tilde{r}_j - r_f] = -\text{Cov}(a_i - b_i \tilde{W}_i, \tilde{r}_j) = b_i \text{Cov}(\tilde{W}_i, \tilde{r}_j)$$

$$\Rightarrow (\frac{a_i}{b_i} - E[\tilde{W}_i]) E[\tilde{r}_j - r_f] = \text{Cov}(\tilde{W}_i, \tilde{r}_j)$$

$$\Rightarrow (\sum_{i=1}^I \frac{a_i}{b_i} - E[\tilde{M}]) E[\tilde{r}_j - r_f] = \text{Cov}(\tilde{M}, \tilde{r}_j)$$

$$\Rightarrow E[\tilde{r}_j - r_f] = (\sum_{i=1}^I \frac{a_i}{b_i} - E[\tilde{M}])^{-1} W_{m_0} \cdot \text{Cov}(\tilde{r}_m, \tilde{r}_j)$$

$$\Rightarrow E[\tilde{r}_m - r_f] = (\sum_{i=1}^I \frac{a_i}{b_i} - E[\tilde{M}])^{-1} W_{m_0} \cdot \text{Var}(\tilde{r}_m)$$

$$\Rightarrow E[\tilde{r}_j - r_f] = \frac{\text{Cov}(\tilde{r}_m, \tilde{r}_j)}{\text{Var}(\tilde{r}_m)} \cdot E[\tilde{r}_m - r_f]$$

- 考慮 A, B 兩個資產, 而且 assume they have the same expected time-1 payoffs
經濟好 \Rightarrow A 好 $\Rightarrow \text{Cov}(\text{Payoff}_M, \text{Payoff}_A) > 0 \Rightarrow \beta_{AM} > 0$

$$\Rightarrow \text{B 不好} \Rightarrow \text{Cov}(\text{Payoff}_M, \text{Payoff}_B) < 0 \Rightarrow \beta_{BM} < 0$$

消費者喜歡 B, 因經濟差時, 還可彌補一下,

\Rightarrow 在 $t = 0$ 時, B 之價錢高於 A 之價錢

$$\Rightarrow E[\tilde{r}_A] > E[\tilde{r}_B]$$

- $\tilde{r}_q = (1 - \beta_{qe})r_f + \beta_{qe}\tilde{r}_e + \tilde{\varepsilon}_{qe}$ (此式也可想成一個 one factor model)

原本 $E[\tilde{\varepsilon}_{je}|\tilde{r}_e] = 0$ 可造成 two monetary fund separation, 接著得到 CAPM.

如果現在改為 $\text{corr}(\tilde{\varepsilon}_{je}, \tilde{\varepsilon}_{ie}) = 0$

只要 assets 夠多, s.t. $\tilde{\varepsilon}'_{je}s$ can be diversified away by forming well-diversified portfolios. 形成 one factor model, \tilde{r}_e is the only factor.

⇒ One (or more) factor model will hold for most of the assets approximately if there is no arbitrage opportunity (in the limit) and there are a large number of assets.

- APT vs. General Equilibrium Method

(1) 不再考慮 state of nature 之 primitive security

(2) 考慮以 risk factor 為主的 primitive security

(3) 非以 General Equilibrium 之 demand 和 supply 的角度來 explain 資產 rate of return 之構成, 而是由已經觀察到之 primitive security 之 rate of return, 來討論其他資產的 rate of return. (General Equilibrium Theories are more ambitious)

General Equilibrium ⇒ No-Arbitrage

≠

- 在 CAPM 中, 由 General Equilibrium 推論出 r_m 為唯一解釋變數.

在 APT 中, 可加入其它之 risk factor.

- APT: identify the sources of systematic risk

: split systematic risk into its fundamental components

- Equilibrium theories aim at providing a complete theory of value on the basis of primitives: preferences, technology, and market structure. However, arbitrage-based theories can only provide a relative theory of value

- APT 之應用

* Given the stochastic behavior of the underlying asset, APT 可算 option price. (如 BS model)

* 由 fundamental security market prices 算出 risk neutral measures. (例如用 complete market 解釋 binomial tree 與 risk neutral valuation)

* 由 complete set of complex securities 之 prices 算出 A-D securities 之 price, 並以此算各種 security or cash flow 之 value

* 總之, APT 是由 market data 來解釋 expected return 與 risk factor 之關係

- Arbitrage Pricing Theory (APT) by Ross (1976)

* An arbitrage opportunity (in the limit) is a sequence of arbitrage portfolios whose expected rates of return are bounded away from zero, while their variances converge to zero. It is “almost” a free lunch.

* 想證明, if there is no arbitrage opportunity, then a linear relation among expected asset returns will hold approximately for most of the assets in a “large” economy.

(where “large” means that the number of assets n is arbitrary large)

$\tilde{r}_j^n = a_j^n + \sum_{k=1}^K \beta_{jk}^n \tilde{\delta}_k^n + \tilde{\varepsilon}_j^n$, $j=1, 2, \dots, n$ (the rates of return on risky assets are generated by a K -factor model)

where $E[\tilde{\varepsilon}_j^n] = 0$

$$E[\tilde{\varepsilon}_j^n \tilde{\varepsilon}_l^n] = 0, \text{ if } l \neq j$$

$$\sigma^2(\tilde{\varepsilon}_j^n) \leq \bar{\sigma}^2$$

(假設 $\tilde{\delta}_k^n$ 為 rates of return on portfolios)

assume $y_j^n = (1 - \sum_{k=1}^K \beta_{jk}^n) r_f + \sum_{k=1}^K \beta_{jk}^n \tilde{\delta}_k^n$ is a portfolio of K factors(folios)

and the riskless asset r_f which replicate the return of asset j (from general equilibrium model, 亦即為 K -factor CAPM:

$$y_j^n - r_f = \sum_{k=1}^K \beta_{jk}^n (\tilde{\delta}_k^n - r_f)$$

(其中在 \tilde{r}_j^n 與 y_j^n 的 β_{jk}^n 是由 \tilde{r}_j^n 與 $\tilde{\delta}_k^n$ 之歷史資料回歸求得, a_j^n 也是由 \tilde{r}_j^n 與 $\tilde{\delta}_k^n$ 之歷史資料回歸求得)

- 希望證明, for most of the assets in a large economy

$$a_j^n + \tilde{\varepsilon}_j^n = (1 - \sum_{k=1}^K \beta_{jk}^n) r_f$$

Case 1: $\tilde{\varepsilon}_j^n = 0$ (trivial)

希望證明 $a_j^n = (1 - \sum_{k=1}^K \beta_{jk}^n) r_f$

$$\begin{cases} \text{if } a_j^n < (1 - \sum_{k=1}^K \beta_{jk}^{n_l}) r_f \\ \text{則買 } y_j^n, 賣 } r_j^n \text{ (free lunch) } \end{cases}$$

$$\begin{cases} \text{if } a_j^n > (1 - \sum_{k=1}^K \beta_{jk}^{n_l})r_f \\ \text{則賣 } y_j^n, \text{ 買 } r_j^n \text{ (free lunch)} \end{cases}$$

$\Rightarrow a_j^n + \tilde{\varepsilon}_j^n = a_j^n = (1 - \sum_{k=1}^K \beta_{jk}^n)r_f$ 得證.

Case 2: $\tilde{\varepsilon}_j^n \neq 0$

想要證明只有最多 \bar{N} 個不滿足 $a_j^n \approx (1 - \sum_{k=1}^K \beta_{jk}^n)r_f$, 表剩下大多都滿足 (4.21.1)

$$\Rightarrow |a_j^n - (1 - \sum_{k=1}^K \beta_{jk}^n)r_f| \geq \varepsilon, j = 1, 2, \dots, N(n) \leq \bar{N}$$

(反證法) 若 \bar{N} 不存在, 則表示若 $\{n_l\} = \{n = K+2, K+3, \dots\}$, if $n_l \rightarrow \infty$, $N(n_l) \rightarrow \infty$ ($N(n_l)$ 表示 n_l risky assets 中, 有 $N(n_l)$ 個違反 linear valuation model.)

(表容許少部份 assets 不滿足 $a_j^n \approx (1 - \sum_{k=1}^K \beta_{jk}^n)r_f$)

we can construct $N(n_l)$ 個 arbitrage portfolios

(i) 若 $a_j^{n_l} > (1 - \sum_{k=1}^K \beta_{jk}^{n_l})r_f$, 則買低賣高, 建立 $\tilde{r}_j^n - y_j^n$ 之 arbitrage portfolio

(ii) 若 $a_j^{n_l} < (1 - \sum_{k=1}^K \beta_{jk}^{n_l})r_f$, 建立 $y_j^n - \tilde{r}_j^n$ 之 arbitrage portfolio

$$\text{rate of return} \begin{cases} \text{(i) } a_j^{n_l} - (1 - \sum_{k=1}^K \beta_{jk}^{n_l})r_f + \tilde{\varepsilon}_j^{n_l} \\ \text{(ii) } (1 - \sum_{k=1}^K \beta_{jk}^{n_l})r_f - a_j^{n_l} - \tilde{\varepsilon}_j^{n_l} \end{cases}$$

** 將此 $N(n_l)$ 個 arbitrage portfolio 以 $\frac{1}{N(n_l)}$ 之 weight 加總, 取 expectation 則 expected rate of return $\geq \varepsilon$

(因 $|a_j^n - (1 - \sum_{k=1}^K \beta_{jk}^n)r_f| \geq \varepsilon$, 且 $E[\tilde{\varepsilon}_j^n] = 0$)

$$\text{variance of rate of return} = \frac{1}{N^2(n_l)} \sum_{j=1}^{N(n_l)} \sigma^2(\tilde{\varepsilon}_j^{n_l}) \leq \frac{\bar{\sigma}^2}{N(n_l)}$$

(因 $\text{corr}(\tilde{\varepsilon}_j^n, \tilde{\varepsilon}_l^n) = 0$)

(當 $N(n_l) \rightarrow \infty$ 時, 期望值 > 0 , variance=0 \Rightarrow 一定會賺錢, 可套利, 此非均衡之狀態)

\Rightarrow 當容許大多數之 asset 都不滿足 $a_j^n \approx (1 - \sum_{k=1}^K \beta_{jk}^n)r_f$, 則存在 arbitrage

opportunity, almost a free lunch

\Rightarrow 若要使此可 arbitrage 情況不存在,

必須 $\exists \bar{N}$, s.t. $N(n) \leq \bar{N}$ for all n

亦即大多數之 asset 都滿足 $|a_j^n - (1 - \sum_{k=1}^K \beta_{jk}^n)r_f| \leq \varepsilon$

\parallel 又由 (4.19.1) 取 $E[\cdot]$ 可得 $E[\tilde{r}_j^n - \sum_{k=1}^K \beta_{jk}^n \tilde{\delta}_k^n] = a_j^n$

代入可得 $|E[\tilde{r}_j^n] - r_f - \sum_{k=1}^K \beta_{jk}^n (E[\tilde{\delta}_k^n] - r_f)| \leq \varepsilon$

- When $n \geq \bar{N}$, a linear relation among expected asset returns holds approximately for most of the assets.

This relation is called Arbitrage Pricing Theory (APT) originated by Ross(1976).

- (4.23)~(4.25) 節說, 當 $n \rightarrow \infty$, 對大多數資產而言, APT 會對, 但對單一資產而言, 到底對不對是個問題, 若假設是對的, 偏離多少也是個問題, 此3小節即要找出偏離的 upperbound.

(Equilibrium rather than arbitrage arguments will be used in the derivation in sections (4.23) to (4.25). The resulting relationship among risky assets is called “equilibrium APT”)

- (4.24) 節, 證明 when $\tilde{\varepsilon}_j \neq 0 \Rightarrow (1 - \sum_{k=1}^K \beta_{jk})r_f - a_j < 0$ for $\forall j$ and

$\alpha_{ij} > 0$ for $\forall i, j$ (α_{ij} 為 individual i 投資於 asset j 之金額)(此部份主要借用了 $\tilde{\varepsilon}_j$ is independent of all the other random variables 之論點 p.111)

- (4.25) 節, $|E[\tilde{r}_j] - r_f - \sum_{k=1}^K (E[\tilde{\delta}_k] - r_f)| \leq \frac{S_j}{I} \cdot \bar{A} \cdot e^{\bar{A}S_j/I} \text{Var}(\tilde{\varepsilon}_j)$
 (當 $u' \geq 0, u'' < 0, u''' \geq 0, A_i(z) \equiv -u_i''(z)/u_i'(z) \leq \bar{A}, E[\tilde{\varepsilon}_j] = 0, \tilde{\varepsilon}_j \geq -1,$
 $(\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_N, \tilde{\delta}_1, \dots, \tilde{\delta}_K)$ are independent)