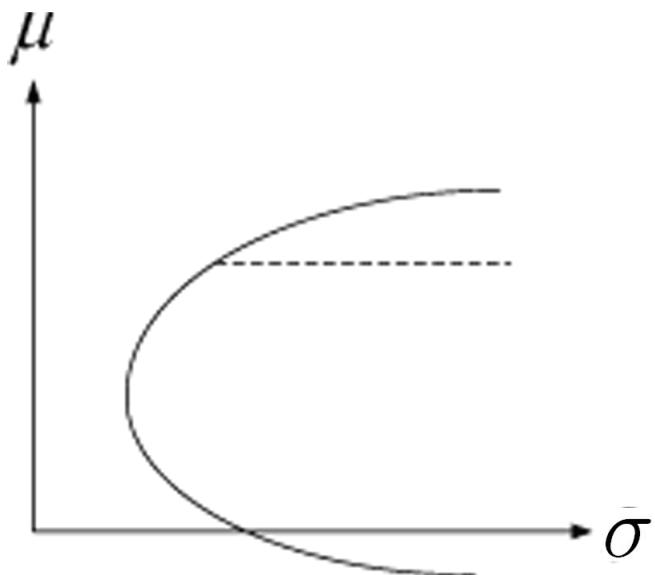


Ch3. Math of the portfolio frontier

- Given expected return μ 下, 找一個 portfolio SSD all other portfolios, 則這個 portfolio 一定擁有最小的 variance



- Expected Utility Hypothesis (expected return + risk averse)
(即使如此, 並不能推到一般常用的 mean-variance model, 進而畫出 mean-variance 之分析圖)

|| 若情況1. expected rate of return 是 normal distribution(arbitrary preferences)
|| 情況2. quadratic utility (arbitrary distributions)
|| 才能推到mean-variance model.

- 其實 mean-variance model 並不是一個完美的 model, 只是在財務上, 其推導較容易, 且可以預測很多事, 所以此章 focus 在此架構下

- $u(\tilde{w}) = u(E[\tilde{w}]) + u'(E[\tilde{w}])(\tilde{w} - E[\tilde{w}]) + \frac{1}{2!}u''(E[\tilde{w}])(\tilde{w} - E[\tilde{w}])^2 + R_3$

其中 $R_3 = \sum_{n=3}^{\infty} \frac{1}{n!} u^{(n)}(E[\tilde{w}]) (\tilde{w} - E[\tilde{w}])^n$

取 $E[\cdot]$

\Rightarrow

$$\begin{aligned} E[u(\tilde{w})] &= u(E[\tilde{w}]) + u'(E[\tilde{w}])(E[\tilde{w}] - E[\tilde{w}]) \\ &\quad + \frac{1}{2!}u''(E[\tilde{w}])E[\tilde{w} - E[\tilde{w}]]^2 + E[R_3] \\ &= u(E[\tilde{w}]) + \frac{1}{2!}u''(E[\tilde{w}])\sigma^2(\tilde{w}) + E[R_3] \end{aligned}$$

其中 $E[R_3] = \sum_{n=3}^{\infty} \frac{1}{n!} u^{(n)}(E[\tilde{w}]) E[\tilde{w} - E[\tilde{w}]]^n$

分析 $E[R_3]$

在情況2時為 0,

在情況1時 third or higher order moment 可用 mean 和 variance 表示,
除此之外, $N+N \rightarrow N$, 雖然 log Normal 之 high moments 也可以用 mean
與 variance 表示, 但 log Normal + log Normal $\not\sim$ log Normal

- If $u'' < 0$, 則上式隱含了希望期望財富越多越好, 但卻厭惡 variance of wealth

- 例: quadratic utility \Rightarrow mean-variance model

If $u(x) = x - \frac{b}{2}x^2$

$$\begin{aligned} \Rightarrow E[u(\tilde{w})] &= u(E[\tilde{w}]) + \frac{1}{2!}u''(E[\tilde{w}])\sigma^2(\tilde{w}) \\ &= E[\tilde{w}] - \frac{b}{2}(E[\tilde{w}])^2 - \frac{b}{2}\sigma^2(\tilde{w}) \\ &= E[\tilde{w}] - \frac{b}{2}E[\tilde{w}^2] \end{aligned}$$

- quadratic utility 之缺點

$$1. R_A(\cdot) = -\frac{u''(\cdot)}{u'(\cdot)} = \frac{b}{1-bx}, \frac{\partial R_A}{\partial x} > 0,$$

由 Ch1 之 proposition 2

\Rightarrow risky assets are inferior goods.

2. utility 非 globally increasing, 只在 $\tilde{w} \leq$ satiation point 才是.

- 例: normal distribution \Rightarrow mean-variance model
(以 futures hedge 為例)

- Futures hedge

$$w_1 = w_0 + \Delta p + Q\Delta f$$

Δp 為現貨價格變化, Δf 為期貨價格變化

$$\begin{cases} w_0 = 0 \\ \Delta p \sim N(\mu_p, \sigma_p^2) \\ \Delta f \sim N(\mu_f, \sigma_f^2) \\ \text{cov}(\Delta p, \Delta f) = \sigma_{pf} \end{cases}$$

$$\Rightarrow w_1 \sim N(\mu, \sigma^2)$$

$$\begin{cases} \mu = \mu_p + Q\mu_f \\ \sigma^2 = \sigma_p^2 + 2Q\sigma_{pf} + Q^2\sigma_f^2 \end{cases}$$

$$(i) \max_Q E[1 - e^{-\alpha w_1}] \quad Q^* \text{會相等嗎?} \quad (ii) \max_Q E[w_1] - \frac{\alpha}{2} \text{var}(w_1)$$

$$(i) \max_Q E[1 - e^{-\alpha w_1}]$$

$$\begin{aligned} E[u(w_1)] &= \int_{-\infty}^{\infty} (1 - e^{-\alpha w_1}) f(w_1) dw_1 \\ &= 1 - \int_{-\infty}^{\infty} e^{-\alpha w_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(w_1-\mu)^2}{2\sigma^2}} dw_1 \end{aligned}$$

$$\begin{aligned}
&= 1 - e^{\frac{(\mu-\alpha\sigma^2)^2-\mu^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(w_1+\alpha\sigma^2-\mu)^2}{2\sigma^2}} dw_1 \\
&= 1 - e^{\frac{(\mu-\alpha\sigma^2)^2-\mu^2}{2\sigma^2}} \\
&= 1 - e^{\frac{\alpha^2\sigma^2}{2}-\alpha\mu} \\
\frac{\partial}{\partial Q} &= \frac{\partial}{\partial \mu} \frac{\partial \mu}{\partial Q} + \frac{\partial}{\partial \sigma} \frac{\partial \sigma}{\partial Q}
\end{aligned}$$

$$\left\| \begin{array}{l} \frac{\partial}{\partial \mu} = -e^{\left(\frac{\alpha^2\sigma^2}{2}-\alpha\mu\right)}(-\alpha) \\ \frac{\partial \mu}{\partial Q} = \mu_f \\ \frac{\partial}{\partial \sigma} = -e^{\left(\frac{\alpha^2\sigma^2}{2}-\alpha\mu\right)}(\alpha^2\sigma) \\ \frac{\partial \sigma}{\partial Q} = \frac{\sigma_{pf}+Q\sigma_f^2}{\sigma} \end{array} \right.$$

$$\begin{aligned}
\text{FOC} &\Rightarrow -e^{\left(\frac{\alpha^2\sigma^2}{2}-\alpha\mu\right)}\alpha[-\mu_f + \alpha(\sigma_{pf} + Q\sigma_f^2)] = 0 \\
&\Rightarrow Q^* = \frac{\mu_f - \alpha\sigma_{pf}}{\alpha\sigma_f^2}
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \max_Q E[w_1] - \frac{\alpha}{2} \text{var}(w_1) \\
&= \mu_p + Q\mu_f - \frac{\alpha}{2}(\sigma_p^2 + 2Q\sigma_{pf} + Q^2\sigma_f^2) \\
\frac{\partial}{\partial Q} &\Rightarrow \mu_f - \alpha\sigma_{pf} - \alpha Q\sigma_f^2 = 0 \\
&\Rightarrow Q^* = \frac{\mu_f - \alpha\sigma_{pf}}{\alpha\sigma_f^2}
\end{aligned}$$

$$\left\| \begin{array}{l} \text{若是 minimum variance model (例如 Hull 中第三章), 則} \\ \min_Q \text{var}(w_1) \\ \frac{\partial}{\partial Q}(\sigma_p^2 + 2Q\sigma_{pf} + Q^2\sigma_f^2) \\ = 2\sigma_{pf} + 2Q\sigma_f^2 = 0 \\ \Rightarrow Q = \frac{\sigma_{pf}}{\sigma_f^2} = \rho_{pf} \frac{\sigma_p}{\sigma_f} \end{array} \right.$$

- 假設 $\tilde{r} \sim N(\mu, \sigma^2)$ 之問題:

因 $\tilde{r} \in (-\infty, \infty)$, 但在財務中

- (1) 損失只可能 -100%, 不可能 - ∞
- (2) 很常 utility 的定義域只在 R^+
- (3) Consumption 沒有負的

- N risky assets: $\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_N$, which are linearly independent, with unequal expectations, finite variances. In addition, unlimited short sales is allowed.

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}, \mathbf{e} = \begin{pmatrix} E[\tilde{r}_1] = e_1 \\ E[\tilde{r}_2] = e_2 \\ \vdots \\ E[\tilde{r}_N] = e_N \end{pmatrix}$$

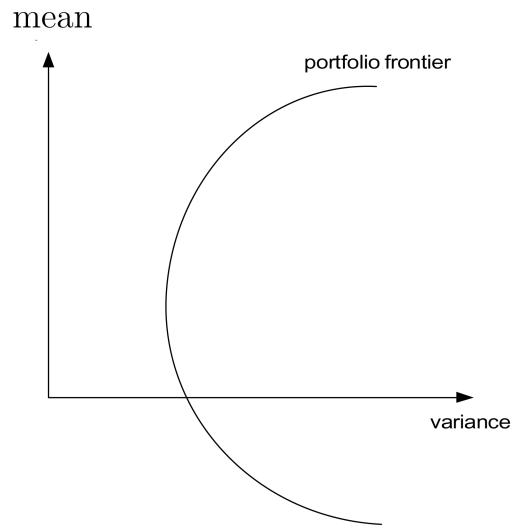
j

$$V = i \begin{pmatrix} & \vdots & \\ \dots & \sigma_{ij} & \dots \\ & \vdots & \end{pmatrix}, \text{ where } \sigma_{ij} = \text{Cov}(\tilde{r}_i, \tilde{r}_j)$$

- portfolio mean = $\mathbf{w}^T \mathbf{e} = E[\tilde{r}_p]$

portfolio variance = $\mathbf{w}^T V \mathbf{w} = \sigma^2(\tilde{r}_p)$

找 portfolio frontier (given $E[\tilde{r}_p]$, find \mathbf{w} as a function of $E[\tilde{r}_p]$ to minimize portfolio variance $\sigma^2(\tilde{r}_p)$)



$$\min_w \frac{1}{2} w^T V w,$$

$$\text{s.t. } w^T e = E[\tilde{r}_p]$$

$$w^T 1_N = 1$$

$$L = \frac{1}{2} w^T V w + \lambda(E[\tilde{r}_p] - w^T e) + \gamma(1 - w^T 1_N)$$

$$\frac{\partial L}{\partial w} \Rightarrow V w_p - \lambda e - \gamma 1_N = 0$$

$$\begin{cases} \frac{\partial \frac{1}{2} w^T V w}{\partial w} = V w \ (N \times 1) \\ \frac{\partial w^T e}{\partial w} = e \ (N \times 1) \\ \frac{\partial w^T 1_N}{\partial w} = 1_N \ (N \times 1) \end{cases}$$

$$\Rightarrow w_p = \lambda(V^{-1}e) + \gamma(V^{-1}1_N)$$

$$\Rightarrow \begin{cases} E[\tilde{r}_p] = w_p^T e = \lambda(e^T V^{-1} e) + \gamma(1_N^T V^{-1} e) \\ 1 = w_p^T 1_N = \lambda(e^T V^{-1} 1_N) + \gamma(1_N^T V^{-1} 1_N) \end{cases}$$

$$\Leftrightarrow A = 1_N^T V^{-1} e = \sum_{ij} \sigma_{ij}^{-1} e_i$$

$$B = \mathbf{e}^T V^{-1} \mathbf{e} = \sum_{ij} e_i e_j \sigma_{ij}^{-1}$$

$$C = \mathbf{1}_N^T V^{-1} \mathbf{1}_N = \sum_{ij} \sigma_{ij}^{-1}$$

(A, B, C 均為 real number, 且 $B > 0, C > 0, D = BC - A^2 > 0$)

$$\Rightarrow \begin{bmatrix} B & A \\ A & C \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} E[\tilde{r}_p] \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} B & A \\ A & C \end{bmatrix}^{-1} \begin{bmatrix} E[\tilde{r}_p] \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{D}(C \cdot E[\tilde{r}_p] - A) \\ \frac{1}{D}(B - A \cdot E[\tilde{r}_p]) \end{bmatrix}$$

代回算 w_p

$$\Rightarrow w_p = \frac{1}{D}(C \cdot E[\tilde{r}_p] - A)(V^{-1} \mathbf{e}) + \frac{1}{D}(B - A \cdot E[\tilde{r}_p])(V^{-1} \mathbf{1}_N) \\ = \frac{1}{D}(BV^{-1} \mathbf{1}_N - AV^{-1} \mathbf{e}) + \frac{1}{D}(CV^{-1} \mathbf{e} - AV^{-1} \mathbf{1}_N)E[\tilde{r}_p]$$

(any portfolio represented by w_p is on the portfolio frontier)

- Define:

$$g = \frac{1}{D}(BV^{-1} \mathbf{1}_N - AV^{-1} \mathbf{e}), g \text{ 為一 } N \times 1 \text{ 的 vector}$$

$$h = \frac{1}{D}(CV^{-1} \mathbf{e} - AV^{-1} \mathbf{1}_N), h \text{ 為一 } N \times 1 \text{ 的 vector}$$

$$\Rightarrow w_p = g + h \cdot 0 = g$$

(where g is a vector of portfolio weight corresponding to a 0 rate of return)

$$w_p = g + h \cdot 1 = g + h$$

(where $g+h$ is a vector of portfolio weight corresponding to a 1 rate of return)

- 要證明所有 w_q on the portfolio frontier, 可由 g 與 $g+h$ 組成

例: $w_q = g + hE[\tilde{r}_q] = (1 - E[\tilde{r}_q])g + E[\tilde{r}_q](g+h)$

- 其實 portfolio frontier 上的任一個投資組合, 都可由另兩個不同之 frontier portfolio 組成

例: 假設存在 α , $E[\tilde{r}_q] = \alpha E[\tilde{r}_{p_1}] + (1 - \alpha)E[\tilde{r}_{p_2}]$

$$\begin{aligned} \alpha w_{p_1} + (1 - \alpha)w_{p_2} &= \alpha(g + hE[\tilde{r}_{p_1}]) + (1 - \alpha)(g + hE[\tilde{r}_{p_2}]) \\ &= g + h(\alpha E[\tilde{r}_{p_1}] + (1 - \alpha)E[\tilde{r}_{p_2}]) \\ &= g + hE[\tilde{r}_q] \\ &= w_q \end{aligned}$$

(linear combination of 2 frontier portfolios (not necessary efficient portfolios) will yield another portfolio)

- Mutual Fund Thorem (Separating Thorem)

Under the above assumptions and the mean-variance framework, we can find 2 portfolios (mutual funds) such that all investors are indifferent in chooseing between portfolios from amongs the original N risky assets or from these two mutual funds

- 要找 $\sigma^2(\tilde{r}_p) = \sum_i \sum_j w_i w_j \sigma_{ij}$ 這個值 (課本 3.11, 但導法不同)

$$L = \frac{1}{2} \sum_i \sum_j w_i w_j \sigma_{ij} + \lambda(E[\tilde{r}_p] - \sum_i w_i e_i) + \gamma(1 - \sum_i w_i)$$

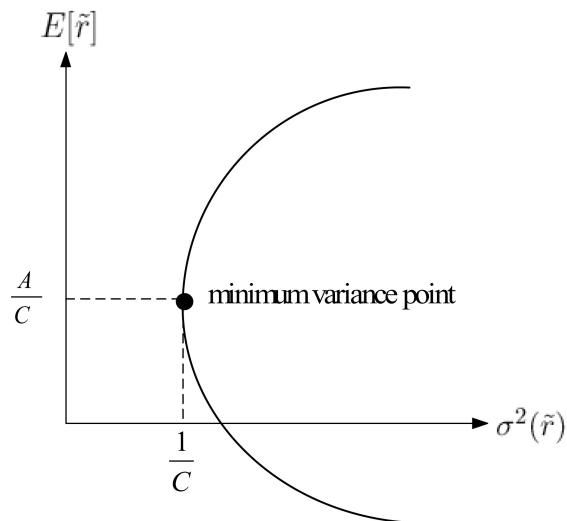
$$\frac{\partial L}{\partial w_i} = \sum_j w_j \sigma_{ij} - \lambda e_i - \gamma = 0$$

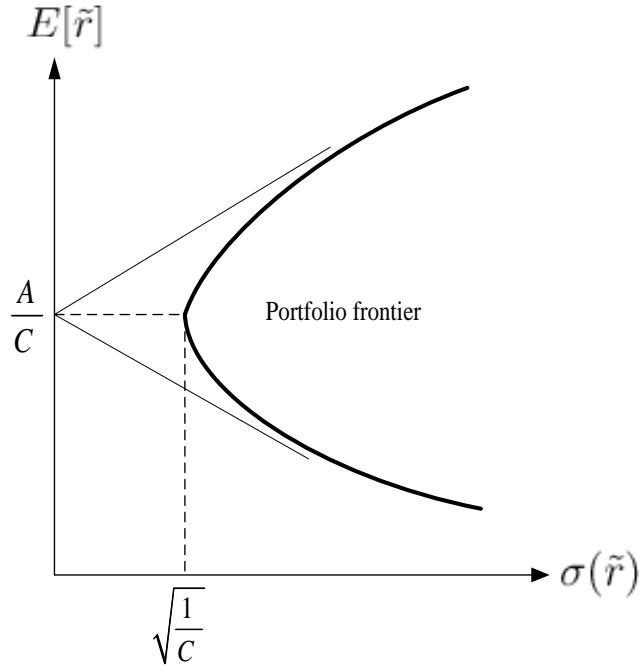
同乘 w_i , 再取 \sum_i $\Rightarrow \sum_i w_i \sum_j w_j \sigma_{ij} - \lambda \sum_i w_i e_i - \gamma \sum_i w_i = 0$

(where $\sum_i w_i \sum_j w_j \sigma_{ij} = \sigma^2(\tilde{r}_p)$, $\sum_i w_i e_i = E[\tilde{r}_p]$, $\sum_i w_i = 1$)

$$\begin{aligned} \Rightarrow \sigma^2(\tilde{r}_p) &= \lambda E[\tilde{r}_p] + \gamma \\ &= \left(\frac{CE[\tilde{r}_p] - A}{D}\right)E[\tilde{r}_p] + \left(\frac{B - AE[\tilde{r}_p]}{D}\right) \quad (\lambda \text{ 與 } \gamma \text{ 代入}) \\ &= \frac{1}{D}(C(E[\tilde{r}_p])^2 - 2A(E[\tilde{r}_p]) + B) \\ &= \frac{1}{C} + \frac{C}{D}(E[\tilde{r}_p] - \frac{A}{C})^2 \end{aligned}$$

(在 N risky assets 時, 在 portfolio frontier 上, $E[\tilde{r}_p]$ 與 $\sigma^2(\tilde{r}_p)$ 之關係) (圖)





- 證明 $\text{Cov}(\tilde{r}_p, \tilde{r}_q) = \mathbf{w}_p^T V \mathbf{w}_q = \frac{C}{D}(E[\tilde{r}_p] - \frac{A}{C})(E[\tilde{r}_q] - \frac{A}{C}) + \frac{1}{C}$
(p.66 3.11.1)

其中 $\mathbf{w}_p = \mathbf{g} + E[\tilde{r}_p]\mathbf{h}$

$$\mathbf{w}_q = \mathbf{g} + E[\tilde{r}_q]\mathbf{h}$$

$$A = \mathbf{1}_N^T V^{-1} \mathbf{e} = \mathbf{e}^T V^{-1} \mathbf{1}_N$$

$$B = \mathbf{e}^T V^{-1} \mathbf{e}$$

$$C = \mathbf{1}_N^T V^{-1} \mathbf{1}_N$$

$$D = BC - A^2$$

$$\mathbf{g} = \frac{1}{D}[B(V^{-1}\mathbf{1}_N) - A(V^{-1}\mathbf{e})]$$

$$\mathbf{h} = \frac{1}{D}[C(V^{-1}\mathbf{e}) - A(V^{-1}\mathbf{1}_N)]$$

$$\mathbf{w}_p^T V \mathbf{w}_q = (\mathbf{g}^T + E[\tilde{r}_p]\mathbf{h}^T)V(\mathbf{g} + E[\tilde{r}_q]\mathbf{h})$$

$$= (\mathbf{g}^T V + E[\tilde{r}_p]\mathbf{h}^T V)(\mathbf{g} + E[\tilde{r}_q]\mathbf{h})$$

$$= \mathbf{g}^T V \mathbf{g} + E[\tilde{r}_q]\mathbf{g}^T V \mathbf{h} + E[\tilde{r}_p]\mathbf{h}^T V \mathbf{g} + E[\tilde{r}_p]E[\tilde{r}_q]\mathbf{h}^T V \mathbf{h}$$

$$\left\| \begin{array}{l} \text{where} \\ \mathbf{g}^T V \mathbf{g} = \frac{B}{D} \\ \mathbf{g}^T V \mathbf{h} = -\frac{A}{D} \\ \mathbf{h}^T V \mathbf{g} = -\frac{A}{D} \\ \mathbf{h}^T V \mathbf{h} = \frac{C}{D} \end{array} \right.$$

$$\begin{aligned} &= \frac{C}{D} E[\tilde{r}_p] E[\tilde{r}_q] - \frac{A}{D} E[\tilde{r}_p] - \frac{A}{D} E[\tilde{r}_q] + \frac{B}{D} \\ &\quad (\because \frac{A^2}{DC} + \frac{1}{C} = \frac{A^2 + BC - A^2}{DC} = \frac{BC}{DC} = \frac{B}{D}) \\ &= \frac{C}{D} (E[\tilde{r}_p] - \frac{A}{C})(E[\tilde{r}_q] - \frac{A}{C}) + \frac{1}{C} \end{aligned}$$

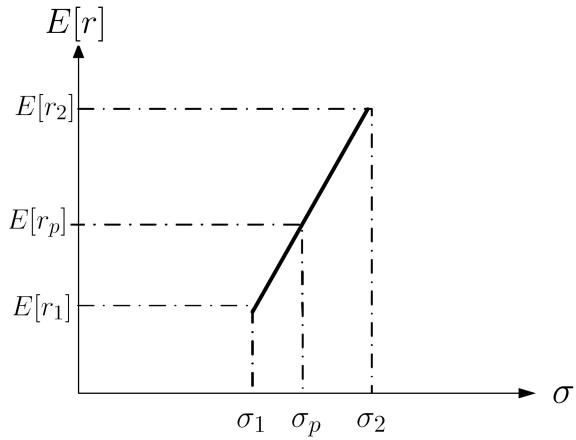
- if $p = q$

$$\begin{aligned} \Rightarrow \text{Cov}(\tilde{r}_p, \tilde{r}_q) = \text{Var}(\tilde{r}_p) &= \sigma^2(\tilde{r}_p) = \frac{C}{D} (E[\tilde{r}_p] - \frac{A}{C})^2 + \frac{1}{C} \\ \Rightarrow \frac{\sigma^2(\tilde{r}_p)}{1/C} - \frac{(E[\tilde{r}_p] - \frac{A}{C})^2}{D/C^2} &= 1 \end{aligned}$$

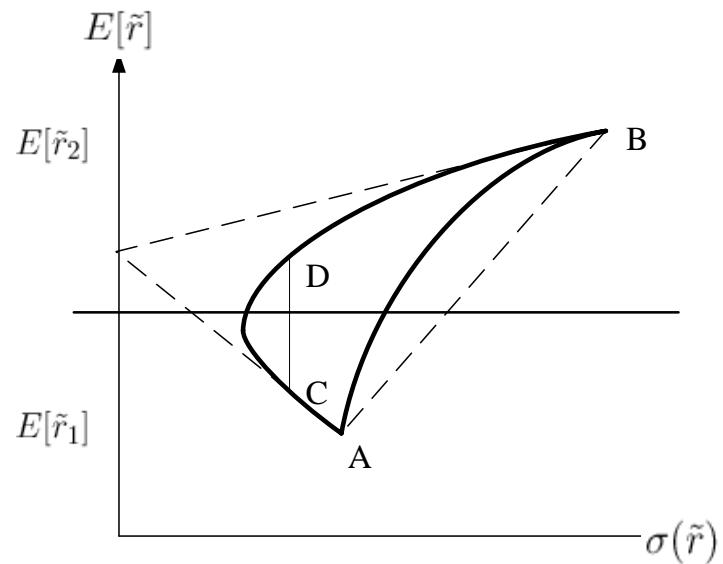
$\Rightarrow \sigma(\tilde{r}_p)$ 與 $E[\tilde{r}_p]$ 之關係為雙曲線

- 考慮用 2 risky assets 來組成出不同之 portfolio frontier

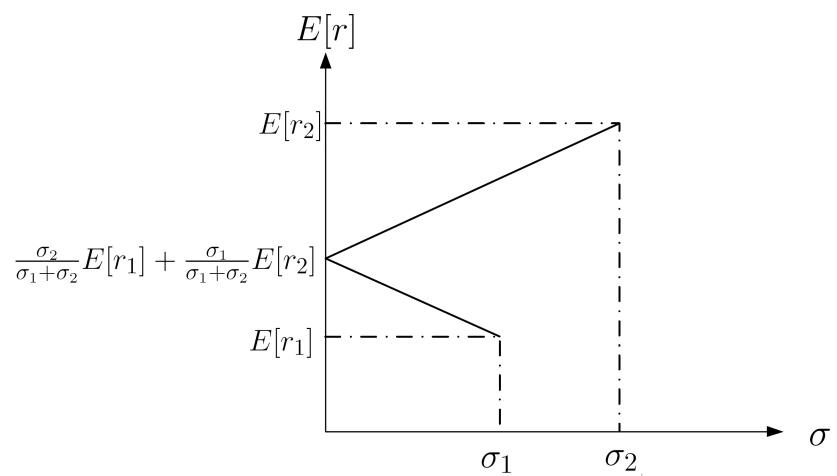
- (i) the 2 assets are perfectly positively correlated (圖)



(ii) imperfectly correlated (圖)



(iii) perfectly negatively correlated (圖)



- 3.12 節, \tilde{r}_{mvp} : $\text{Cov}(\tilde{r}_p, \tilde{r}_{mvp}) = \text{Var}(\tilde{r}_{mvp})$, 其中 p 為任何 portfolio (未必要在 frontier 上)

$$\min_a \text{Var}(a\tilde{r}_p + (1-a)\tilde{r}_{mvp}) \quad (a^* \text{ 應為 } 0)$$

$$\Rightarrow \text{目標式} = a^2\sigma^2(\tilde{r}_p) + 2a(1-a)\text{Cov}(\tilde{r}_p, \tilde{r}_{mvp}) + (1-a)^2\sigma^2(\tilde{r}_{mvp})$$

$$\frac{\partial}{\partial a} \Rightarrow 2a\sigma^2(\tilde{r}_p) + 2(1-2a)\text{Cov}(\tilde{r}_p, \tilde{r}_{mvp}) - 2(1-a)\sigma^2(\tilde{r}_{mvp}) = 0$$

$$a = 0 \text{ 代入} \Rightarrow \text{Cov}(\tilde{r}_p, \tilde{r}_{mvp}) = \text{Var}(\tilde{r}_{mvp})$$

- 3.13 節, 在 \tilde{r}_{mvp} 以上, 叫 efficient frontier, 且 efficient frontier + efficient frontier 還會是 efficient frontier (因 portfolio frontier 符合 Mutual Fund Thorem)(但 mutual funds 之 weight 不可是負, 參考 p.69(3.13.1))

- For any p on the efficient frontier, except for the minimum variance portfolio, there exists a unique portfolio on the portfolio frontier, which has a zero covariance with p .

$$\text{Cov}(\tilde{r}_p, \tilde{r}_{zc(p)}) = \frac{C}{D}((E[\tilde{r}_p] - \frac{A}{C})(E[\tilde{r}_{zc(p)}] - \frac{A}{C}) + \frac{D}{C^2}) = 0$$

$$\Rightarrow E[\tilde{r}_{zc(p)}] = \frac{A}{C} - \frac{\frac{D}{C^2}}{E[\tilde{r}_p] - \frac{A}{C}}$$

(given $E[\tilde{r}_p]$, then $E[\tilde{r}_{zc(p)}]$ can be derived, and $w_{zc(p)}$ is obtained uniquely)

參考 p.71 Figure 3.15.1

- 3.15 節 p 與 $zc(p)$ 在 $\sigma(\tilde{r}) - E[\tilde{r}]$ 與 $\sigma^2(\tilde{r}) - E[\tilde{r}]$ 平面上之幾何關係.
(p.71, Figure 3.15.1, p.72, Figure 3.15.2)

- 若 p 不在 minimal variance frontier 上 (or portfolio frontier), 則 $q = zc(p)$ 會在 p 與 mvp 所形成之 frontier 上

w_q is the solution of

$$\min_{w_q} \frac{1}{2} w_q^T V w_q$$

$$\text{s.t. } w_q^T V w_p = 0$$

$$w_q^T 1_N = 1$$

$$\text{解可得 } w_q = \frac{1}{1-C\sigma^2(\tilde{r}_p)} w_p - \frac{C\sigma^2(\tilde{r}_p)}{1-C\sigma^2(\tilde{r}_p)} \cdot \frac{V^{-1}1_N}{1_N^T V^{-1}1_N}$$

$$\left\| w_{mvp} = g + hE[\tilde{r}_{mvp}] = g + h \frac{A}{C} = \frac{V^{-1}1_N}{C} = \frac{V^{-1}1_n}{1_N^T V^{-1}1_N} \right.$$

$$\Rightarrow w_q = \frac{1}{1-C\sigma^2(\tilde{r}_p)} w_p - \frac{C\sigma^2(\tilde{r}_p)}{1-C\sigma^2(\tilde{r}_p)} \cdot w_{mvp}$$

(q 由 p 與 mvp 線性組合而成, 亦即 q 在 p 與 mvp 所組成的 portfolio frontier 上)

$$\begin{aligned} \Rightarrow E[\tilde{r}_q] &= w_q^T \cdot e = \frac{1}{1-C\sigma^2(\tilde{r}_p)} w_p^T \cdot e - \frac{C\sigma^2(\tilde{r}_p)}{1-C\sigma^2(\tilde{r}_p)} \cdot \frac{1_N^T V^{-1}e}{1_N^T V^{-1}1_N} \\ &= \frac{E[\tilde{r}_p]}{1-C\sigma^2(\tilde{r}_p)} - \frac{C\sigma^2(\tilde{r}_p)}{1-C\sigma^2(\tilde{r}_p)} \cdot \frac{A}{C} \\ &= \frac{E[(\tilde{r}_p)] - A\sigma^2(\tilde{r}_p)}{1-C\sigma^2(\tilde{r}_p)} \end{aligned}$$

$E[\tilde{r}_q]$ 剛好是 p 與 mvp 在 $\sigma^2(\tilde{r}_p) - E[\tilde{r}_p]$ 平面上的連線之截距 (p.74 Figue 3.15.3)

- 考慮 N risky asset, 1 riskless asset

$$\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}^T V \mathbf{w}$$

$$\text{s.t. } \mathbf{w}^T \mathbf{e} + (1 - \mathbf{w}^T \mathbf{1}_N) r_f = E[\tilde{r}_p] (\text{or } r_f + \mathbf{w}^T (\mathbf{e} - r_f \mathbf{1}_N) = E[\tilde{r}_p])$$

$$\text{FOC} \Rightarrow V \mathbf{w}_p = \lambda (\mathbf{e} - r_f \mathbf{1}_N)$$

$$\Rightarrow \mathbf{w}_p = \lambda V^{-1} (\mathbf{e} - r_f \mathbf{1}_N), \text{ 代回限制式}$$

$$\Rightarrow \lambda (\mathbf{e} - r_f \mathbf{1}_N)^T V^{-1} (\mathbf{e} - r_f \mathbf{1}_N) = E[\tilde{r}_p] - r_f$$

$$\Rightarrow \lambda = \frac{E[\tilde{r}_p] - r_f}{H}, \text{ 代回 } \mathbf{w}_p$$

$$\left\| \begin{aligned} \text{其中 } H &= (\mathbf{e} - \mathbf{1}_N \cdot r_f)^T V^{-1} (\mathbf{e} - \mathbf{1}_N \cdot r_f) \\ &= \sum_{ij} (\mathbf{e}_i - r_f) \sigma_{ij}^{-1} (\mathbf{e}_j - r_f) \\ &= \sum_{ij} \mathbf{e}_i \sigma_{ij}^{-1} \mathbf{e}_j - r_f \sum_{ij} \sigma_{ij}^{-1} \mathbf{e}_j - r_f \sum_{ij} \mathbf{e}_i \sigma_{ij}^{-1} + r_f^2 \sum_{ij} \sigma_{ij}^{-1} \\ &= B - 2A r_f + C r_f^2 > 0 \\ &(> 0 : \text{因 } C > 0, \text{ 且 } A^2 - BC < 0) \end{aligned} \right.$$

$$\Rightarrow \mathbf{w}_p = \left(\frac{E[\tilde{r}_p] - r_f}{H} \right) V^{-1} (\mathbf{e} - r_f \mathbf{1}_N)$$

(vs. all risky asset 時, $\mathbf{w}_p = g + h \cdot E[\tilde{r}_p]$)

繼續看 σ_p^2 與 $E[\tilde{r}_p]$ 的關係

$$\text{FOC of } w_i \Rightarrow \sum_j w_j \sigma_{ij} - \lambda (e_i - r_f) = 0$$

$$\text{同乘 } w_i, \text{ 再取 } \sum_i \Rightarrow \sum_{ij} w_i w_j \sigma_{ij} - \lambda \sum_i w_i (e_i - r_f) = 0$$

$$\text{where } \sum_{ij} w_i w_j \sigma_{ij} = \sigma_p^2,$$

$$\sum_i w_i (e_i - r_f) = \sum_i w_i e_i - \sum_i w_i r_f$$

$$\left\| \begin{aligned} (\text{因 } \mathbf{w}^T \mathbf{e} + (1 - \mathbf{w}^T \mathbf{1}_N) r_f = E[\tilde{r}_p] \Rightarrow \mathbf{w}^T \mathbf{e} - \mathbf{w}^T \mathbf{1}_N r_f = E[\tilde{r}_p] - r_f \\ \Rightarrow \sum_i W_i e_i - \sum_i W_i r_f = E[\tilde{r}_p] - r_f,) \\ \text{再加上 } \lambda = \frac{E[\tilde{r}_p] - r_f}{H} \end{aligned} \right.$$

$$\Rightarrow \sigma_p^2 = \frac{(E[\tilde{r}_p] - r_f)^2}{H}$$

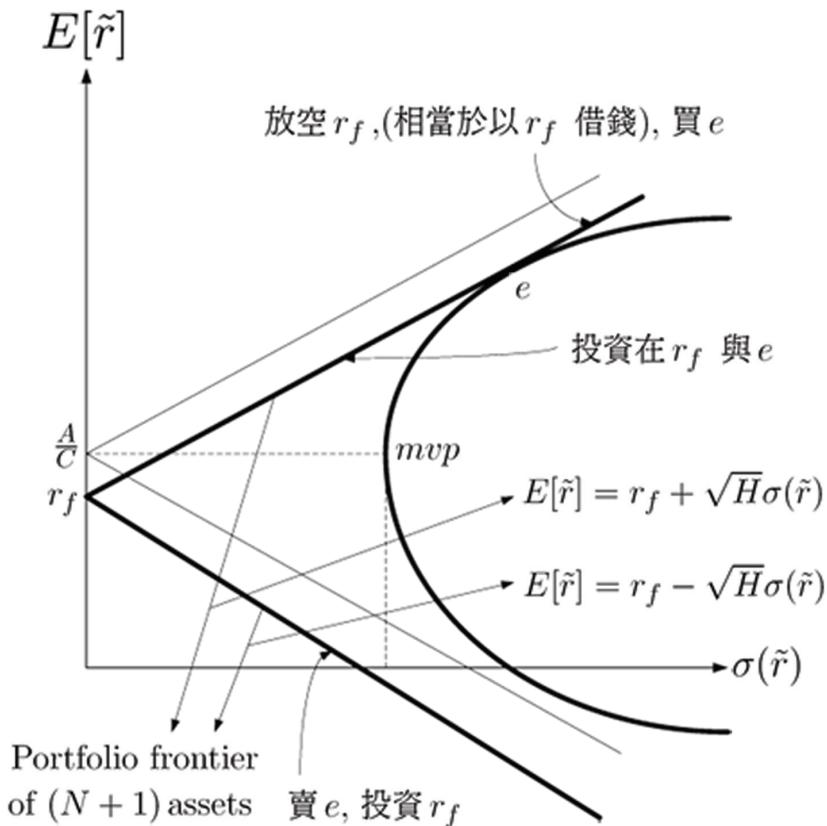
$$\begin{aligned}
& \text{課本之導法} \\
& \left\| \begin{aligned}
& \sigma^2(\tilde{r}_p) = \mathbf{w}_p^T V \mathbf{w}_p \\
& = (e - r_f \cdot \mathbf{1}_N)^T \cdot V^{-1} \cdot \frac{E[\tilde{r}_p] - r_f}{H} \cdot V \cdot V^{-1} (e - r_f \cdot \mathbf{1}_N) \frac{E[\tilde{r}_p] - r_f}{H} \\
& (\text{因 } H = (e - r_f \mathbf{1}_N)^T V^{-1} (e - r_f \mathbf{1}_N)) \\
& = \frac{(E[\tilde{r}_p] - r_f)^2}{H}
\end{aligned} \right.
\end{aligned}$$

在 N risky assets, 1 risk free asset 下, $E[\tilde{r}_p]$ 與 $\tilde{\sigma}_p$ 之間的關係.

$$\Rightarrow \sigma(\tilde{r}_p) = \begin{cases} \frac{E[\tilde{r}_p] - r_f}{\sqrt{H}} & \text{if } E[\tilde{r}_p] \geq r_f \\ -\frac{E[\tilde{r}_p] - r_f}{\sqrt{H}} & \text{if } E[\tilde{r}_p] < r_f \end{cases}$$

所有符合上述關係之 p , 即形成 N risky assets, 1 risk free asset 之 portfolio frontier

- Case 1: $r_f < \frac{A}{C}$ (圖)



要證明 e 點是切點，希望證明 (i) 與 (ii) 同時成立

$$(i) \frac{E[\tilde{r}_e] - r_f}{\sigma(\tilde{r}_e)} = \sqrt{H}$$

(ii) 且其中 e 點在 all risky assets 之 portfolio frontier 上

其中 (ii) 表示 e 點亦即符合 3.11.2a 或 3.11.2b

$$\Rightarrow \sigma^2(\tilde{r}_e) = \frac{1}{D}(CE[\tilde{r}_e]^2 - 2AE[\tilde{r}_e] + B) \quad (3.11.2b)$$

↓ by 配方法

$$\begin{aligned} & \frac{C}{D}[(E[\tilde{r}_e] - \frac{A}{C})^2 + \frac{D}{C^2}] \\ & = \frac{C}{D}(E[\tilde{r}_e] - \frac{A}{C})^2 + \frac{1}{C} \quad (3.11.2a) \end{aligned}$$

將 $\sigma(\tilde{r}_e)$ 想成 x , $E[\tilde{r}_e]$ 想成 y

$$\Rightarrow x^2 = \frac{C}{D}(y - \frac{A}{C})^2 + \frac{1}{C}$$

$$\Rightarrow x^2 - \frac{C}{D}(y - \frac{A}{C})^2 = \frac{1}{C}$$

過 $(\sigma(\tilde{r}_e), E[\tilde{r}_e])$ 之切線方程式

$$\sigma(\tilde{r}_e)x - \frac{C}{D}(E[\tilde{r}_e] - \frac{A}{C})(y - \frac{A}{C}) = \frac{1}{C}$$

當 $x = 0$ 時, 可求得相對應之 $y = \frac{A}{C} - \frac{\frac{D}{C^2}}{E[\tilde{r}_e] - \frac{A}{C}}$ (*)

若 $y = r_f \Rightarrow r_f = \frac{A}{C} - \frac{\frac{D}{C^2}}{E[\tilde{r}_e] - \frac{A}{C}}$ (意即是先有 r_f , 再決定 $E[\tilde{r}_e]$ 與 $\sigma(\tilde{r}_e)$ 之切點位置)

$$\Rightarrow E[\tilde{r}_e] = \frac{A}{C} - \frac{\frac{D}{C^2}}{r_f - \frac{A}{C}},$$

且 $\sigma^2(\tilde{r}_e) = \frac{1}{D}(CE[\tilde{r}_e]^2 - 2AE[\tilde{r}_e] + B)$

$$= \frac{H}{C^2(r_f - \frac{A}{C})^2} \quad (\text{參考下面說明})$$

$$\begin{aligned}
\sigma^2(\tilde{r}_e) &= \frac{1}{D}(CE[\tilde{r}_e]^2 - 2AE[\tilde{r}_e] + B) \\
&= \frac{C}{D}(E[\tilde{r}_e] - \frac{A}{C})^2 + \frac{1}{C} \\
&= \frac{C}{D}(\frac{D/C^2}{r_f-A/C})^2 + \frac{1}{C} \quad (\text{因 } E[\tilde{r}_e] = \frac{A}{C} - \frac{D/C^2}{r_f-A/C}) \\
&= \frac{\frac{D}{C^3}}{(r_f-A/C)^2} + \frac{1}{C} = \frac{\frac{D}{C}+C(r_f-A/C)^2}{C^2(r_f-A/C)^2} \\
&= \frac{\frac{D}{C}+Cr_f^2-2Ar_f+\frac{A^2}{C}}{C^2(r_f-A/C)^2} = \frac{Cr_f^2-2Ar_f+\frac{D+A^2}{C}}{C^2(r_f-A/C)^2} \\
&= \frac{Cr_f^2-2Ar_f+\frac{BC-A^2+A^2}{C}}{C^2(r_f-A/C)^2} = \frac{Cr_f^2-2Ar_f+B}{C^2(r_f-A/C)^2} \\
&= \frac{H}{C^2(r_f-A/C)^2} \\
\Rightarrow \sigma(\tilde{r}_e) &= -\frac{\sqrt{H}}{C(r_f-A/C)} \\
(\text{取負號是因為 } r_f &< \frac{A}{C}, \text{ 但 } \sigma(\tilde{r}_e) \text{ 需為正})
\end{aligned}$$

if $\frac{E[\tilde{r}_e]-r_f}{\sigma(\tilde{r}_e)} = \sqrt{H}$, 則得證

$$\begin{aligned}
E[\tilde{r}_e] \text{ 與 } \sigma(\tilde{r}_e) \text{ 代入} \Rightarrow \frac{E[\tilde{r}_e]-r_f}{\sigma(\tilde{r}_e)} &= \left(\frac{A}{C} - \frac{\frac{D}{C^2}}{r_f-\frac{A}{C}} - r_f\right)\left(-\frac{C(r_f-\frac{A}{C})}{\sqrt{H}}\right) \\
&= \left(-\frac{H}{Cr_f-A}\right)\left(-\frac{Cr_f-A}{\sqrt{H}}\right) = \sqrt{H} \\
(\text{其中, } H &= B - 2Ar_f + Cr_f^2)
\end{aligned}$$

根據課本(3.11.1),

$$\text{Cov}(\tilde{r}_p, \tilde{r}_q) = \mathbf{w}_p^T V \mathbf{w}_q = \frac{C}{D}(E[\tilde{r}_p] - \frac{A}{C})(E[\tilde{r}_q] - \frac{A}{C}) + \frac{1}{C}$$

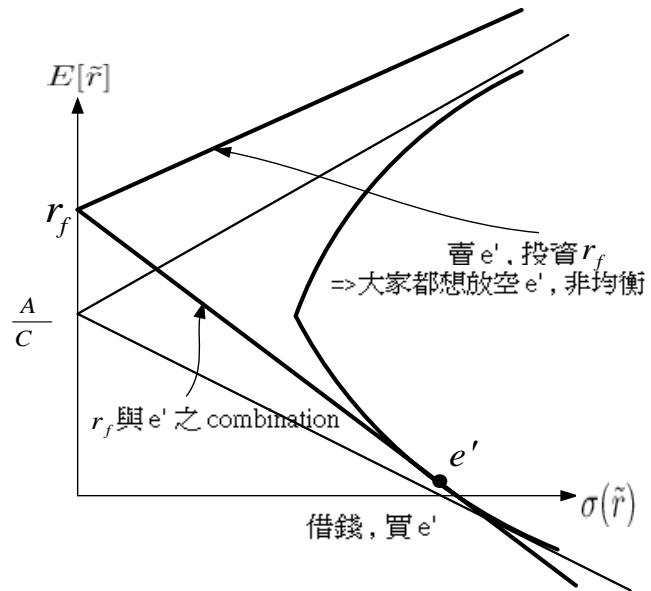
若找 $q = zc(p)$, s.t. $\text{Cov}(\tilde{r}_p, \tilde{r}_{zc(p)}) = 0$

$$\Rightarrow E[\tilde{r}_{zc(p)}] = \frac{A}{C} - \frac{\frac{D}{C^2}}{E[\tilde{r}_p] - \frac{A}{C}}$$

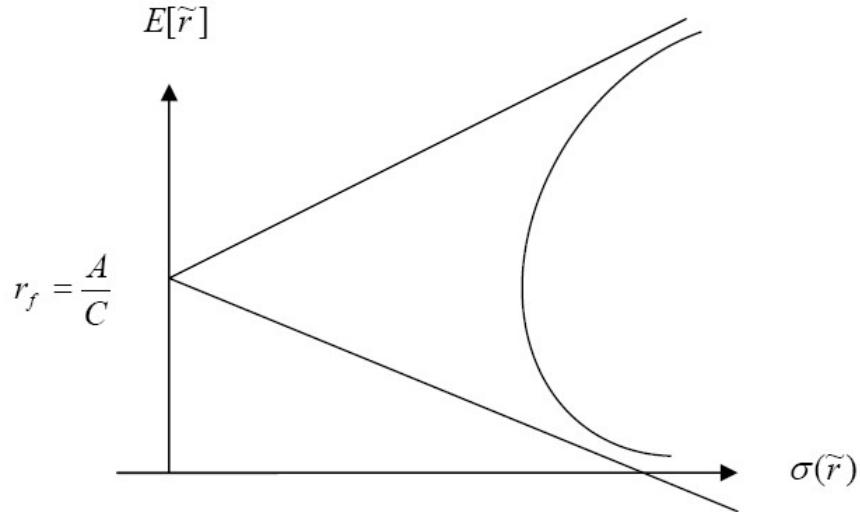
$E[\tilde{r}_{zc(p)}]$ 與上面的的(*)式剛好相等

如此也可看出 p 與 $zc(p)$ 在 $E[\tilde{r}] - \sigma(\tilde{r})$ 上之幾何關係

Case 2: $r_f > \frac{A}{C}$ (不會是均衡) (圖)



Case3: $r_f = \frac{A}{C}$ (不會是均衡) (圖)



$$\begin{aligned}
 &\text{follow the definition of } H = B - 2Ar_f + Cr_f^2 \\
 &= B - 2A\left(\frac{A}{C}\right) + C\left(\frac{A}{C}\right)^2 \\
 &= \frac{BC - A^2}{C} = \frac{D}{C} > 0
 \end{aligned}$$

⇒ 又在 all risky case 中, 兩條漸近線之方程式為:

$$E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}}\sigma(\tilde{r}_p) \quad (\text{由課本 (3.11.2a) 式可得})$$

⇒ 恰好與 N risky assets + 1 riskless asset

$$\sigma(\tilde{r}_p) = \begin{cases} \frac{E[\tilde{r}_p] - r_f}{\sqrt{H}}, & E[\tilde{r}_p] \geq r_f \\ -\frac{E[\tilde{r}_p] - r_f}{\sqrt{H}}, & E[\tilde{r}_p] < r_f \end{cases} = \begin{cases} \frac{E[\tilde{r}_p] - \frac{A}{C}}{\sqrt{\frac{D}{C}}}, & E[\tilde{r}_p] \geq \frac{A}{C} \\ -\frac{E[\tilde{r}_p] - \frac{A}{C}}{\sqrt{\frac{D}{C}}}, & E[\tilde{r}_p] < \frac{A}{C} \end{cases}$$

之 portfolio frontier 一樣 (因 $r_f = \frac{A}{C}$ 時, $H = \frac{D}{C}$)

又因為沒有切點, 所以 portfolio frontier 並非由 riskless asset 與 a portfolio on the portfolio frontier of all risky assets 來組成

那 portfolio frontier 到底爲何呢？

$$\begin{aligned} \text{將 } r_f = \frac{A}{C} \text{ 代入(3.18.1) } w_p &= V^{-1}(e - r_f 1_N)(\frac{E[\tilde{r}_p] - r_f}{H}) \text{ 並前乘 } 1_N \\ \Rightarrow 1_N^T w_p &= 1_N^T V^{-1}(e - \frac{A}{C} 1_N)(\frac{E[\tilde{r}_p] - r_f}{H}) \\ &= (A - \frac{A}{C} C)(\frac{E[\tilde{r}_p] - r_f}{H}) = 0 \end{aligned}$$

(where $1_N^T V^{-1} e$ 為 A, $1_N^T V^{-1} 1_N$ 為 C)

- $1_N^T w_p = 0$, 表 $w_0 = 1 \Rightarrow$ 投資全部在 r_f , 但並不表示沒投資在風險資產, 而是風險資產的部位有正有負, 但 weight 之合爲 0, 只要符合上述條件均爲 portfolio frontier

- 3.16節 q 為任一 portfolio; p 為 frontier portfolio.

$$\begin{aligned} \text{Cov}(\tilde{r}_p, \tilde{r}_q) &= w_p^T V w_q = (\lambda(V^{-1}e) + \gamma(V^{-1}1_N))^T V w_q \\ (w_p &= (\lambda(V^{-1}e) + \gamma(V^{-1}1_N)): \text{ FOC when there are } N \text{ risky assets}) \\ &= \lambda e^T V^{-1} V w_q + \gamma 1_N^T V^{-1} V w_q \\ &= \lambda e^T w_q + \gamma 1_N^T w_q \\ &= \lambda E[\tilde{r}_q] + \gamma \\ \Rightarrow E[\tilde{r}_q] &= \frac{\text{Cov}(\tilde{r}_p, \tilde{r}_q) - \gamma}{\lambda} \end{aligned}$$

$$\left\| \begin{aligned} &\text{又由之前的 FOC (3.9.3a) (3.9.3b)} \\ &\Rightarrow \lambda = \frac{CE[\tilde{r}_p] - A}{D}, \gamma = \frac{B - AE[\tilde{r}_p]}{D} \end{aligned} \right.$$

$$\begin{aligned} &= \frac{AE[\tilde{r}_p] - B}{CE[\tilde{r}_p] - A} + \text{Cov}(\tilde{r}_p, \tilde{r}_q) \frac{D}{CE[\tilde{r}_p] - A} \\ &= \frac{A}{C} - \frac{\frac{D}{C^2}}{E[\tilde{r}_p] - \frac{A}{C}} + \frac{\text{Cov}(\tilde{r}_p, \tilde{r}_q)}{\sigma^2(\tilde{r}_p)} \left[\frac{1}{C} + \frac{[E[\tilde{r}_p] - \frac{A}{C}]^2}{\frac{D}{C}} \right] \frac{D}{CE[\tilde{r}_p] - A} \\ &= E[\tilde{r}_{zc(p)}] + \beta_{qp} \left(\frac{\frac{D}{C^2}}{E[\tilde{r}_p] - A} + E[\tilde{r}_p] - \frac{A}{C} \right) \\ &= E[\tilde{r}_{zc(p)}] + \beta_{qp} (E[\tilde{r}_p] - E[\tilde{r}_{zc(p)}]) \\ &= (1 - \beta_{qp}) E[\tilde{r}_{zc(p)}] + \beta_{qp} E[\tilde{r}_p] \end{aligned}$$

- 3.17 節, $\tilde{r}_q = \beta_0 + \beta_1 \tilde{r}_{zc(p)} + \beta_2 \tilde{r}_p + \tilde{\varepsilon}_q$, where \tilde{r}_p , $\tilde{r}_{zc(p)}$, $\tilde{\varepsilon}_q$ are uncorrelated
- 3.19 節 q 為任一 portfolio; p 為 frontier portfolio 且 $E[\tilde{r}_p] \neq r_f$

$$\text{Cov}(\tilde{r}_q, \tilde{r}_p) = \mathbf{w}_q^T V \mathbf{w}_p$$

$$\left\| \begin{array}{l} \text{因 } E[\tilde{r}_q] = (\mathbf{1}_N^T \mathbf{w}_q) r_f + \mathbf{w}_q^T \mathbf{e} \\ \text{or } r_f + \mathbf{w}_q^T (\mathbf{e} - r_f \mathbf{1}_N) = E[\tilde{r}_q] \\ \Rightarrow \mathbf{w}_q^T (\mathbf{e} - r_f \mathbf{1}_N) = E[\tilde{r}_q] - r_f \\ \text{因 } p \text{ 在 portfolio frontier} \\ \Rightarrow \mathbf{w}_p = V^{-1} (\mathbf{e} - r_f \mathbf{1}_N)^{\frac{(E[\tilde{r}_p] - r_f)}{H}} \end{array} \right.$$

$$\Rightarrow \mathbf{w}_q^T V \mathbf{w}_p = \mathbf{w}_q^T V V^{-1} (\mathbf{e} - r_f \mathbf{1}_N)^{\frac{(E[\tilde{r}_p] - r_f)}{H}} = \frac{(E[\tilde{r}_q] - r_f)(E[\tilde{r}_p] - r_f)}{H}$$

$$\Rightarrow E[\tilde{r}_q] - r_f = \frac{\text{Cov}(\tilde{r}_q, \tilde{r}_p)}{(E[\tilde{r}_p] - r_f)^2 / H} (E[\tilde{r}_p] - r_f)$$

(where $\frac{(E[\tilde{r}_p] - r_f)^2}{H} = \sigma^2(\tilde{r}_p)$, 因 portfolio p 在 N risky assets+ 1 riskless asset 之 portfolio frontier 上)

$$\Rightarrow E[\tilde{r}_q] - r_f = \beta_{qp} (E[\tilde{r}_p] - r_f), \text{ 其中 } \beta_{qp} = \frac{\text{Cov}(\tilde{r}_q, \tilde{r}_p)}{\sigma^2(\tilde{r}_p)}$$

$$\tilde{r}_q = (1 - \beta_{qp}) r_f + \beta_{qp} \tilde{r}_p + \tilde{\varepsilon}_{qp} \text{ with } \text{Cov}(\tilde{r}_p, \tilde{\varepsilon}_{qp}) = E[\tilde{\varepsilon}_{qp}] = 0$$

(CAPM 之先驅)