

Ch2. Stochastic Dominance

- 兩個風險資產的風險性比較

ω_i	1	2	3
$P(\omega_i)$	0.4	0.4	0.2
\tilde{r}_X	10%	50%	50%
\tilde{r}_Y	10%	50%	100%

$$E[\tilde{r}_X] = 34\%, \sigma_{\tilde{r}_X} = 19.60\%$$

$$E[\tilde{r}_Y] = 44\%, \sigma_{\tilde{r}_Y} = 33.23\%$$

如果只看 mean 與 variance, 無法分辨 X 好或 Y 好, 但如果細看 X 與 Y 之 return distribution, 所有人都會覺得 Y 好

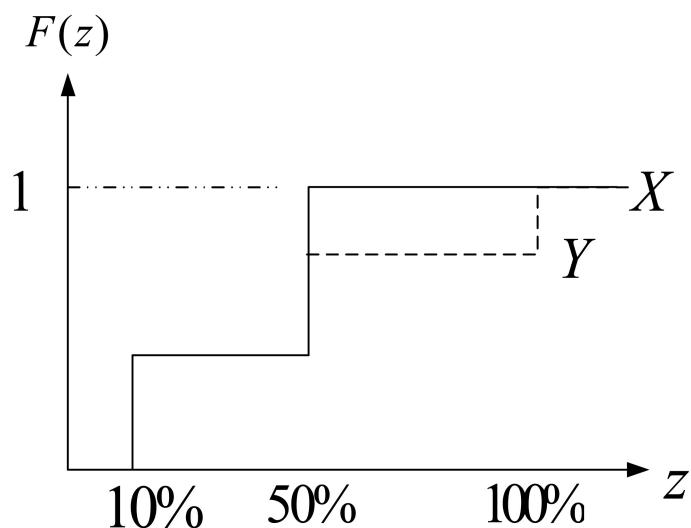
- Let $F_X(\cdot)$, $F_Y(\cdot)$ denote cumulative distribution functions of the rates of return on X and Y

$F_i(\bar{z}) = \text{prob}(z \leq \bar{z}), \bar{z} \in [L, U]$ ($L=0, U=1$ in this chapter)

(圖 $\Rightarrow Y \geq X$)

FSDX: 因 higher returns are more likely)

(FSD: First Degree Stochastic Dominance)



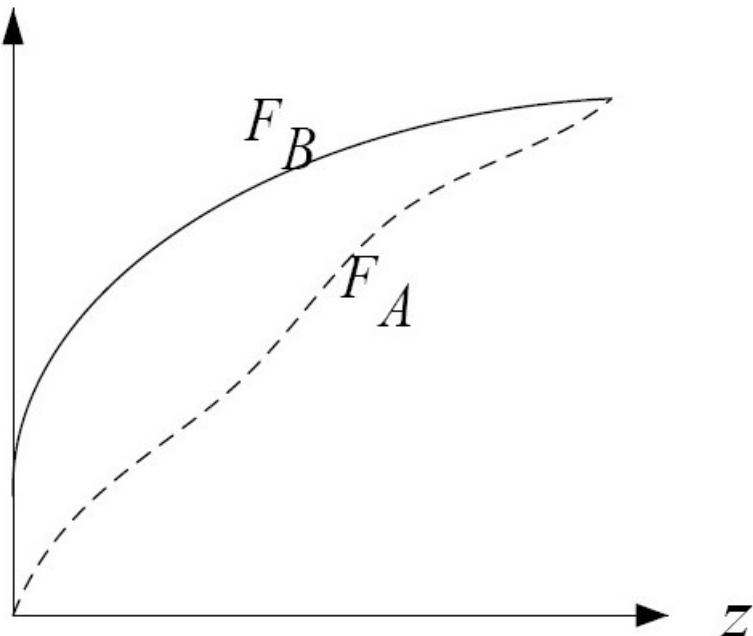
$A \stackrel{FS}{\geq} B$ iff $F_A(z) \leq F_B(z), \forall z$ (圖)

$\iff E[u(1 + \tilde{r}_A)] \geq E[u(1 + \tilde{r}_B)],$ 亦即

$$\int_{[L,U]} u(1 + z) dF_A(z) \geq \int_{[L,U]} u(1 + z) dF_B(z),$$

where u is increasing and continuous

prob



$$(\implies) E[u(1 + \tilde{r}_A)] \geq E[u(1 + \tilde{r}_B)]$$

$$\Rightarrow \int_{[0,1]} u(1 + z) dF_A(z) \geq \int_{[0,1]} u(1 + z) dF_B(z)$$

$$\Rightarrow u(1)F_A(0) + \int_0^1 u(1 + z) dF_A(z) \geq u(1)F_B(0) + \int_0^1 u(1 + z) dF_B(z)$$

$$\Rightarrow u(1)(F_A(0) - F_B(0)) + \int_0^1 u(1 + z) d[F_A(z) - F_B(z)]$$

$$\stackrel{\text{integration by parts}}{=} u(1)(F_A(0) - F_B(0)) + u(1 + z)(F_A(z) - F_B(z)) \Big|_0^1 \\ - \int_0^1 [F_A(z) - F_B(z)] du(1 + z)$$

$$\begin{aligned}
& \text{因 } F_A(1) = F_B(1) = 1 \\
& \Rightarrow u(1+z)(F_A(z) - F_B(z))|_0^1 \\
& = u(2)(F_A(1) - F_B(1)) - u(1)(F_A(0) - F_B(0)) \\
& = 0 - u(1)(F_A(0) - F_B(0))
\end{aligned}$$

$$= - \int_0^1 [F_A(z) - F_B(z)] du(1+z) \geq 0$$

因 $du(1+z) \geq 0$, 且 $F_A(z) - F_B(z) \leq 0$, 此式 ≥ 0 成立

得証當 $F_A(z) \leq F_B(z) \Rightarrow E[u(1+\tilde{r}_A)] \geq E[u(1+\tilde{r}_B)]$

(\Leftarrow) 用反証法, 當 $E[u(1+\tilde{r}_A)] \geq E[u(1+\tilde{r}_B)]$ 成立,

但 $\exists x, F_A(x) > F_B(x)$

假設存在區間 $[x, c] \in [0, 1]$ s.t. $F_A(z) > F_B(z), \forall z \in [x, c]$

(參考 p. 43 Figure 2.3.1)

找一個 utility (continuous and increasing), 使得 $E[u(1+\tilde{r}_A)] \geq E[u(1+\tilde{r}_B)]$ 不成立

假設 $u(1+z) = \int_0^z 1_{[1+x, 1+c]}(1+t) dt$ (參考 p. 44 Figure 2.3.2)

$$\Rightarrow u'(1+z) = 1_{[1+x, 1+c]}(1+z) =$$

$$\begin{cases} 1 & \text{if } 1+x \leq 1+z \leq 1+c \\ 0 & \text{otherwise} \end{cases}$$

($\Rightarrow u' \geq 0$ 成立)

此時 $E[u(1+\tilde{r}_A)] - E[u(1+\tilde{r}_B)]$

$$= - \int_0^1 [F_A(z) - F_B(z)] du(1+z) \quad (\text{課本 2.3.1})$$

$$= - \int_x^c [F_A(z) - F_B(z)] dz < 0 \quad (\text{因 } F_A(z) > F_B(z) \text{ in } [x, c])$$

$\rightarrow \leftarrow$, 所以當 $E[u(1+\tilde{r}_A)] \geq E[u(1+\tilde{r}_B)] \Rightarrow F_A(z) \leq F_B(z)$

得証

- 1. $A \stackrel{FSD}{\geq} B \Leftrightarrow E[u(1 + \tilde{r}_A)] \geq E[u(1 + \tilde{r}_B)]$, u is increasing
- 2. $F_A(z) \leq F_B(z), \forall z \in [L, U]$
- 3. $\tilde{r}_A \stackrel{d}{=} \tilde{r}_B + \tilde{\alpha}, \tilde{\alpha} > 0$

這三個 statements 是 equivalent

1與2等價之前証過了

$$3 \Rightarrow 1 \quad E[u(1 + \tilde{r}_A)] = E[u(1 + \tilde{r}_B + \tilde{\alpha})] \geq E[u(1 + \tilde{r}_B)]$$

(因 $\tilde{\alpha} > 0$, 且 $u' \geq 0$)

1 \Rightarrow 3 課本沒證明, 不過直覺上應該是對的

由 3 可知 $A \stackrel{FSD}{\geq} B \Rightarrow E[\tilde{r}_A] = E[\tilde{r}_B] + E[\tilde{\alpha}] \Rightarrow E[\tilde{r}_A] \geq E[\tilde{r}_B]$ (反之則未必)

(此外, $E[\tilde{r}_A] \geq E[\tilde{r}_B]$ 與 $E[u(1 + \tilde{r}_A)] \geq E[u(1 + \tilde{r}_B)]$ 意思不同)

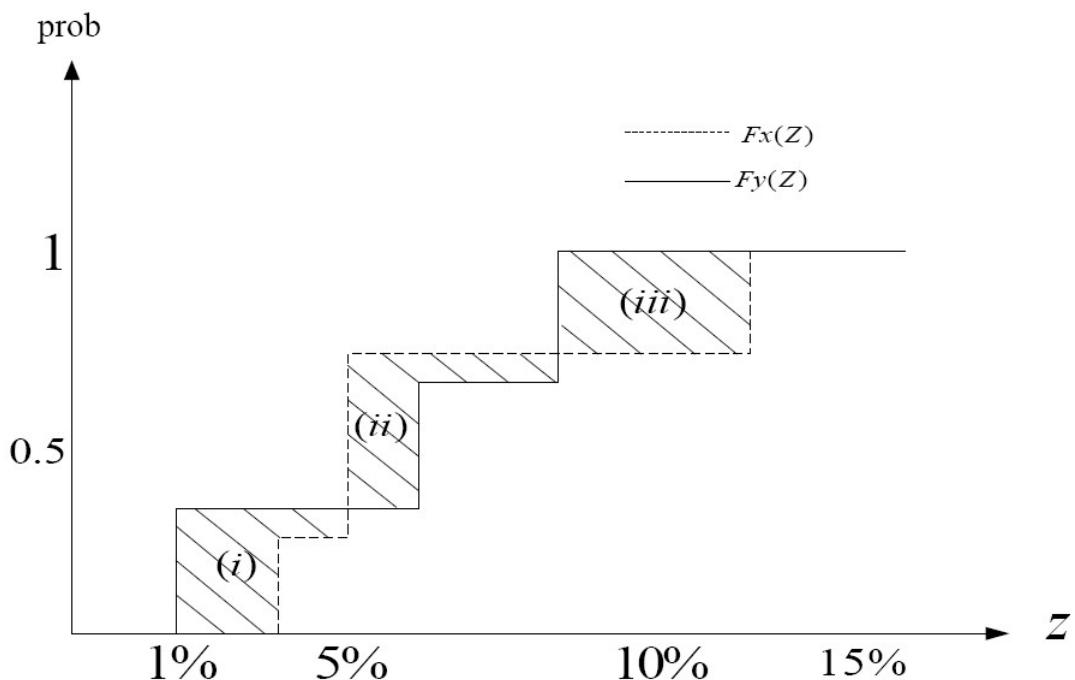
- 上面所述皆在 u is increasing (individual is nonsatiable) 之情況下, 試想一個 risk averse (未必需 increasing), 到底會 prefers A or B 呢?

- 考慮一個例子

	prob	0.25	0.5	0.25	prob	$1/3$	$1/3$	$1/3$
	r_X	4%	5%	12%	r_Y	1%	6%	8%

$$E[r_X] = 6.5\%, \sigma_{r_X} = 3.2\%, E[r_Y] = 5\%, \sigma_{r_Y} = 2.9\%$$

(此時一般人會選擇 X , 但由圖無法看出 $F_X(z) \leq F_Y(z), \forall z$, 亦即並非 $X \stackrel{FSD}{\geq} Y \Rightarrow$ 此時 FSD 不能用)



$$X - Y$$

$$(i) = -\frac{13}{12}\%$$

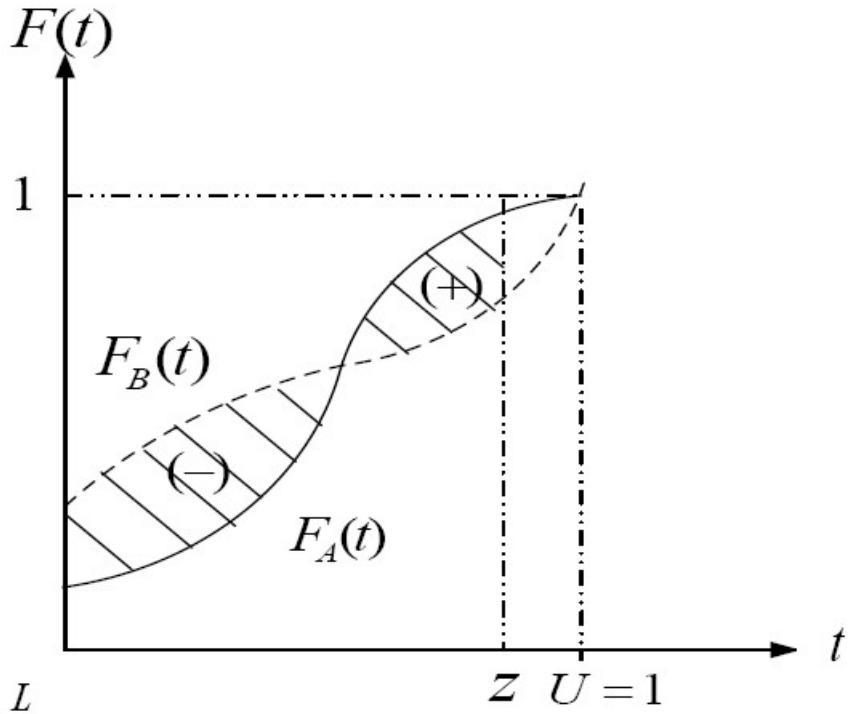
$$(ii) = \frac{7}{12}\%$$

$$(iii) = -1\%$$

$$\Rightarrow (i) + (ii) = -\frac{1}{2}\%$$

$$\Rightarrow (i) + (ii) + (iii) = -\frac{3}{2}\%$$

- A $\overset{>}{SSD}$ B $\Leftrightarrow E[\tilde{r}_A] = E[\tilde{r}_B], S(z) = \int_L^z (F_A(t) - F_B(t))dt \leq 0, \forall z \in [L, U]$
 $\Leftrightarrow E[u(1 + \tilde{r}_A)] \geq E[u(1 + \tilde{r}_B)], u$ is concave ($u'' < 0$)



- 證明 if A $\overset{>}{SSD}$ B $\Rightarrow E[u(1 + \tilde{r}_A)] \geq E[u(1 + \tilde{r}_B)],$
given $E[\tilde{r}_A] = E[\tilde{r}_B]$ and $u'' < 0$
from (2.3.1)

$$\begin{aligned}
&\Rightarrow \int_{[0,1]} u(1+z)d[F_A(z) - F_B(z)] \\
&= - \int_0^1 [F_A(z) - F_B(z)]du(1+z) \\
&= - \int_0^1 [F_A(z) - F_B(z)]u'(1+z)dz \\
&= - \int_0^1 u'(1+z)[F_A(z) - F_B(z)]dz \\
&= - \int_0^1 u'(1+z)dS(z)
\end{aligned}$$

$$\stackrel{\text{integration by parts}}{=} -u'(1+z)S(z) \Big|_0^1 + \int_0^1 S(z)du'(1+z)$$

$$\begin{aligned} & \text{因 } S(0) = \int_{[0,0]} [F_A(z) - F_B(z)] dz = 0 \\ & S(1) = \int_{[0,1]} [F_A(z) - F_B(z)] dz \stackrel{S(0)=0}{=} \int_0^1 [F_A(z) - F_B(z)] dz \\ & \stackrel{\text{integration by parts}}{=} [F_A(z) - F_B(z)]z \Big|_0^1 - \int_0^1 z d[F_A(z) - F_B(z)] \\ & = [F_A(1) - F_B(1)] \cdot 1 - [F_A(0) - F_B(0)] \cdot 0 \\ & \quad - \int_0^1 z dF_A(z) + \int_0^1 z dF_B(z) \\ & = 0 - 0 - E[\tilde{r}_A] + E[\tilde{r}_B] = 0 \end{aligned}$$

$$= \int_0^1 S(z)du'(1+z) = \int_0^1 S(z)u''(1+z)dz \geq 0$$

- 證明 $E[u(1+\tilde{r}_A)] \geq E[u(1+\tilde{r}_B)] \Rightarrow A \underset{SSD}{\geq} B$ (亦即 $S(z) < 0, \forall z$)

反證法, assume $\exists z, S(z) > 0$, 看是否會造成 $E[u(1+\tilde{r}_A)] < E[u(1+\tilde{r}_B)]$,
如果會 \Rightarrow 產生矛盾 $\Rightarrow S(z) < 0, \forall z$

assume $\exists [a, b]$ s.t. $S(z) > 0, \forall z \in [a, b]$

define $u''(1+z) = -1_{[1+a, 1+b]}(1+z)$

$$\Rightarrow u'(1+z) = \int_0^z -1_{[1+a, 1+b]}(1+t)dt$$

$$\stackrel{\text{往上平移(b-a)}}{\Rightarrow} u'(1+z) = \int_z^1 1_{[1+a, 1+b]}(1+t)dt$$

$$\text{from (2.6.3)} \Rightarrow \int_{[0,1]} u(1+z)d[F_A(z) - F_B(z)]$$

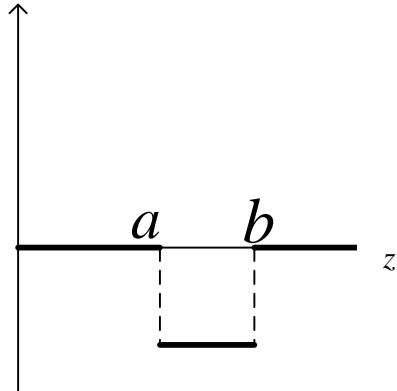
$$= \int_0^1 S(z)du'(1+z)$$

$$= \int_0^1 S(z)u''(1+z)dz$$

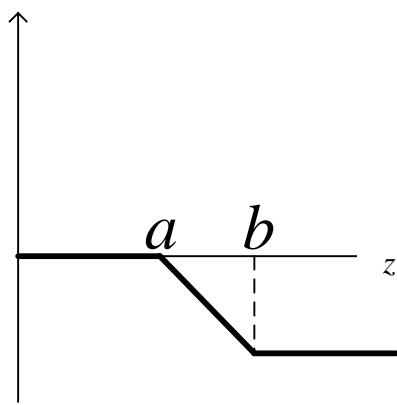
$$= - \int_{[a,b]} S(z)dz < 0$$

$\rightarrow \leftarrow$, 由反證法得證 $S(z) < 0, \forall z$

$$u''(1+z) = -1_{[1+a,1+b]}(1+z)$$

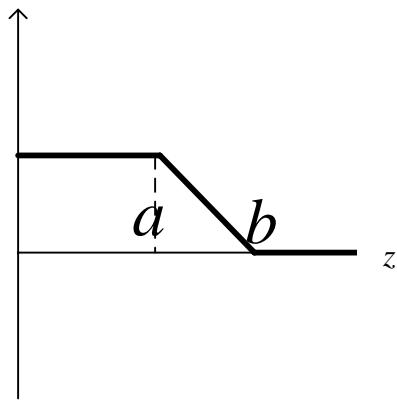


$$u'(1+z) = \int_0^z -1_{[1+a,1+b]}(1+t) dt$$



$$u'(1+z) \leftarrow u'(1+z) + (b - a)$$

$$u'(1+z) = \int_z^1 1_{[1+a,1+b]}(1+t) dt$$

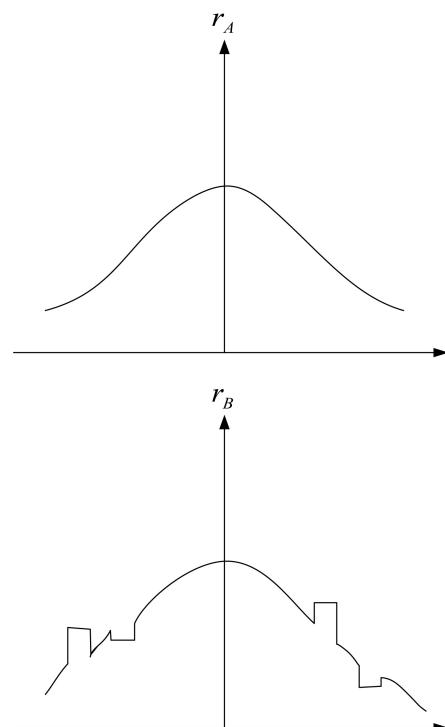


- 例子: (續)

$z(\%)$	$F_X(t)$	$\int_0^z F_X(t)dt(\%)$	$F_Y(t)$	$\int_0^z F_Y(t)dt(\%)$	$S(z)(\%)$
0	0	0	0	0	0
1	0	0	1/3	1/3	-1/3
2	0	0	1/3	2/3	-2/3
3	0	0	1/3	1	-1
4	0.25	0.25	1/3	4/3	-13/12
5	0.75	1	1/3	5/3	-2/3
6	0.75	1.75	2/3	7/3	-7/12
7	0.75	2.5	2/3	3	-1/2
8	0.75	3.25	1	4	-3/4
9	0.75	4	1	5	-1
10	0.75	4.75	1	6	-5/4
11	0.75	5.5	1	7	-3/2
12	1	6.5	1	8	-3/2
13	1	7.5	1	9	-3/2

(where $S(z) = \int_0^z (F_X(t) - F_Y(t))dt$)

- 1. $A \geq_{SSD} B \Leftrightarrow E[u(1 + \tilde{r}_A)] \geq E[u(1 + \tilde{r}_B)]$, 而且 $E[\tilde{r}_A] = E[\tilde{r}_B]$, u is concave.
 - 2. $E[\tilde{r}_A] = E[\tilde{r}_B]$ and $S(z) \leq 0, \forall z \in [L, U]$
 - 3. $\tilde{r}_B \stackrel{d}{=} \tilde{r}_A + \tilde{\varepsilon}$, with $E[\tilde{\varepsilon}|\tilde{r}_A] = 0$ (圖)
- 以上三個 statements 是 equivalent



B 比 A 風險高

- 1 與 2 等價, 之前已經證明過了
- $3 \Rightarrow 1$, 亦即希望證明當 $\tilde{r}_B \stackrel{d}{=} \tilde{r}_A + \tilde{\varepsilon}$, 且 $E[\tilde{\varepsilon}|\tilde{r}_A] = 0$
 $\Rightarrow E[\tilde{r}_B] = E[\tilde{r}_A]$, 且 $E[u(1 + \tilde{r}_B)] \leq E[u(1 + \tilde{r}_A)]$, $u'' < 0$
證明如下: $E[u(1 + \tilde{r}_B)] = E[u(1 + \tilde{r}_A + \tilde{\varepsilon})] = E[E[u(1 + \tilde{r}_A + \tilde{\varepsilon})|\tilde{r}_A]]$
 - || Jensen's inequality
if u is concave $\Rightarrow E[u(\cdot)] \leq u(E[\cdot])$
$$\begin{aligned} &\leq E[u(E[1 + \tilde{r}_A + \tilde{\varepsilon}|\tilde{r}_A])] = E[u(1 + \tilde{r}_A + E[\tilde{\varepsilon}|\tilde{r}_A])] \\ &= E[u(1 + \tilde{r}_A)], \text{ 故得証} \end{aligned}$$
- $1 \Rightarrow 3$ 很複雜, 請參考 Rothschild and Stiglitz (1970)
- 由 statement 3 可以知道 if $A \geq_{SSD} B \Rightarrow E[\tilde{r}_A] = E[\tilde{r}_B]$ and $\text{Var}(\tilde{r}_A) \leq \text{Var}(\tilde{r}_B)$
 - (i) $E[\tilde{\varepsilon}] = E[E(\tilde{\varepsilon}|\tilde{r}_A)] = E[0] = 0 \Rightarrow E[\tilde{r}_A] = E[\tilde{r}_B]$
 - (ii) $\text{Cov}(\tilde{r}_A, \tilde{\varepsilon}) = E[\tilde{r}_A \cdot \tilde{\varepsilon}] - E[\tilde{r}_A]E[\tilde{\varepsilon}]$
 - || 因 $E[\tilde{r}_A \cdot \tilde{\varepsilon}] = E[E[\tilde{r}_A \cdot \tilde{\varepsilon}|\tilde{r}_A]]$
 $= E[\tilde{r}_A E[\tilde{\varepsilon}|\tilde{r}_A]] = E[0] = 0$
 - $= 0$

因此 $\text{Var}(\tilde{r}_B) = \text{Var}(\tilde{r}_A) + \text{Var}(\tilde{\varepsilon})$
 $\Rightarrow \text{Var}(\tilde{r}_A) \leq \text{Var}(\tilde{r}_B)$
- $A \geq_{SSD}^M B \Leftrightarrow E[\tilde{r}_A] \geq E[\tilde{r}_B]$, $S(z) \leq 0, \forall z \in [L, U]$
 $\Leftrightarrow E[u(1 + \tilde{r}_A)] \geq E[u(1 + \tilde{r}_B)]$, u is nonsatiable ($u' > 0$)
and risk averse ($u'' < 0$)
- 1. $A \geq_{SSD}^M B \Leftrightarrow E[u(1 + \tilde{r}_A)] \geq E[u(1 + \tilde{r}_B)]$, u is concave and increasing
2. $E[\tilde{r}_A] \geq E[\tilde{r}_B]$ and $S(z) \leq 0, \forall z \in [L, U]$
3. $\tilde{r}_B \stackrel{d}{=} \tilde{r}_A + \tilde{\varepsilon}$, with $E[\tilde{\varepsilon}|\tilde{r}_A] \leq 0$
以上三個 statements 是 equivalent

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prob	0.25	0.5	0.25	prob	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
r_X	1%	7%	12%	r_Y	3%	5%	8%

$$E[r_X] = 6.75\%, \sigma_{r_X} = 3.90\%, E[r_Y] = 5.33\%, \sigma_{r_Y} = 2.055\%$$

Neither FSD nor SSD

(Stochastic dominance is useful in comparing the riskiness of risky assets. However, it does not define a complete order among risky assets.)

- All risk-averse agents will prefer SSD assets

FSD \Rightarrow SSD

FSD $\not\Rightarrow$ SSD

- 2.10 節講, A $\overset{\geq}{SSD}$ B 下, 當 risky asset 之 riskiness 上升 (從 A \rightarrow B), 投資人投資在 risky asset 之比率 a 未必下降. 除非, $R_R(z) < 1$, $R'_R(z) > 0$, 且 $R'_A(z) < 0$ ($\Rightarrow u''' > 0$)

(1.20 節與 1.26 節之結果隱含的是當人變的更 risk averse 時, a 會下降)

- Suppose A $\overset{\geq}{SSD}$ B, 且對某個 individual with u 而言,

他會投資 a 單位在資產 A 上

$$\Rightarrow E[u'((1 + r_f) + a(\tilde{r}_A - r_f))(\tilde{r}_A - r_f)] = 0 \quad (1)$$

若將 risky asset 由 A 改為 B, 則 individual 會投資 $< a$ 在資產 B 嗎?

$$\text{亦即 } E[u'((1 + r_f) + a(\tilde{r}_B - r_f))(\tilde{r}_B - r_f)] \leq 0 \quad (2)$$

Define $u'((1 + r_f) + a(z - r_f))(z - r_f) = V(z)$

$$(1) \Rightarrow \int_{[0,1]} V(z) dF_A(z) = 0$$

$$(2) \Rightarrow \int_{[0,1]} V(z) dF_B(z) \leq 0 \quad (\leq 0 \text{ 要成立, 需 } V(z) \text{ is concave},$$

此時 RRA 會 <1 , RRA 是 increasing, 且 ARA 是 decreasing.)

參考課本 p.51 (2.10.5)

- Arrow-Ptatt 在 Ch1 證明, 若由投資人之 risk averse attitude 來看, 若 $R_A^i(z) \geq R_A^k(z), \forall z$, 則 i 投資在 risky asset 之比率一定低於 k 投資在 risky asset 之比率, 若 i, k 都投資全部之 wealth 在 risky asset, 則 i 所需的最小 risk premium 會大於 k 所需的 risk premium. (當兩個 asset 都是 risky asset, 則上述不成立)
- Suppose $\tilde{r}_A = \tilde{r}_B + \tilde{z}$, 且 $E[\tilde{z} | \tilde{r}_B] \geq 0$
 $\Rightarrow E[\tilde{r}_A] > E[\tilde{r}_B]$ (因 $E[\tilde{z}] = E[E[\tilde{z} | \tilde{r}_B]] \geq 0$)
 且 $\text{Var}(\tilde{r}_A) > \text{Var}(\tilde{r}_B)$ if \tilde{z} 與 \tilde{r}_B independent (亦即 A 比 B 更 risky 且同時也擁有較高之報酬)

Example

$$\tilde{z} = \begin{cases} 2, \text{with prob } = \frac{1}{2} \\ -1, \text{with prob } = \frac{1}{2} \end{cases} \Rightarrow E[\tilde{z}] = E[\tilde{z} | \tilde{r}_B] = \frac{1}{2} > 0$$

$$\tilde{r}_B = \begin{cases} 1, \text{with prob } = \frac{1}{2} \\ 0, \text{with prob } = \frac{1}{2} \end{cases}$$

p	\tilde{z}	$1 + \tilde{r}_B + a\tilde{z}$	$a^* = \frac{1}{4}$
$\frac{1}{4}$	2	$2 + 2a$	$\frac{5}{2}$
$\frac{1}{4}$	2	$1 + 2a$	$\frac{3}{2}$
$\frac{1}{4}$	-1	$2 - a$	$\frac{7}{4}$
$\frac{1}{4}$	-1	$1 - a$	$\frac{3}{4}$

given $u'_k(\frac{5}{2}) = 0, u'_k(\frac{7}{4}) = 2, u'_k(\frac{3}{2}) = 3, u'_k(\frac{3}{4}) = 4$

u' is decreasing $\Rightarrow u$ is concave ($u'' < 0$)

$\frac{\partial}{\partial a} \Rightarrow E[u'_k(1 + \tilde{r}_B + a\tilde{z})\tilde{z}] = 0 \Rightarrow a^* = \frac{1}{4}$ (由上面 u'_k 之設定與 \tilde{z} 和 \tilde{r}_B 之設定可以看出)

(表 $\frac{1}{4}$ 投到風險高的資產 A)

Let G be a concave function

$$G'(u_k(\frac{5}{2})) = 0, G'(u_k(\frac{7}{4})) = 0, G'(u_k(\frac{3}{2})) = 10, G'(u_k(\frac{3}{4})) = 10,$$

$u_k(\cdot)$ is increasing, G' is decreasing ($\Rightarrow G$ is concave, $G'' < 0$),

又 $u_i = G(u_k)$, 表 i 比 k 更 risk averse, 應投資更少在 \tilde{z} .

$$E[u'_i(1 + \tilde{r}_B + \frac{1}{4}\tilde{z})\tilde{z}] = E[G'(u_k(1 + \tilde{r}_B + \frac{1}{4}\tilde{z}))u'_k(1 + \tilde{r}_B + \frac{1}{4}\tilde{z})\tilde{z}] = 5 > 0$$

(> 0 表示想要投資更多到 A) (與 Arrow-Pratt 相衝突)

- 2.12 ~ 2.14 節, 改用 Ross 的 definition of strongly more risk aversion, 則上述的兩個 risky 之問題可解決.

(i) 2.12 證明 Ross 之第一點可推到 AP 之第一點.

(ii) 2.13 證明 Ross 之第二點與第一點是等價的.

(iii) 2.14 證明若 i is strongly more risk averse than k , 則當 A 比 B 更 risky 時, i 投資在 A 之比例較小.

- 証 Ross 之第一點可推到 AP 之第一點

from Ross, i is strongly more risk averse than k

$$\text{if } \inf_z \underbrace{\left(\frac{u''_i(z)}{u''_k(z)}\right)}_{(1)} \geq \sup_z \underbrace{\left(\frac{u'_i(z)}{u'_k(z)}\right)}_{(2)}$$



滿足 (1) 條件下界的 z 滿足 (2) 條件上界的 z

$$\Rightarrow \text{for any } z, \frac{u''_i(z)}{u''_k(z)} \geq \frac{u'_i(z)}{u'_k(z)} \Rightarrow -\frac{u''_i(z)}{u'_i(z)} \geq -\frac{u''_k(z)}{u'_k(z)}$$

$$\Rightarrow R_A^i \geq R_A^k$$

- 但 AP 之第一點不可推到 Ross 之第一點

例: $u_i(z) = -e^{(-az)}$, $u_k(z) = -e^{-(bz)}$, and $a > b$

$$\Rightarrow \frac{u'_i(z_1)}{u'_k(z_1)} = \frac{a}{b} e^{-(a-b)z_1}, \frac{u''_i(z_2)}{u''_k(z_2)} = (\frac{a}{b})^2 e^{-(a-b)z_2}$$

當 $z_2 - z_1$ 很大時, 可造成 $\frac{u''_i(z_2)}{u''_k(z_2)} < \frac{u'_i(z_1)}{u'_k(z_1)}$

- 證明 Ross 的第一點與第二點是 equivalent

$$2 \Rightarrow 1$$

if $u_i(z) = \lambda \cdot u_k(z) + G(z)$, if $\lambda > 0$, and $G' < 0, G'' < 0$

$$\Rightarrow u'_i(z) = \lambda u'_k(z) + G'(z) \leq \lambda u'_k(z) (\Rightarrow \frac{u'_i(z)}{u'_k(z)} \leq \lambda)$$

and $u''_i(z) = \lambda u''_k(z) + G''(z) \leq \lambda u''_k(z) (\Rightarrow \frac{u''_i(z)}{u''_k(z)} \geq \lambda)$

$$\Rightarrow \frac{u''_i(z)}{u''_k(z)} \geq \lambda \geq \frac{u'_i(z)}{u'_k(z)}$$

$$1 \Rightarrow 2$$

1 成立 $\Rightarrow u'_i(z) - \lambda u'_k(z) \leq 0$ (找 $G'(z) = u'_i(z) - \lambda u'_k(z) \leq 0$)

$u''_i(z) - \lambda u''_k(z) \leq 0$ (找 $G''(z) = u''_i(z) - \lambda u''_k(z) \leq 0$)

則如此定義之 G' 與 G'' 可滿足 $u_i(z) = \lambda u_k(z) + G(z)$

- 證明 if i is strongly more risk averse than k , 則當 A 比 B 更 risky 時, i 投資在 A 之比例小

Suppose $E[u'_k(1 + \tilde{r}_B + a\tilde{z})\tilde{z}] = 0$ ($\tilde{z} = \tilde{r}_A - \tilde{r}_B$)

$$E[u'_i(1 + \tilde{r}_B + a\tilde{z})\tilde{z}]$$

$$= E[\lambda u'_k(1 + \tilde{r}_B + a\tilde{z})\tilde{z} + G'(1 + \tilde{r}_B + a\tilde{z})\tilde{z}]$$

$$= E[G'(1 + \tilde{r}_B + a\tilde{z})\tilde{z}]$$

$$= E[E[G'(1 + \tilde{r}_B + a\tilde{z})\tilde{z} | \tilde{r}_B]]$$

$$= E[\text{Cov}(G'(1 + \tilde{r}_B + a\tilde{z}), \tilde{z} | \tilde{r}_B) + E[G'(1 + \tilde{r}_B + a\tilde{z}) | \tilde{r}_B]E[\tilde{z} | \tilde{r}_B]]$$

$$\leq E[\text{Cov}(G'(1 + \tilde{r}_B + a\tilde{z}), \tilde{z} | \tilde{r}_B)] (\text{因 } G' < 0, E[\tilde{z} | \tilde{r}_B] \geq 0)$$

$$\leq 0 \quad (\text{因 } G'' < 0 \Rightarrow \tilde{z} \uparrow, G' \downarrow \Rightarrow \text{Cov}(\tilde{z}, G') \leq 0)$$

\Rightarrow 對 i 而言, 對於投資 A, 其最佳之選擇是 $< a$, which is the optimal choice of k .

- A \geq_{TSD} B $\Leftrightarrow E[u(1 + \tilde{r}_A)] \geq E[u(1 + \tilde{r}_B)],$
 for all decreasing $R_A(z)$ utility function.
 $\Leftrightarrow E[\tilde{r}_A] = E[\tilde{r}_B], T(t) = \int_L^t S(z)dz \leq 0, \forall t \in [L, U]$