財務理論

Ch1. Preference Representation and Risk Aversion

- (Preface) The main focus of this book is on individual's consumption and investment (or portfolio) decisions under uncertainty and their implication for the valuation of securities.

- 各章內容

  Ch1: Utility, risk aversion; one risky asset and one riskless asset

  Ch2: Stochastic dominance

  Ch3: Portfolio theory, mean-variance model

  Ch4: CAPM and APT

  Ch5: Pareto optimal and competitive equilibrium

  Ch6: Pricing contingent claims

  Ch7: 多期之 Pareto optimal equilibrium

  Ch8: 多期之 APT, martingale 與 risk neutral valuation

  Ch9: Heterogeneous agents with different information

  Ch10: CAPM之計量相關分析與驗證

- Expected utility hypothesis:

  $\widetilde{x}$ 比 $\widetilde{y}$ 好 iff $\mathrm{E}[\mathrm{u}(\widetilde{x})] \geq \mathrm{E}[\mathrm{u}(\widetilde{y})]$

  where $\mathrm{E}[\cdot]$ is the individual's subjective expectation.

- Under expected utility hypothesis.

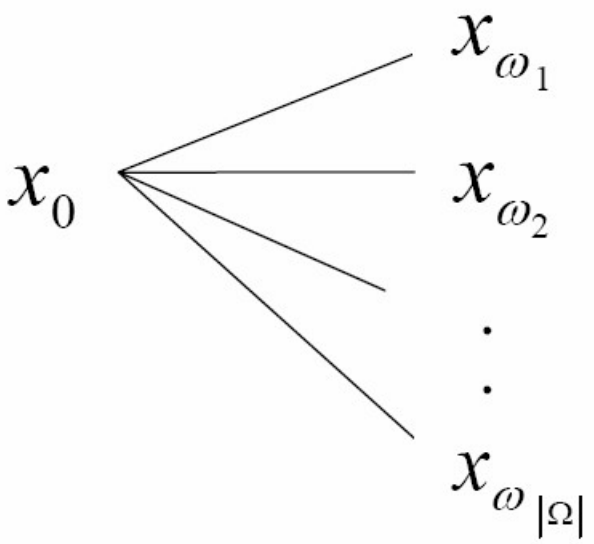
  1. Determine the probability of possible asset payoffs.

  2. Assign an index to each possible consumption outcome.

  3. Choose the consumption and investment policy to maximize the expected value of the index.

- Possible states of nature: Ω, $\omega \in \Omega$.

- Consumption plan is a specification of the number of units of the single consumption good in different states of nature. (Table 1.2.1)
  (圖: a consumption plan x, can be viewed as a r.v. $\tilde{x}$)



- It is hoped to represent an individual's preference by a utility function on consumption plan, H.
  Individual prefers x to x' iff $H(x) \geq H(x')$,
  where $H(x) = H(x_0, x_{\omega_1}, x_{\omega_2}, \ldots, x_{\omega_{|\Omega|}})$
  if $|\Omega| \uparrow, H$ 很複雜.

- 提出 $u(x_\omega)$ 之觀念, 再用 state 之可能發生機率, 做 weight, 得 expected utility, where x_w is certain outcome
  prefer $x \text{to} x' \Leftrightarrow \int_\Omega u(x_\omega) dP(\omega) \geq \int_\Omega u(x'_\omega) dP(\omega) \Leftrightarrow \mathrm{E}[u(\tilde{x})] \geq \mathrm{E}[u(\tilde{x}')]$

- 另一種 certain outcome 之想法,a consumption plan is certain if $x_\omega = z$, $\forall\, \omega \in \Omega$.亦即 certain consumption plan 之 utility 用機率做 weighted average.(此時, u compares consumption plans that are certain)

- 並非所有之 preference 都可用 expected utility representation
  方法1: 客觀機率 (von Neumann and Morgenstern(1953))
  方法2: 主觀機率 (Savage(1972))
  In this book, the function u defined on sure things is a von Neumann-Morgenstern utility function (1953), 其中 $P(\omega)$ 爲客觀機率.

- X: collections of consumption plans.
  Binary relation R on X: a collection of pairs $(x, y) \in R$, x, $y \in X$.
  $\succeq$ is defined as a binary relation: if x is preferred to $y \Rightarrow x \succeq y$
  (or $(x, y) \in R$)
  Complete: either $x \succeq y$ or $y \succeq x$.
  Transitive: $x \succeq y$, $y \succeq z$, then $x \succeq z$.

- A preference relation is a binary relation that is transitive and complete.

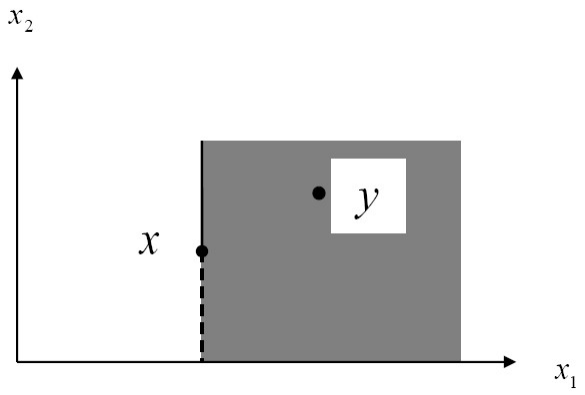
- $x \succ y \longleftrightarrow x \succeq y$ and $y \not\succeq x$, 即 $(x, y) \in R$, $(y, x) \notin R$.
  $x \sim y \longleftrightarrow x \succeq y$ and $y \succeq x$ (indifferent).

- 若 X 有 finite or countable elements, a preference relation $\succeq$ can always be represented by H, 例 p.5 (若有 uncountable elements, 則未必)

  $H(x_n) \geq H(x_m)$ iff $x_n \succeq x_m$

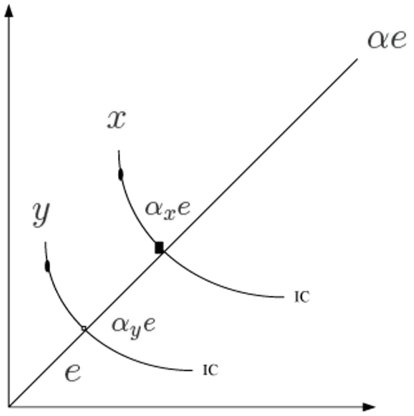
- Lexicographic preference (圖) (uncountably infinite number of the collections of consumption plans)

$$x \succeq y \Leftrightarrow \begin{cases} x_1 > y_1 \\ \text{if } \ x_1 = y_1, x_2 \geq y_2 \end{cases}$$



- Utility representation theorem: (多加條件, 使得 continuous utility function 存在)
  continuity: $\{y|y \succeq x\}$ and $\{y|x \succeq y\}$ are closed
  strong monotonicity: If $y \succeq x$ $(y_1 \geq x_1, y_2 \geq x_2, ..., y_n \geq x_n)$, then $y \succ x$.

- Suppose a preference relation is continuous and strong monotonic on R^m_+, then there is a continuons utility function h that represents the preference.
  $(x \succeq y \Longleftrightarrow h(x) \geq h(y))$
  h: $R^m_+ \to$ R, and $e \in R^m_+$ (圖)
  因爲 continuity $\Rightarrow \{\alpha | \alpha e \succeq x, \alpha \geq 0\}$ and $\{\alpha | \alpha e \preceq x, \alpha \geq 0\}$ are closed.
  加上 strong monotonicity $\Rightarrow$ there exists a unique α_x s.t $\alpha_x e \sim x$
  where $\alpha_x = h(x)$
  $(\Leftarrow) h(x) > h(y) \Rightarrow h(x)e \succ h(y)e \Rightarrow \alpha_x e \succ \alpha_y e \Rightarrow x \succ y$
  $(\Rightarrow)\ x \succ y \Rightarrow \alpha_x e \succ \alpha_y e \Rightarrow h(x)e \succ h(y)e \Rightarrow h(x) > h(y)$



- 以下開始講 expected utility

- When Ω has uncountably infinite elements, a probability is actually defined not on Ω (若如此, 則 $P(\omega)$ 均爲 0), but rather on a collection of subsets of Ω that satisfies a certain structure.
  $F_x(z) \equiv P\{\omega \in \Omega : x_\omega \leq z\}$ $(z \in R)$
  $E[u(\widetilde{x})] = \int_{-\infty}^{\infty} u(z) dF_x(z)$
  (對R積分, 而非之前對 Ω 積分 $\int_\Omega u(x_\omega) dP(\omega)$)

- $F_x(z)$ 對於描述 consumption plan 很重要, 其實人的不同 preference 就是指對不同的 prob. distribution 的喜好程度, 但並非 $F_x(z)$ 一樣, 每個 state 的 consumption payoff 都相同. 例:

	ω_1	ω_2	ω_3	ω_4	ω_5	$P(\omega_i) = \frac{1}{5}$
x	2	3	1	8	0	
y	0	1	2	3	8	

  $\Rightarrow F_x(z) = F_y(z)$, 但每個 state 的 consumption payoff 均不同.

- 考慮 state space 是 finite

$$\begin{cases} x_\omega \in Z, \forall \omega \in \Omega, \forall x \in X \\ p(z) \geq 0 \text{ for all } z \in Z \text{ and } \sum_{z \in Z} p(z) = 1 \end{cases}$$

  $F_x(z') = \sum_{z \leq z'} p(z)$

  and $E[u(\widetilde{x})] = \sum_{z \in Z} u(z)p(z)$

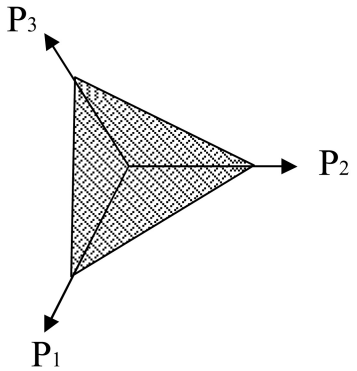
- 可將 consumption plan 想成 lottery, 其 prizes space is Z (which is a fix and finite set).

  A lottery $x \in X$.

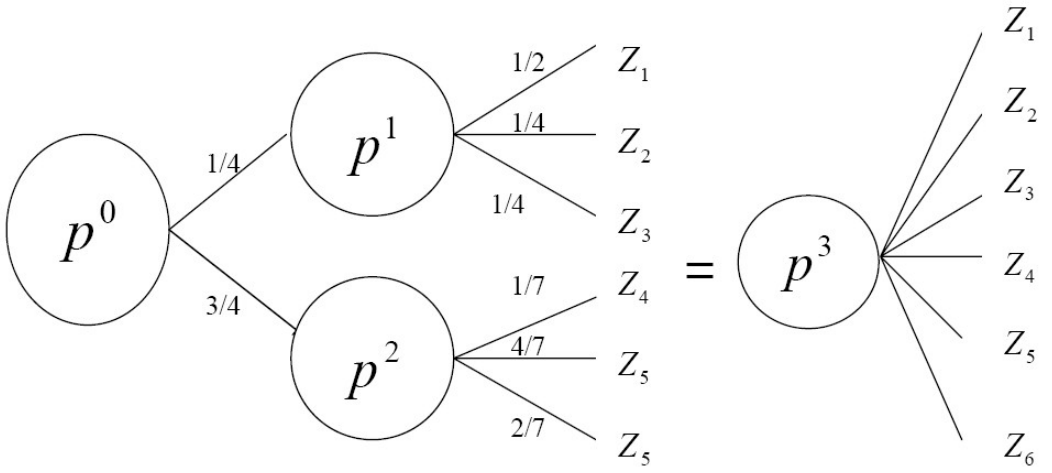
  $x = [p_1, p_2, ..., p_N; z_1, z_2, ..., z_N]$, 因爲 $Z = [z_1, ..., z_N]$ is fixed.

  所以不同之 x, 只在於 P=$[p_1, p_2, ..., p_N]$ 之不同 (P 可代表 lottery)

  (圖: p_1+p_2+$p_3 = 1$ 的平面上任一點, 均是 one lottery.)



- Compound lottery
  (圖: p^0, p^1, p^2, p^3: vector of prob. , which are lotteries)



- Binary relation on P 要滿足 Axiom 0∼3, 才會有相對應的 expected utility representation

  Axiom 0: 只看結果, 不看過程

  Axiom 1: $\succeq$ is a preference relation on P

  Axiom 2: substitution axiom (independent axiom)

  $\forall\ p, q, r \in P$, if $p \succ q$,

  then $ap + (1-a)r \succ aq + (1-a)r$, $a \in (0,1]$

  Axiom 3: Archimedean axiom

  $\forall\ p, q, r \in P$, if $p \succ q \succ r$, then

  $\exists\ a, b \in (0,1)$ s.t $ap + (1-a)r \succeq q \succeq bp + (1-b)r$

  (means $p \neq \infty$, $r \neq -\infty$, 亦即沒有 p 如此好, 沒有r如此差)

- 若 $\succeq$ is a binary relation on P, 滿足上述 Axiom $1 \sim 3$, 有 6 個 properties (p.9).

- Theorem: If lottery space X satisfies Axiom $0 \sim 3$,

  $\exists$ a function u: $Z \to R$ s.t. for p, $p' \in P$,

  $$p \succeq p' \quad \text{iff} \quad \sum_{i=1}^{N} p_i u(z_i) \geq \sum_{i=1}^{N} p'_i u(z_i)$$

  $\sum_{i=1}^{N} p_i u(z_i) \to E(u(x))$, 其中 $x = [p_1, p_2, ..., p_N; z_1, z_2, ..., z_N]$

  $\sum_{i=1}^{N} p'_i u(z_i) \to E(u(x'))$, 其中 $x' = [p'_1, p'_2, ..., p'_N; z_1, z_2, ..., z_N]$

  (亦即當 $\succeq$ on P 能用 expected utility 來表示 iff $\succeq$ on P 符合 Axiom $1 \sim 3$.)

- 證明 ($\Rightarrow$), 滿足 Axiom $0 \sim 3$ 之 binary relation $\succeq$ 可用 expected utility framework 來表示 (不只是用 utility H 來表示).

  $$\text{define } P_z(z') = \begin{cases} 1 & \text{if } z' = z \\ 0 & \text{if } z' \neq z \end{cases}$$

  $\max\ z^0 \Rightarrow P_{z^0} \succeq \forall\ p \in \text{P}$

  $\min\ z_0 \Rightarrow P_{z_0} \preceq \forall\ p \in \text{P}$

  case 1: $P_{z^0} \sim P_{z_0}$, trivial, $p \sim q$, $\forall\ p$, $q \in$ P, any $u(z) = k$

  can be a utility for sure things

  因 $E[u(x)] = \sum_{i=1}^{N} p_i u(z_i) = k$

  $E[u(x')] = \sum_{i=1}^{N} q_i u(z_i) = k$

  無論機率分配 p, q 如何都一樣喜好, 表示對於任何 sure prize 之 utility 都一樣.

  (例: $u(1) = 3$, $u(2) = 3$, $u(3) = 3$, $u(4) = 3$,...)

  case 2: $P_{z^0} \succ P_{z_0}$, for $\forall\ p \in$ P, 且 $aP_{z^0} + (1-a)P_{z_0} \sim p$,

  define $H(p) = a$, 亦即每個 lottery, 可想成最好跟最壞的組合.

By property 1 in p.9: 先定義 $H(p) = a$, $H(q) = b$,
$0 \leq a, b \leq 1$爲$\succeq$ 相對應之 utility.
$p \succeq q \Longleftrightarrow H(p)P_{z^0} + (1 - H(p))P_{z_0} \succeq H(q)P_{z^0} + (1 - H(q))P_{z_0}$
$\Longleftrightarrow H(p) \geq H(q)$,
where $H(p) = a$ and $H(q) = b$
$$bP_{z^0} + (1 - b)P_{z_0} \sim q,\ aP_{z^0} + (1 - a)P_{z^0} \sim p$$

想證明 $\exists u(\cdot)\text{on}Z s.t. H(p) = \sum_{z \in Z} u(z)p(z)$
(想証明utility 與 expected utility 可相對應)

先證H is linear,
亦即$H(ap + (1 - a)q) = aH(p) + (1 - a)H(q)$
(參考 p.10 (1.10.1) 以下)

define $u(z) = H(P_z)$, $\forall\ z \in Z$ (此時$u(z)$是實數, 且$\in$[0,1]),
where $u(\cdot)$is the utility on sure things

證明如下:

$\because p \sim \sum_{z \in Z} p(z)P_z$

其中 $p = [P_1, P_2, ..., P_{|z|}]$

$\therefore p(z)$ 爲 z 出現的機率.

$$\begin{aligned} H(p) &= H(\sum_{z \in Z} p(z)P_z) \\ &= \sum_{z \in Z} p(z)H(P_z)(\text{by the fact that H is linear}) \\ &= \sum_{z \in Z} p(z)u(z) \end{aligned}$$

(當然還存在其它之 von Neumann-Morgenstern utility $\hat{u}$, 但一定是u之 strictly positive linear transformation, 亦即$\hat{u} = cu + d$,where $c > 0$)(Exercise 1.4)

($\Leftarrow$) 如果$\succeq$有相對應之 expected utility representation, 亦即$p \succeq q$ iff $\sum_{z \in Z} u(z)p(z) \geq \sum_{z \in Z} u(z)q(z)$,

$\Rightarrow \succeq$ Axiom 0 $\sim$ 3成立 (Exercise 1.4)

- 若 Z is an infinite set, 例: 正實數, 則只符合 axiom $0 \sim 3$ 不夠, 還需要 Axiom 4, the sure thing principle:

  if p is concentrated on a set $B \in Z$, 其中 every point in B is at least as good as q, than p must be at least as good as q, 可參考 Fishburn (1970).

- 考慮多期, Z^{T+1} is $(Z_0, Z_1, ..., Z_T)$, $z \in Z^{T+1} \Rightarrow z = (z_0, z_1, ..., z_T)$

  where $z_0, z_1, ..., z_T$ 爲 sure things

$$\text{A probability } P \text{ on } Z^{T+1} \text{ (assume } Z^{T+1} \text{ is finite)} \quad \begin{cases} p(z) \in [0,1], \forall z \in Z^{T+1}, \\ \text{(其中 } z \text{ 爲一條path)} \\ \\ \sum\limits_{z \in Z^{T+1}} p(z) = 1. \\ \text{(所有 path 之機率合爲 1)} \end{cases}$$

- $\succeq$ is a preference relation satisfies axion $0 \sim 3$ iff

  $\exists$ a von Neumarn-Morgenstenn utility function on sure things (意即存在 expected utility presentation)

  $\Rightarrow$ for all p, $q \in$ P,

$$p \succeq q \text{ iff } \sum_{z \in Z^{T+1}} u(z_0, z_1, ..., z_T) p(z = z_0, ..., z_T) \geq \sum_{z \in Z^{T+1}} u(z_0, z_1, ..., z_T) q(z = z_0, ..., z_T)$$

  if <u>time-additive</u> $\Rightarrow u(z_0, ..., z_T) = \sum\limits_{t=0}^{T} u_t(z_t)$

  $\downarrow$

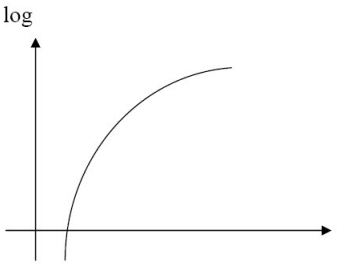
* What an individual consumes at one time will not have any effect on his desire to consume at any other time.

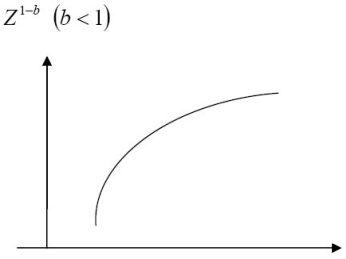
* 且沒法考慮 path-dependent:

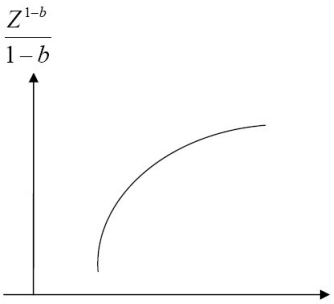
$u(90, 110, 90, 95) - u(120, 110, 90, 95)$ 會等於 $u_0(90) - u_0(120)$ 嗎?

- von Neumann-Morgenstren utility 需 bounded (即使是面對 unbounded consumption level)
  (因爲要符合 Archimedean axiom)

- 但很多 function 是 unbounded (from above or from below)
  例:







- 如果Z is finite, von Neumann-Morgenstern utility 不會有 unbounded 之問題

- $u(z) \leq u(b) + u'(b)(z-b)$ (if u is concave)
  $E[u(z)] \leq u(b) + u'(b)(E[z]-b)$
  expected utility is bounded if u is concave and $E[x]$ is bounded even while u is unbounded
- Allais Paradox
  Z=[\$0, \$1m, \$5m], P=[P_1, P_2, P_3]

$$P_1 = \{ \text{ \$1m, with prob} = 1 \quad (0,1,0)$$

$$P_2 = \begin{cases} \text{\$5m, with prob} = 0.1 \\ \text{\$1m, with prob} = 0.89 \\ \text{\$0m, with prob} = 0.01 \end{cases} \quad (0.01, 0.89, 0.1)$$

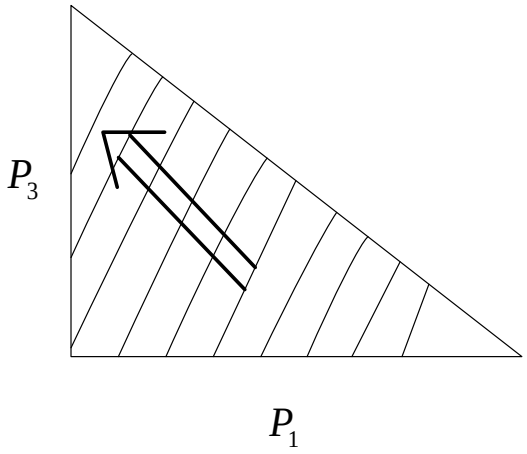
$$P_3 = \begin{cases} \text{\$5m, with prob} = 0.1 \\ \text{\$0, with prob} = 0.9 \end{cases} \quad (0.9, 0, 0.1)$$

$$P_4 = \begin{cases} \text{\$1m, with prob} = 0.11 \\ \text{\$0, with prob} = 0.89 \end{cases} \quad (0.89, 0.11, 0)$$

實驗結果, $P_1 \succ P_2$, $P_3 \succ P_4$
$P_1 \sim 0.11(\$1\text{m}) + 0.89(\$1\text{m})$
$P_2 \sim 0.11(\frac{1}{11}(\$0) + \frac{10}{11}(\$5\text{m})) + 0.89(\$1\text{m})$
因 $P_1 \succ P_2$
$\Rightarrow 0.11(\$1\text{m}) + 0.89(\$1\text{m}) \succ 0.11(\frac{1}{11}(\$0) + \frac{10}{11}(\$5\text{m})) + 0.89(\$1\text{m})$
$\$1\text{m} \succ \frac{1}{11}(\$0) + \frac{10}{11}(\$5\text{m})$
By substitution
$0.11(\$1\text{m}) + 0.89(\$0\text{m}) \succ 0.11(\frac{1}{11}(\$0) + \frac{10}{11}(\$5\text{m})) + 0.89(\$0)$
$\Rightarrow P_4 \succ P_3$

- if N=3

$Z_1 < Z_2 < Z_3,\ P_2 = 1 - P_1 - P_3$



P_1不變, P_3增加, P_2減少 $\Rightarrow$ preference 增加

$E[u(z)] = P_1u(z_1) + P_2u(z_2) + P_3u(z_3)$

$=P_1u(z_1) + (1 - P_1 - P_3)u(z_2) + P_3u(z_3)$

$=u(z_2) + P_1(u(z_1) - u(z_2)) + P_3(u(z_3) - u(z_2))$

Suppose $E[u(z)]$爲 constant

$\Rightarrow \frac{dP_3}{dP_1}|_{E[u(z)]}$ is constant$=\frac{u(z_2)-u(z_1)}{u(z_3)-u(z_2)}>0$, 其中$\frac{u(z_2)-u(z_1)}{u(z_3)-u(z_2)}$爲 IC 的斜率

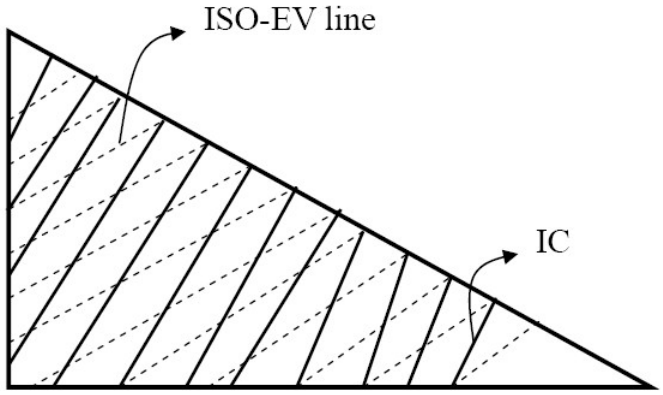
$E[z] = P_1z_1 + P_2z_2 + P_3z_3$

$\Rightarrow \frac{dP_3}{dP_1}|_{E[z]}$ is constant$=\frac{z_2-z_1}{z_3-z_2}> 0$

其中$\frac{z_2-z_1}{z_3-z_1}$爲 ISO-expected value lines 之斜率, 若$z_2=\frac{z_1+z_3}{2}$, 則斜率爲1

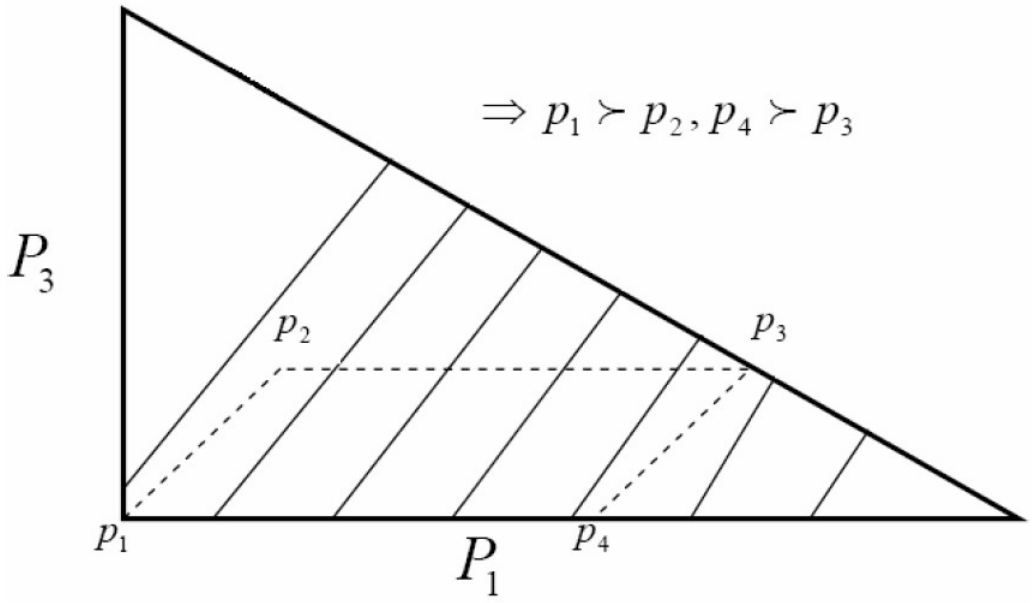
- Considering the case of risk aversion,then $u(\cdot)$is concave

$$\Rightarrow \frac{u(z_2)-u(z_1)}{z_2-z_1} > \frac{u(z_3)-u(z_2)}{z_3-z_2} \Rightarrow \frac{u(z_2)-u(z_1)}{u(z_3)-u(z_2)} > \frac{z_2-z_1}{z_3-z_2}$$



- 可以用上述的分析法, 來看 Allais Paradox. Allais Paradox 相當於 $[z_1, z_2, z_3] = [\$0, \$1\text{m}, \$5\text{m}]$

  (圖: $\Rightarrow p_1 \succ p_2$ 會推得 $p_4 \succ p_3$)



- Tversky-Kahneman 提出 Framing Effects
  600病人,4種治療方式 (由醫生決定)
  (1)

$$\text{A: 200人活} \succ \text{B} \begin{cases} 0, \text{ with prob} = \frac{2}{3} \\ 600, \text{ with prob} = \frac{1}{3} \end{cases} \text{ (怕2/3的機率, 大家都死)}$$

  (2)

$$\text{C: 400人死} \prec D \begin{cases} 600, \text{ with prob} = \frac{2}{3} \\ 0, \text{ with prob} = \frac{1}{3} \end{cases} \text{ (希望1/3的機率, 大家都活)}$$

  實際 (1)(2) 應一樣 (亦即 A$\succ$B $\Rightarrow$ C$\succ$D)
  Axiom 2 (independent axiom) 很常被違反

- Machina Paradox

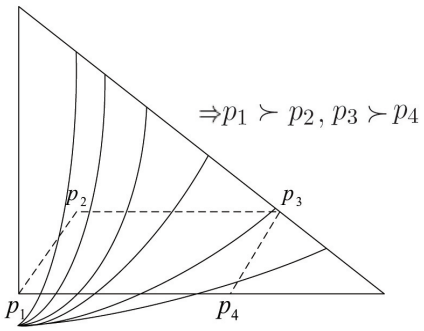
$$\underset{\text{staying home}}{(z_1)} \prec \underset{\text{movies about Venice}}{(z_2)} \prec \underset{\text{trip to Venice}}{(z_3)}$$

$$\text{A:} \begin{cases} \text{Z}_3, \text{with prob} = 0.99 \\ \text{Z}_2, \text{with prob} = 0.01 \end{cases}$$

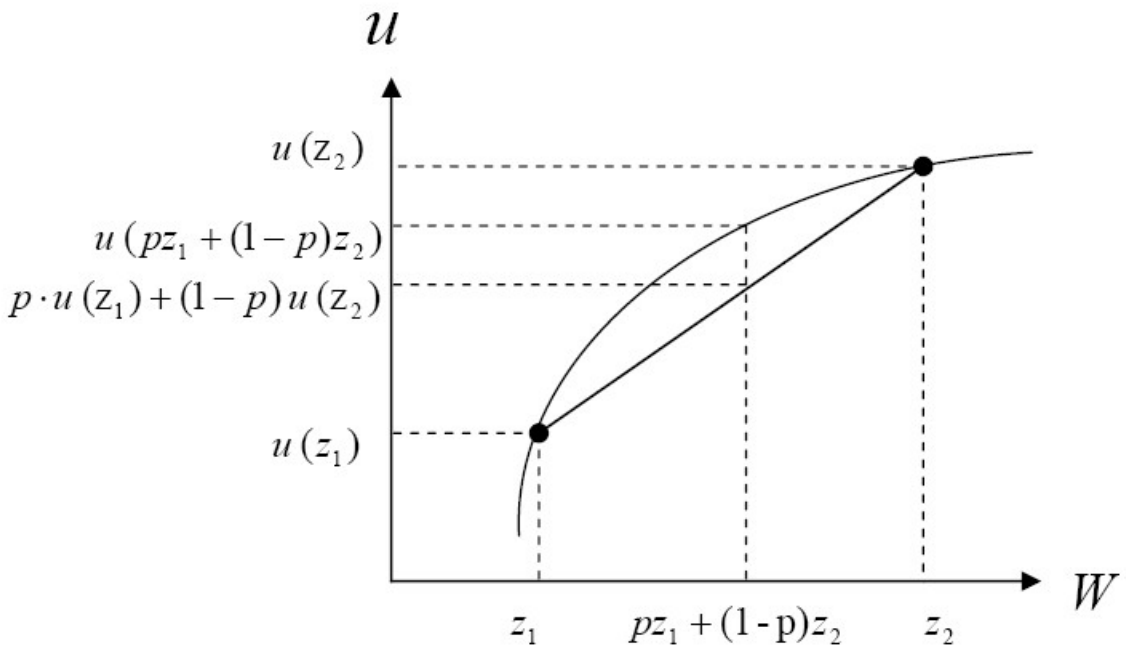
$$\text{B:} \begin{cases} \text{Z}_3, \text{with prob} = 0.99 \\ \text{Z}_1, \text{with prob} = 0.01 \end{cases}$$

  由 independent axiom: A $\succ$ B
  但實際上 A $\prec$ B (因爲 disappointment aversion)

- Non-expected Utility (may be nonlinear)
  (圖: 可滿足 $p_1 \succ p_2$, $p_3 \succ p_4$)



- Risk aversion (圖)



$$\begin{cases} z_1, \text{with prob} = p \\ z_2, \text{with prob} = 1-p \end{cases} \prec pz_1 + (1-p)z_2$$

$$\Rightarrow pu(z_1)+(1\text{-}p)u(z_2) < u(pz_1 + (1-\text{p})z_2)$$

- 不接受 fair game⇒ risk aversion ⇒ concave function
- risk aversion 其實只是邊際效用遞減

- An agent A is more risk averse than an agent B (Globally)

Arrow-Pratt Theorem	Ross Thorem 1981 (2.12 $\sim$ 2.14)
(處理一個 risky asset, 一個 riskless asset)	(A is strongly more risk averse than B, 可處理兩個 risky assets) (人之風險態度與資產之風險程度同時考慮進來)
(1) $-\frac{A''(w)}{A'(w)} \geq -\frac{B''(w)}{B'(w)}$, $\forall w$ (absolute risk aversion)	(1) $\exists \lambda > 0$, s.t. $\forall w_1$, w_2 , $\frac{A''(w_1)}{B''(w_1)} \geq \lambda \geq \frac{A'(w_2)}{B'(w_2)}$ (if $w_1 = w_2$, 則回歸到 AP) $A \overset{\supseteq}{R}$ B $\Rightarrow A \supseteq$ B $\not\Leftarrow$ (⇒)當$W_1 = W_2$可得証 (⇐)$A = -e^{-aw}$ $B = -e^{-bw}$ 若$a > b \Rightarrow A \overset{\supseteq}{A} P\ B$ 是否可以找到 W_1, W_2 $s.t.$ $\frac{A''(W_1)}{B''(W_1)} < \frac{A'(W_2)}{B'(W_2)}$ $(\frac{a}{b})^2 e^{(b-a)W_1} < \frac{a}{b} e^{(b-a)W_2}$ $e^{(b-a)(W_1-W_2)} < \frac{a}{b}$ 只要$W_1 - W_2$夠大即可
⇕	⇕
(2)$\exists G, G' \geq 0, G'' < 0$ (which means G is strictly concave)	(2)$\exists \Phi, \Phi' \leq 0, \Phi'' \leq 0,$ 且$\lambda > 0$, s.t. $A = \lambda B + \Phi$
A=$G(B)$	
⇕	⇕
(3) $\forall w$, $\tilde{\varepsilon}$, $E[\tilde{\varepsilon}] = 0$, $E[\tilde{\varepsilon}^2] = \sigma^2$ $E[A(w + \tilde{\varepsilon})] = A(w - \pi_A)$ $E[B(w + \tilde{\varepsilon})] = B(w - \pi_B)$ 等式兩邊, 都對 utility 做 Taylor expansion 可得 $\pi_A \approx [-\frac{A''(w)}{A'(w)}] \cdot \frac{\sigma^2}{2}$ $\pi_B \approx [-\frac{B''(w)}{B'(w)}] \cdot \frac{\sigma^2}{2}$ $\pi_A \geq \pi_B$	(3) $\forall \tilde{W}, \tilde{\varepsilon}, E[\tilde{\varepsilon}\|\tilde{W}] = 0$ $E[A(\tilde{W} + \tilde{\varepsilon})] = E[A(\tilde{W} - \pi_A)]$ $E[B(\tilde{W} + \tilde{\varepsilon})] = E[B(\tilde{W} - \pi_B)]$ $\pi_A \geq \pi_B$

- Arrow-Pratt Measure of Risk Aversion

  1 risky asset + 1 riskless asset

  $\tilde{w} = W_0(1 + r_f) + a(\tilde{r} - r_f)$

  $\max_a E[u(\tilde{W})] = E[u(W_0(1 + r_f) + a(\tilde{r} - r_f))]$

  $\frac{\partial}{\partial a} : E[u'(\tilde{W})(\tilde{r} - r_f)] = 0$

  Proposition 1: $E[\tilde{r} - r_f] > 0 \Rightarrow a^* > 0$

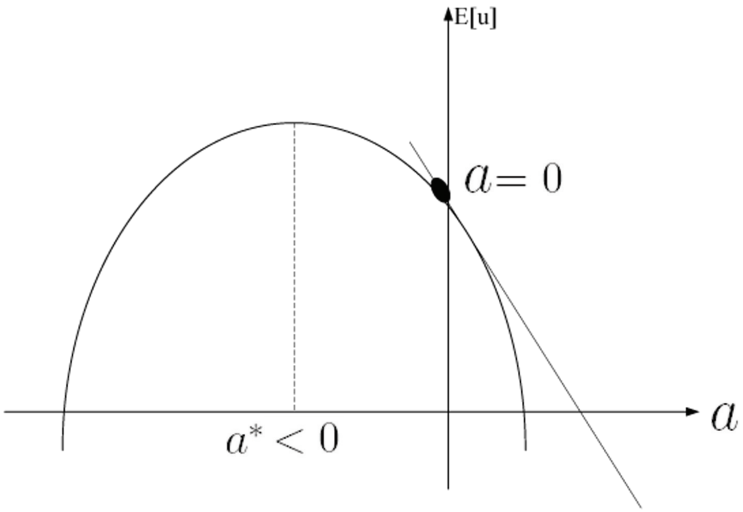
  pf: 反證法,

  assume the optimal $a^* \leq 0$ when $E[\tilde{r} - r_f] > 0$ (圖)

  $\Rightarrow E[u'(W_0(1 + r_f))(\tilde{r} - r_f)] \leq 0$ (因爲 $a = 0$ 時, 一階微分 ≤ 0)

  $\Rightarrow u'(W_0(1 + r_f))E[\tilde{r} - r_f] \leq 0$, 因爲 $u' > 0 \Rightarrow E[\tilde{r} - r_f] < 0$

  $\Rightarrow$ 矛盾 $\Rightarrow E[\tilde{r} - r_f] > 0 \Rightarrow a^* > 0$



Proposition 2: $\frac{dR_A(z)}{dz} < 0, \forall\ z \Rightarrow \frac{da^*}{dW_0} > 0, \forall\ W_0$

當人是 DARA $\Rightarrow$ 越有錢, 越願意買 risky asset, 投資越多在 $\tilde{r}$ ($\tilde{r}$ is a normal good)

Proposition 3: $\frac{dR_R(z)}{dz} > 0, \forall\ z \Rightarrow \eta = \frac{\frac{da^*}{a^*}}{\frac{dW_0}{W_0}} < 1$

(其中$R_A(z) = -\frac{u''(z)}{u'(z)}$, $R_R(z) = -\frac{u''(z)z}{u'(z)}$)

當人是 IRRA $\Rightarrow$ 越有錢, risky asset 佔總資產之比例減少

- Ross 將上述理論發展成 many risky assets

  $\tilde{W} = (W_0 - \sum a_j)(1 + r_f) + \sum a_j(1 + \tilde{r}_j)$

  $= W_0(1 + r_f) + \sum a_j(\tilde{r}_j - r_f)$

  a_j: amount invested in jth risky asset

  $\max_{a_j}$ E[u($\tilde{W}$)]

  FOC: $E[u'(\tilde{W})(\tilde{r}_j - r_f)] = 0\ \forall\ j$, 得 a_j^*

  Proposition 1: If $\exists\ j'$ s.t. $E(\tilde{r}_{j'} - r_f) > 0$, then $a_j^* > 0$

  (in the case of one risky asset, $j = j'$)

  (要投資 risky assets, 至少有一個 risky assets 之期望報酬大於無風險利率)

  Proof of proposition 1: (圖)

  if $\forall\ a_j^* \leq 0$ (不是沒投資, 就是放空)

  for all $a_j = 0$, $E[u'(W_0(1 + r_f))(\tilde{r}_j - r_f)] \leq 0, \forall\ j$

  (如果 $a_j \uparrow$, 會更負)

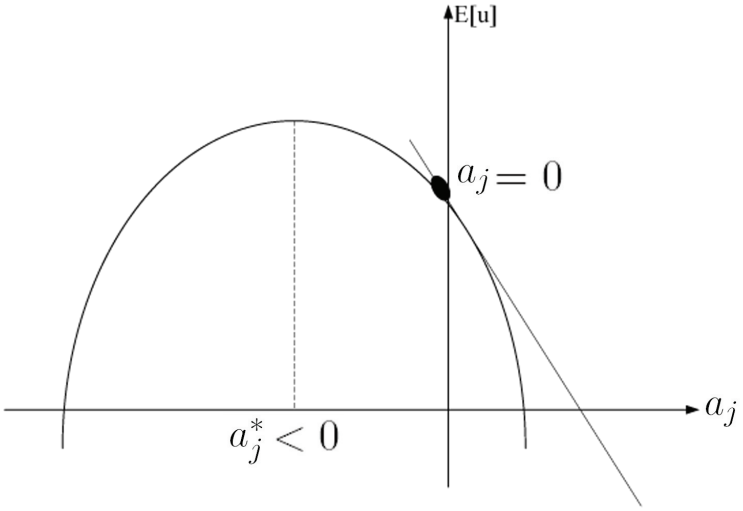
  $\Rightarrow u'(W_0(1 + r_f))E[\tilde{r}_j - r_f] \leq 0$

  因 $u' > 0 \Rightarrow E[\tilde{r}_j - r_f] < 0$

  $\Rightarrow$ if $\forall\ a_j \leq 0 \Rightarrow E[\tilde{r}_j - r_f] \leq 0\ \forall\ j$

  反之, 如果有任一個資產之超額報酬 > 0,

  則一定存在一個 risky asset 之投資額 $a_j > 0$



Proof of proposition 2: 考慮 one risky asset

$\mathrm{E}[u'(\tilde{W})(\tilde{r} - r_f)] = 0$, where $\tilde{W} = W_0(1 + r_f) + a(\tilde{r} - r_f)$

(全微分)

$E[u''(\tilde{W})(\tilde{r} - r_f)^2]da + E[u''(\tilde{W})(1 + r_f)(\tilde{r} - r_f)]dW_0 = 0$

$\Rightarrow \frac{da}{dW_0} = \frac{E[u''(\tilde{W})(\tilde{r} - r_f)](1+r_f)}{-E[u''(\tilde{W})(\tilde{r} - r_f)^2]}$, $u' > 0$, $u'' < 0$

$\mathrm{sign}\{\frac{da}{dW_0}\} = \mathrm{sign}\{E[u''(\tilde{W})(\tilde{r} - r_f)]\}$

(i) for $\tilde{r} \geq r_f \Rightarrow \tilde{W} \geq W_0(1 + r_f)$

$\Downarrow$因 $\frac{dR_A(z)}{dz} < 0$

$R_A(\tilde{W}) \leq R_A(W_0(1 + r_f))$

$\Downarrow$ 同乘 $-u'(\tilde{W})(\tilde{r} - r_f) \leq 0$

$u''(\tilde{W})(\tilde{r} - r_f) \geq -R_A(W_0(1 + r_f))u'(\tilde{W})(\tilde{r} - r_f)$

(ii) for $\tilde{r} < r_f \Rightarrow \tilde{W} < W_0(1 + r_f)$

$\Rightarrow R_A(\tilde{W}) > R_A(W_0(1 + r_f))$

$\Rightarrow u''(\tilde{W})(\tilde{r} - r_f) > -R_A(W_0(1 + r_f))u'(\tilde{W})(\tilde{r} - r_f)$

$\because$ (i)(ii) $\Rightarrow$

$E[u''(\tilde{W})(\tilde{r} - r_f)] > -R_A(W_0(1 + r_f))E[u'(\tilde{W})(\tilde{r} - r_f)]$

where $E[u'(\tilde{W})(\tilde{r}-r_f)]$ 爲 FOC $=0$

$\Rightarrow E[u''(\tilde{W})(\tilde{r}-r_f)]>0$

$\Rightarrow \frac{da}{dW_0}>0$

$(\frac{R_A(z)}{dz}\leq 0 \Rightarrow u'''>0$, p.23, (1.21.5))

Proof of proposition 3:

the elasticity of the demand for the risky asset

$\eta=\frac{\frac{da}{a}}{\frac{dW_0}{W_0}}=\frac{da}{dW_0}\frac{W_0}{a}=1+\frac{\frac{da}{dW_0}W_0-a}{a}$

$\eta=1+\frac{W_0(1+r_f)E[u''(\tilde{W})(\tilde{r}-r_f)]+aE[u''(\tilde{W})(\tilde{r}-r_f)^2]}{-aE[u''(\tilde{W})(\tilde{r}-r_f)^2]}$($\frac{da}{dW_0}$ 代入)

$\Rightarrow \eta=1+\frac{E[u''(\tilde{W})(\tilde{r}-r_f)\tilde{W}]}{-aE[u''(\tilde{W})(\tilde{r}-r_f)^2]}$

sign$\{\eta-1\}$=sign$\{E[u''(\tilde{W})(\tilde{r}-r_f)\tilde{W}]\}$

(under increasing relative risk aversion)

$$R_R(W_0(1+r_f)+a(\tilde{r}-r_f))\begin{cases}\geq R_R(W_0(1+r_f)), \text{if } \tilde{r}\geq r_f\\ < R_R(W_0(1+r_f)), \text{if } \tilde{r}<r_f\end{cases}$$

⇓同乘 $-u'(\tilde{W})(\tilde{r}-r_f)$

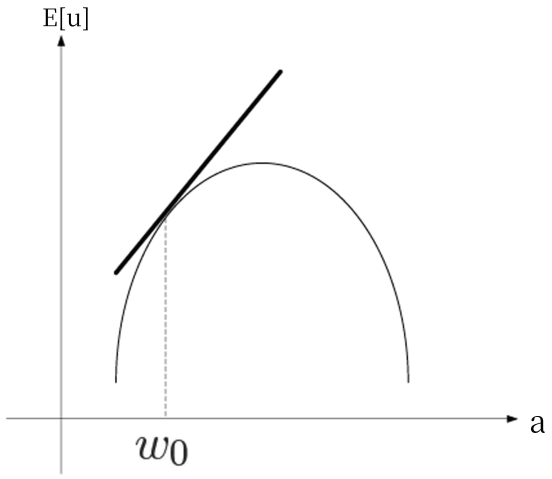
$$u''(\tilde{W})\tilde{W}(\tilde{r}-r_f)\begin{cases}\leq -R_R(W_0(1+r_f))u'(\tilde{W})(\tilde{r}-r_f), \text{if } \tilde{r}\geq r_f\\ < -R_R(W_0(1+r_f))u'(\tilde{W})(\tilde{r}-r_f), \text{if } \tilde{r}<r_f\end{cases}$$

$E[u''(\tilde{W})(\tilde{r}-r_f)\tilde{W}]<E[R_R(W_0(1+r_f))u'(\tilde{W})(\tilde{r}-r_f)]$

因爲 $E[u'(\tilde{W})(\tilde{r}-r_f)]=0\Rightarrow \eta-1<0$

- p.25 1.23節, 用 quadratic utility, negative exponential, narrow power, extended power 做例子看$\frac{dR_A(z)}{dz}$, $\frac{dR_R(z)}{dz}$

- p.20 1.20節, 將所有財產W_0投資到 $\tilde{r}$, 所需之 equity premium
  $E[u'(W_0(1+\tilde{r}))(\tilde{r}-r_f)] \geq 0$ (全部投資在$\tilde{r}$, 但還不夠滿意)
  Taylor series expansion
  $\Rightarrow E[u'(W_0(1+\tilde{r}))(\tilde{r}-r_f)] \approx u'(W_0(1+r_f))E[\tilde{r}-r_f]$
  $+u''(W_0(1+r_f))E[(\tilde{r}-r_f)^2]W_0 \geq 0$
  $\Rightarrow u'(W_0(1+r_f))E[\tilde{r}-r_f] \geq -u''(W_0(1+r_f))E[(\tilde{r}-r_f)^2]W_0$
  $\Rightarrow E[\tilde{r}-r_f] \geq R_A(W_0(1+r_f))E[(\tilde{r}-r_f)^2]W_0$
  ($R_A \uparrow$, 需越高的 risk premium, 才願意全部投資在$\tilde{r}$)



- 之前分析都在於 risk is small, 且用 local risk aversion (around $W_0(1+r_f)$) 來分析, 現在要看global risk aversion

  若 $R_A^i(z) \geq R_A^k(z)\ \forall\ z \Rightarrow$ individual i is more risk averse than k globally, 當這 兩個人之 initial wealth W_0 相同時, 要 i 將所有財富投資到 risky asset 所須之 risk premium, 會高於 k 將所有財富投資到 risky asset 所須之 risk premium

- Assume 對 k 而言, investing all wealth on the risky asset lets his utility be maximized

  $\Rightarrow E[u_k'(W_0(1+\tilde{r}))(\tilde{r}-r_f)]=0$

  此時, $E[\tilde{r}-r_f]$代表了對 k 而言, 將所有 wealth 投資到 risky asset 所須之最小之 risk premium

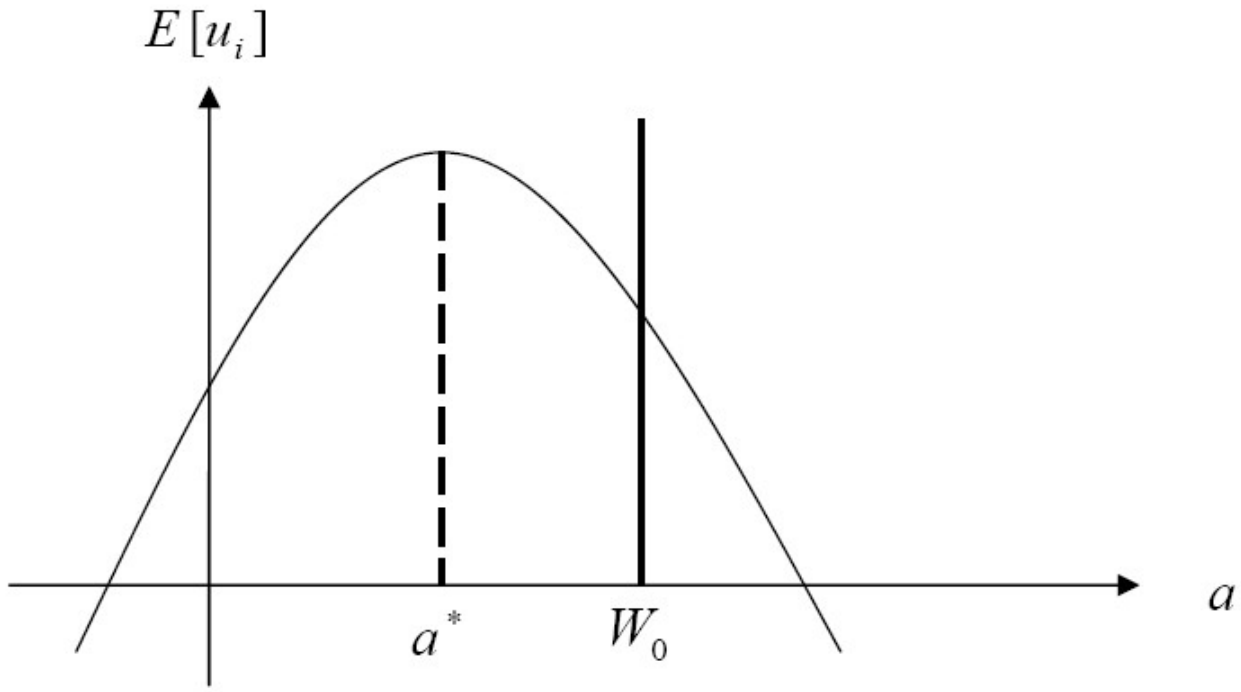
  只需證明, 對一個更 risk averse 之 i 而言

  $E[u_i'(W_0(1+\tilde{r}))(\tilde{r}-r_f)]\leq 0$

  (即給定了 $E[\tilde{r}-r_f]$下, 若全部投資在 $\tilde{r}$, 非最佳決策)

  (圖: $a^*<W_0$ 才是最佳決策)

  (此結果隱含當投資人之風險趨避態度不同, a^*也不同; 越風險趨避, a^*越小)



- 證明如下

  $u_i = G(u_k)$, $G' > 0$, $G'' < 0$ (convave transformation)

  where $G' > 0, G'' < 0$ iff $R_A^i(z) \geq R_A^k(z), \forall\ z$

  (上式之證明參考 p.29 1.25節)

  $E[u_i'(W_0(1+\tilde{r}))(\tilde{r} - r_f)] = E[G'(u_k(W_0(1+\tilde{r})))u_k'(W_0(1+\tilde{r}))(\tilde{r} - r_f)]$

  (let $G'(u_k(W_0(1+\tilde{r})))u_k'(W_0(1+\tilde{r}))(\tilde{r}-r_f) = \Delta$)

  $= E[\Delta] = E[\Delta|\tilde{r} - r_f \geq 0]P(\tilde{r} - r_f \geq 0) + E[\Delta|\tilde{r} - r_f < 0]P(\tilde{r} - r_f) < 0$

  (i) $E[\Delta|\tilde{r}-r_f \geq 0] \leq G'(u_k(W_0(1+r_f)))E[u_k'(W_0(1+\tilde{r}))(\tilde{r}-r_f)|(\tilde{r}-r_f) \geq 0]$

  (因 $G'(u_k(W_0(1+\tilde{r}))) < G'(u_k(W_0(1+r_f))), u_k' > 0, G'' < 0, \tilde{r} - r_f \geq 0.$)

  (ii) $E[\Delta|\tilde{r}-r_f < 0] \leq G'(u_k(W_0(1+r_f)))E[u_k'(W_0(1+\tilde{r}))(\tilde{r}-r_f)|(\tilde{r}-r_f) < 0]$

  (因 $G'(u_k(W_0(1+\tilde{r}))) > G'(u_k(W_0(1+r_f)), u_k' > 0, G'' < 0, \tilde{r} - r_f < 0)$

  $\Rightarrow E[\Delta] \leq G'(u_k(W_0(1+r_f)))E[u_k'(W_0(1+\tilde{r}))(\tilde{r} - r_f)] = 0,$

  (where $E[u_k'(W_0(1+\tilde{r}))(\tilde{r} - r_f)] = 0$ is FOC of k.)

  $\Rightarrow$ 得證

- 多個 assets 時, proposition 2 和 proposition 3, 未必對, 因 proposition 2 說, $\frac{dR_A(z)}{dz} < 0 \Rightarrow \frac{da}{dW_0} > 0$ for $\forall\ W_0$, 但多個資產時, 未必所有 $\frac{da_j}{dW_0} > 0$

  同理, proposition 3 說, $\frac{dR_R(z)}{dz} > 0 \Rightarrow \frac{\frac{da}{a}}{\frac{dW_0}{W_0}} < 1$, 但多個資產時, W_0 之變動會造成 portfolio 之重組, 但非所有 $\frac{\frac{da_j}{a_j}}{\frac{dW_0}{W_0}} < 1$

- 但若投資之 $\tilde{r}_j$ 形成一個固定比例之portfolio $\tilde{r}_p$, 且只在 $\tilde{r}_p$ 與 r_f 間選擇比率, 則之前的 proposition 2 與 proposition 3 會再度成立, 此時個人之 optimal portfolio 是 riskless asset 和 a risky asset mutual fund 的 linear combination, 這個現象稱爲 "two fund monetary separation"

- Cass and Stiglitz (1970) 證明某些 utility function 可使得 two fund monetary separation 成立

two fund monetary separation$\Leftrightarrow u'(z)=(A+Bz)^C$ or $u'(z)=Ae^{Bz}$ (選取適當之 ABC 使得 $u'>0, u''<0$)

- 1.28節, 證明, 若 $u'(z)=(A+BZ)^C$ or $u'(z)=Ae^{BZ}$, 則只有 α 變, b_j 不變

$\max_{\{\alpha,b_j\}} \mathcal{L}=E[u(\tilde{W})]+\lambda[1-\sum_j b_j]$

其中, $\tilde{W}=W_0(1+\alpha r_f+(1-\alpha)\sum_j b_j\tilde{r}_j)$

$\frac{\partial}{\partial\alpha}\Rightarrow E[u'(\tilde{W})W_0(r_f-\sum_j b_j\tilde{r_j})]=0$

$$\Rightarrow E[u'(\tilde{W})W_0 r_f]=E[u'(\tilde{W})W_0\sum_j b_j\tilde{r_j}] \quad (1)$$

$$\frac{\partial}{\partial b_j}\Rightarrow E[u'(\tilde{W})W_0(1-\alpha)\tilde{r_j}]=\lambda \quad (2)$$

$$\overset{\sum_j b_j}{\Rightarrow} E[u'(\tilde{W})W_0(1-\alpha)\sum_j b_j\tilde{r_j}]=\sum_j b_j\lambda=\lambda \quad (3)$$

由 (2)=(3)

$$\Rightarrow E[u'(\tilde{W})W_0\sum_j b_j\tilde{r_j}]=E[u'(\tilde{W})W_0\tilde{r_j}] \quad (4)$$

(4)−(1)

$\Rightarrow E[u'(\tilde{W})W_0(\tilde{r_j}-r_f)]=0$ (課本 (1.28.4) 是直接兩邊除W_0)

假設 $u'(z)=(A+Bz)^c$代入

$\Rightarrow E[(A+BW_0(1+r_f+(1-\alpha)\sum_j b_j(\tilde{r_j}-r_f)))^C(\tilde{r_j}-r_f)]=0$ (1.28.5)

假設同一個投資人, 但財富爲w_0', 且投資比重爲α', b_j'

$\Rightarrow E[(A+Bw_0'(1+r_f+(1-\alpha')\sum_j b_j'(\tilde{r_j}-r_f)))^C(\tilde{r_j}-r_f)]=0$ (1.28.6)

假設$w_0'(1-\alpha')b_j'=\frac{A+Bw_0'(1+\alpha' r_f)}{A+BW_0(1+\alpha r_f)}W_0(1-\alpha)b_j$成立 (即 (1.28.7) 式)

代入 (1.28.6)

$$E[(A+\underbrace{Bw_0'(1+r_f)}+Bw_0'(1-\alpha')\sum b_j'\tilde{r_j}\underbrace{-Bw_0'(1-\alpha')\sum b_j'r_f}{}^{C}(\tilde{r_j}-r_f)]=0$$

$$\searrow \qquad\qquad \swarrow$$

$$Bw_0'(1+r_f-r_f+\alpha' r_f)$$

$$\Rightarrow E[(A+Bw_0'(1+\alpha' r_f)+\underbrace{Bw_0'(1-\alpha')\sum b_j'\tilde{r_j}}{}^{C}(\tilde{r_j}-r_f)]=0$$

$$\downarrow$$

$$B\tfrac{A+Bw_0'(1+\alpha' r_f)}{A+BW_0(1+\alpha r_f)}W_0(1-\alpha)\sum b_j\tilde{r_j}$$

等號兩邊同乘$(\frac{A+BW_0(1+\alpha r_f)}{A+Bw_0'(1+\alpha' r_f)})^C$

$$\Rightarrow E[(A+BW_0(1+\alpha r_f)+BW_0(1-\alpha)\sum b_j\tilde{r_j}^{C}(\tilde{r_j}-r_f)]=0$$

(即爲 (1.28.5))

$\Big\|$ $\because (\frac{A+BW_0(1+\alpha r_f)}{A+Bw_0'(1+\alpha' r_f)})^c=\frac{u'(W_0(1+\alpha r_f))}{u'(w_0'(1+\alpha' r_f))}$, 且$u'>0, u''<0$

$\Big\|$ $\Rightarrow$ 此項不可能爲0, 所以才能同乘此項

- 1.29節, 舉例滿足 (1.27.1) 與 (1.27.2) 之 utility