Chapter 18

Risk Measurement for a Single Facility

Introduction

- To quantify credit risk, it is wished to
  - Obtain the probability distribution of the losses from a credit portfolio
  - Measure the contribution of each loan to the portfolio loss
- In this chapter, quantifying the risk for individual facilities is considered
  - One-year default case
  - Downgrades considered
  - Multiple-year case

Determining Losses Due to Default

- Define
  - Exposure at Default (EAD) = *E*
  - Loss In the Event of Default (LIED) = *S*
  - Probability of Default (PD) = *P*
  - Default Indicator Variable = *I*

$$L = I \times S \times E$$

$$L = \begin{cases} S \times E & \text{if default} \\ 0 & \text{o/w} \end{cases}$$

- 考慮E與S在一年內為constant

$$EL = \overline{L} = P(S \times E) + (1-P)(0) = P \times E \times S$$

$$UL^2 = P(E \times S - \overline{L})^2 + (1-P)(0 - \overline{L})^2$$

$$= (P - P^2)E^2S^2$$

$$UL = \sqrt{P - P^2} \times E \times S$$

★ A Bernoulli variable can only equal one or zero and the variance of a Bernoulli variable is *P(1-P)*

- 若S與E非constant，且其間有相關性

$$EL=\overline{L}=\sum_{I=0,1}p(I)\int_S\int_E ISE\ pr(S,E\mid I)dEdS$$

$$=P\int_S\int_E SE\ pr(S,E\mid I=1)dEdS$$

$$=P(\overline{S}\,\overline{E}+\sigma^2_{S,E})$$

★與之前不一樣的地方，除了由常數改為變數的平均，還多了一項covariance

$$UL^2 = \sum_{I=0,1} p(I) \int_S \int_E (ISE - \overline{L})^2 \; pr(S, E \mid I) dEdS$$

$$= \left(\sum_{I=0,1} p(I) \int_S \int_E (ISE)^2 \; pr(S, E \mid I) dEdS \right)$$

$$- \left(2\overline{L} \sum_{I=0,1} p(I) \int_S \int_E (ISE) \; pr(S, E \mid I) dEdS \right)$$

$$+ \left(\overline{L}^2 \sum_{I=0,1} p(I) \int_S \int_E \; pr(S, E \mid I) dEdS \right)$$

$$= \left(\sum_{I=0,1} p(I) \int_S \int_E (ISE)^2 \; pr(S, E \mid I) dEdS \right) - \overline{L}^2$$

$$= P \int_S \int_E (SE)^2 \; pr(S, E \mid I = 1) dEdS - \overline{L}^2$$

- If *S*, *E* is uncorrelated

$$UL^2 = P\int_S\int_E (SE)^2\ pr(S)\,pr(E)\,dE\,dS - \overline{L}^2$$

$$= P\int_S S^2\ pr(S)\,dS\ \int_E E^2\ pr(E)\,dE - \overline{L}^2$$

$$= P(\sigma_S^2 + \overline{S}^2)(\sigma_E^2 + \overline{E}^2) - (P\overline{S}\,\overline{E})^2$$

$$= (P - P^2)\overline{S}^2\,\overline{E}^2 + P(\sigma_S^2\overline{E}^2 + \sigma_E^2\overline{S}^2 + \sigma_S^2\sigma_E^2)$$

★與之前不一樣的地方，除了由常數改為變數的平均，還多了後面那一項

Determining Losses Due to Both Default and Downgrades

- 上小節只考慮了default，沒考慮credit migration的影響，此小節同時考慮default與downgrades帶來的損失，亦即要算考慮了default與downgrades的EL與UL
- Credit rating (p.270 Table 19-1) and Credit migration (p.267 Table 18-A1 and p.259 Table 18-1)
- 舉一BBB rating之loan，本金$100，3年到期一次還本，再給定prob. matrix of credit migration的情況下，算一年後的EL與UL (p.260~p.262)

$$EL = \sum_{G} P_G L_G$$

$$UL = \sqrt{\sum_{G} P_G (L_G - EL)^2}$$

- 比較只考慮default的EL與UL

$$EL = P_D L_D$$

$$UL = \sqrt{P_D (L_D - EL)^2 + (1 - P_D)(0 - EL)^2} = \sqrt{P_D - P_D^{\ 2}} L_D$$

★發現EL會與同時考慮default與downgrades的EL少很多，但UL卻差不多，因為UL大多被極端值所決定

- 若考慮L_D非constant，則要用模擬來計算EL與UL，先根據prob. matrix of credit migration來模擬一年後可能的credit rating，如果是除了default以外的rating，則L_G的計算與Table 18.3一樣，但若是default，則隨機產生L_D

Determining Default Probabilities Over Multiple Years

- 考慮多年期的loan，亦即需考慮多年期之credit migration (p.262~p.266)
  - Cumulative probability of default：0 ~ T破產機率 (p.265 Figure 18-1 and Table 18-5)
  - Marginal probability of default：T-1 ~ T破產機率

$$P_{D,Marginal,T} = P_{D,Cumulative,T} - P_{D,Cumulative,T-1}$$

  - Conditional probability of default：(p.266 Figure 18-2)

$$P_{D,Conditional,T} = \frac{P_{D,Cumulative,T} - P_{D,Cumulative,T-1}}{1 - P_{D,Cumulative,T-1}}$$

$$= \frac{P_{D,Marginal,T}}{1 - P_{D,Cumulative,T-1}}$$

- For a single-A rated bond, the probability of default in the first year is 4 basis points. The probability of defaulting by the end of the second year is given as follows:

$$
\begin{aligned}
P_{D,2} &= \sum_G P_G P_{D|G} \\
&= P_{AAA}P_{D|AAA} + P_{AA}P_{D|AA} + P_A P_{D|A} \\
&\quad + P_{BBB}P_{D|BBB} + P_{BB}P_{D|BB} + P_B P_{D|B} \\
&\quad + P_{CCC}P_{D|CCC} + P_D P_{D|D} \\
&= 0.07\% \times 0\% + 2.25\% \times 0.01\% + 91.76\% \times 0.04\% \\
&\quad + 5.19\% \times 0.22\% + 0.49\% \times 0.98\% + 0.2\% \times 5.3\% \\
&\quad + 0.01\% \times 21.94\% + 0.04\% \times 100\% \\
&= 0.11\%
\end{aligned}
$$

- This is painful to do with the above algebra, but is easy to do with matrix multiplication if we treat the migration matrix (*M*) as a state transition matrix.
- Define G to be a vector giving the probability of being in each grade and follow the above example, then

$$G_{T=0} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

and

$$G_{T=1} = MG_{T=0}$$

$$G_{T=2} = MG_{T=1} = MMG_{T=0} = M^2 G_{T=0}$$

$$\vdots$$

$$G_{T=N} = M^N G_{T=0}$$