Chapter 7

Value-at-Risk Contribution

Introduction

- The reports from a VaR calculator
  - Total VaR for the trading operation
  - Stand-alone VaR for each subportfolio
  - Stand-alone VaR for each risk factor
  - Sensitivity to each risk factor
  - VaR Contribution for each subportfolio
  - VaR Contribution for each risk factor

Introduction

- The problems with the stand-alone VaR
  - The stand-alone VaR for a subportfolio is the VaR that this portfolio have if we ignored the rest portfolios of the bank
  - Similarly, the stand-alone VaR for each risk factor is calculated by setting the standard deviation on all the other risk factors equal to zero
  - The main problem with the stand-alone VaR is that the sum of the stand-alone VaRs in general does not equal the total VaR (This is because the stand-alone VaR ignores the correlation with the rest of the portfolio)

VaRC (Value at Risk Contribution)

- VaRC is constructed so that the sum of VaRC for all subportfolios equals the total VaR for the portfolio
- VaRC gives us a measure of risk for each individual subportfolio that includes the interportfolio correlation effects
- VaRC is also useful for allocating the bank's capital to those units causing the risk and for setting limits on the amount of risk that individual traders may take

- 原本之VaR $P = A + B$

$$\Rightarrow \text{VaR}_P = 2.32\sigma_P = 2.32\sqrt{\sigma_A^2 + 2\rho_{AB}\sigma_A\sigma_B + \sigma_B^2}$$

$$= \sqrt{\text{VaR}_A^2 + 2\rho_{AB}\text{VaR}_A\text{VaR}_B + \text{VaR}_B^2}$$

- 現在希望 $\text{VaR}_p = \text{VaRC}_A + \text{VaRC}_B$

$$\sigma_P^2 = \sigma_A^2 + 2\rho_{AB}\sigma_A\sigma_B + \sigma_B^2$$

$$\sigma_P^2 = \sigma_A\left(\sigma_A + \rho_{AB}\sigma_B\right) + \sigma_B\left(\sigma_B + \rho_{AB}\sigma_A\right)$$

$$\sigma_P = \sigma_A\left(\frac{\sigma_A + \rho_{AB}\sigma_B}{\sigma_P}\right) + \sigma_B\left(\frac{\sigma_B + \rho_{AB}\sigma_B}{\sigma_P}\right)$$

$$\text{VaR}_P = 2.32 \cdot \sigma_P = 2.32 \cdot \sigma_A \cdot \left(\frac{\sigma_A + \rho_{AB}\sigma_B}{\sigma_P}\right) + 2.32 \cdot \sigma_B\left(\frac{\sigma_B + \rho_{AB}\sigma_A}{\sigma_P}\right)$$

If define $W_A = \dfrac{\sigma_A + \rho_{AB}\sigma_B}{\sigma_P}$, and $W_B = \dfrac{\sigma_B + \rho_{AB}\sigma_A}{\sigma_P}$

$\Rightarrow \text{VaR}_P = \text{VaRC}_A + \text{VaRC}_B$

where $\text{VaRC}_A = 2.32 \cdot \sigma_A \cdot \left(\dfrac{\sigma_A + \rho_{AB}\sigma_B}{\sigma_P} \right) = 2.32 \cdot \sigma_A \cdot W_A$

$$\text{VaRC}_B = 2.32 \cdot \sigma_B \cdot \left(\frac{\sigma_B + \rho_{AB}\sigma_A}{\sigma_P} \right) = 2.32 \cdot \sigma_B \cdot W_B$$

* W_A 與 W_B 是VaRC factor

If $\rho_{AB} = 1, W_A = W_B$

$\rho_{AB} = -1, W_A = -W_B$

- VaRC: 不只能用資產分，也能用risk factors來分

  假設影響有 risk factor 1 (with standard deviation σ_1) 和 risk factor 2 (with standard deviation σ_2)

  Define $d_i = \dfrac{\partial P}{\partial r_i}$

$$\because \ \Delta P = d_1 \Delta r_1 + d_2 \Delta r_2$$

$$\therefore \ \sigma_P^2 = d_1^2 \sigma_1^2 + 2\rho_{12} d_1 \sigma_1 d_2 \sigma_2 + d_2^2 \sigma_2^2$$

$$\Rightarrow \ \sigma_P = d_1 \sigma_1 \left(\frac{d_1 \sigma_1 + \rho_{12} d_2 \sigma_2}{\sigma_P} \right) + d_2 \sigma_2 \left(\frac{d_2 \sigma_2 + \rho_{12} d_1 \sigma_1}{\sigma_P} \right)$$

$$\Rightarrow \text{VaRC}_1 = 2.32 \times d_1 \sigma_1 \frac{\left(d_1 \sigma_1 + \rho_{12} d_2 \sigma_2 \right)}{\sigma_P}$$

$$\text{VaRC}_2 = 2.32 \times d_2 \sigma_2 \frac{\left(d_2 \sigma_2 + \rho_{12} d_1 \sigma_1 \right)}{\sigma_P}$$

- (i) 一般寫法 (N risky asset with N risk factor)

$$\text{VaRC}_1 = 2.32 \cdot d_1\sigma_1 \cdot \left(\frac{d_1\sigma_1 + \rho_{1,2}d_2\sigma_2 + \cdots + \rho_{1,N}d_N\sigma_N}{\sigma_p} \right)$$

$$\vdots$$

$$\text{VaRC}_j = 2.32 \cdot d_j\sigma_j \cdot \left(\frac{\rho_{j,1}d_1\sigma_1 + \rho_{j,2}d_2\sigma_2 + \cdots + d_j\sigma_j + \cdots + \rho_{j,N}d_N\sigma_N}{\sigma_p} \right)$$

(ii)用 Σ 來表示

$$\sigma_P = \sum_{i=1}^{N} d_i\sigma_i \times \frac{\sum_{j=1}^{N}\rho_{i,j}d_j\sigma_j}{\sigma_P}, \text{其中 } \sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{i,j}\left(d_j\sigma_j\right)\left(d_i\sigma_i\right)$$

$$\text{VaRC}_i = 2.32 \cdot d_i\sigma_i \times \frac{\sum_{j=1}^{N}\rho_{i,j}d_j\sigma_j}{\sigma_P}$$

(iii) 用matrix來表示

$$D = [d_1 \quad d_2] = D_1 + D_2 \qquad D_1 = [d_1 \quad 0] \qquad D_2 = [0 \quad d_2]$$

$$C = \begin{bmatrix} \sigma_1^2 & \rho_{1,2}\sigma_1\sigma_2 \\ \rho_{1,2}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$\sigma_P^2 = D\,C\,D^T = (D_1 + D_2)C\,D^T = D_1 C\,D^T + D_2\,C\,D^T$$

$$\Rightarrow \sigma_p = \frac{DCD^T}{\sqrt{DCD^T}} = \frac{D_1CD^T}{\sqrt{DCD^T}} + \frac{D_2CD^T}{\sqrt{DCD^T}}$$

$$\Rightarrow \text{VaR}_P = \text{VaRC}_1 + \text{VaRC}_2 = 2.32\frac{D_1CD^T}{\sqrt{DCD^T}} + 2.32\frac{D_2CD^T}{\sqrt{DCD^T}}$$

where $D_1C\,D^T = [d_1 \;\; 0]\begin{bmatrix} \sigma_1^2 d_1 + \rho_{12}\sigma_1\sigma_2 d_2 \\ \rho_{12}\sigma_1\sigma_2 d_1 + \sigma_2^2 d_2 \end{bmatrix} = \sigma_1^2 d_1^2 + \rho_{12}\sigma_1 d_1\sigma_2 d_2 = \sigma_1 d_1(\sigma_1 d_1 + \rho_{12}\sigma_2 d_2)$

$$D_2C\,D^T = [0 \;\; d_2]\begin{bmatrix} \sigma_1^2 d_1 + \rho_{12}\sigma_1\sigma_2 d_2 \\ \rho_{12}\sigma_1\sigma_2 d_1 + \sigma_2^2 d_2 \end{bmatrix} = \rho_{12}\sigma_1 d_1\sigma_2 d_2 + \sigma_2^2 d_2^2 = \sigma_2 d_2(\rho_{12}\sigma_1 d_1 + \sigma_2 d_2)$$

- 例子：美國公司持有英鎊債券與英鎊現金

$$D_{Bond} = \left[d_{Bound,FX} \quad d_{Bond,r_P} \right]$$

$$D_{cash} = \left[d_{cash,FX} \quad 0 \right]$$

$$D = D_{Bond} + D_{cash} = \left[d_{Bond,FX} + d_{cash,FX} \qquad d_{Bond,r_P} \right]$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$d_{FX} \qquad\qquad d_{r_P}$$

$$= [d_{FX} \quad 0] + [0 \quad d_{r_P}]$$

$$= D_{FX} + D_{r_P}$$

$$D_{Bond} = \begin{bmatrix} 74.7 & -563 \end{bmatrix}$$

$$D_{cash} = \begin{bmatrix} 100 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 174.7 & -563 \end{bmatrix}$$

$$D_{FX} = \begin{bmatrix} 174.7 & 0 \end{bmatrix}$$

$$D_{r_p} = \begin{bmatrix} 0 & -563 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.0004 & -0.00006 \\ -0.00006 & 0.000025 \end{bmatrix}$$

$$\text{VaR} = 13.12$$

$$\text{VaRC}_{Bond} = 2.32 \cdot \frac{D_{Bond}\, C\, D^T}{\sqrt{DCD^T}} = 8.86$$

$$\text{VaRC}_{cash} = 2.32 \cdot \frac{D_{cash}\, C\, D^T}{\sqrt{DCD^T}} = 4.26$$

} 13.12

$$\text{VaRC}_{FX} = 2.32 \cdot \frac{D_{FX}\, C\, D^T}{\sqrt{DCD^T}} = 7.4347$$

$$\text{VaRC}_{r_P} = 2.32 \cdot \frac{D_{r_P}\, C\, D^T}{\sqrt{DCD^T}} = 5.6760$$

} 13.12

- 之前講的都是用Parametric VaR來看VaRC，要如何用Monte Carlo or Historical simulation求VaRC呢？
  - If we run 5000 scenarios, the 99% VaR would be defined by the 50th-worst result
  - VaRC can be calculated using the worst 50 cases

$$\%\text{Contribution of Position}_i = \frac{\sum_{50}[\text{Loss}_i|\text{Loss}_P > \text{VaR}]}{\sum_{50}[\text{Loss}_P|\text{Loss}_P > \text{VaR}]}$$

$$\text{VaRC}_i = \text{VaR}_P \times \%\text{Contribution of Position}_\text{i}$$

(但我認為公式應該是 $\%\text{Contribution of Position}_\text{i} = \frac{1}{50}\sum_{50}\frac{[\text{Loss}_i|\text{Loss}_P > \text{VaR}]}{[\text{Loss}_P|\text{Loss}_P > \text{VaR}]}$