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## **Chapter 6**

# The Three Common Approaches for Calculating Value at Risk

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# Introduction

- The VaR is started in 1993, when it was endorsed by the Group of Thirty (G-30) as part of “best practices” for dealing with derivatives
  - The methodology behind VaR is the combination of finance theory and statistics
    - Finance theory: pricing and calculating the sensitivity of financial instruments
    - Statistics: studying the behavioral of the distributions of risk factors
  - VaR is viewed as the best single measure to assess market risk because it combines information on the sensitivity of the value to changes in market-risk factors with information on the probable amount of change in those factors
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# Introduction

- Definition 1: VaR is the calculation of the level of loss (outcome) that is so bad that there is only a 1 in 100 chance of there being a loss (outcome) worse than the calculated VaR
- Definition 2: VaR is reported as the deviation between the mean and the percentile
  - \* If the time horizon is very short, the average return is close to zero and can be ignored, so the two definitions usually give similar values
- Three approaches to calculate VaR:
  - Parametric VaR (Definition 2)
  - Historical Simulation (Definition 1)
  - Monte Carlo Simulation (Definition 1)

# Parametric VaR

- 又稱Linear VaR, Variance-Covariance VaR, Greek-normal VaR, Delta-Normal VaR, Delta-Gamma Normal VaR

1. Define the set of risk factors

2. Find  $\frac{\partial \text{Portfolio } (p)}{\partial \text{Risk Factor } (r_i)}$

3. Using historical data of the risk factors to calculate the standard deviation of the changes and the correlations between them

4. 
$$\sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial p}{\partial r_i} \frac{\partial p}{\partial r_j} \rho_{i,j} \sigma_{r_i} \sigma_{r_j}$$

5. Assume the loss distribution is normally distributed, then

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$$\text{VaR}_{99\%} = 2.32 \times \sigma_p$$

- Example 1 : stock ( $S$ ), p.106 ~107
  - 2 : bond ( $r$ ), p.107 ~108
  - 3 : foreign bond ( $r, FX$ ), p.108 ~ 112
  - 4 : foreign bond ( $r, FX$ ) + foreign currency ( $FX$ ), p.113 ~ 114
  - 5 : foreign bond ( $r, FX$ ) (考慮 returns 為 normal distributions) p.132 ~ 133
- Exponentially Weighted Moving Average (EWMA) p. 134 ~135

The covariance is calculated recursively using yesterday's estimate of the covariance and today's market change.

$$\sigma_{i,j,t} = (1 - \lambda)x_{i,t}x_{j,t} + \lambda\sigma_{i,j,t-1}^2$$

\*In the RiskMetrics model of J. P. Morgan,  $\lambda$  is estimated to be 0.94 for daily data or 0.97 for monthly data.

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- 優點

1. 速度最快
2. 可計算出 VaR 之 contribution (explained in Ch7)

- 缺點

1. if  $p(r_i)$  不是 linear function，則用 sensitivity 預估，會誤差很大
2. 沒法描述 extreme tail events, such as crises (因假設 normal distribution)
3. assume variance-covariance between risk factors is stable and constant over time

- VaR Calculator (p.114 Figure 6-2)
- The VaR is typically run overnight, and ideally, at the start of the day, senior management should get a report of VaR to show how much could be lost today, and the main causes of such a loss
- Intraday calculation of VaR:
  - To manage some fast-moving positions
  - To let the trader know his VaR limit which is the important information for him to decide the next trade
  - \* Incremental calculation: 通常將新交易的derivative vectors加上前一日derivative vectors，再用前一日算出的covariance matrix，來計算intraday VaR (p.116 Figure

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# Historical-Simulation VaR

1. Calculate the percentage change of the risk factor on each day (usually the data of the last 250 trading days is taken as input)
2. Each percentage change is then multiplied by today's market values to represent 250 scenarios for tomorrow's values
3. For each of these scenarios, the portfolio is valued using full, nonlinear pricing models
4. The third-worst scenarios is selected as being the 99% VaR

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\*p.116~117, Table 6-2



- 優點

1. 最簡單計算VaR的方法
2. 容易解釋給別人聽
3. 不需假設分配 or stable correlation
4. 花的時間比parametric VaR長，比Monte-Carlo短

- 缺點

1. 結果會被幾次近期的大危機所dominate，而且這些從前的危機，並不一定能代表未來的危機
2. Unpleasant “window effect”：當250天的 window不斷的shift，某一次嚴重的危機，可能過了250天後，就不再列入考慮了，尤其是當這個嚴重的危機要掉出window前後，VaR會有極劇烈的變化

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# Monte Carlo Simulation VaR

1. Randomly creating many scenarios for future rates of the risk factors (The scenarios start from today's market condition and go one day forward to give possible values at the end of that day)
2. Using nonlinear pricing model to estimate the change in value for each end-of-day scenario
3. Calculating VaR according to the 1-percentile worst result

\*p.119 Figure 6-4

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- 優點

1. 比起parametric好，因為是套用完整的nonlinear pricing model，而非用線性來approximation投資組合價值的變化
2. 比起historical VaR好，因為可generate infinite number of scenarios

- 缺點

1. 模擬risk factor上千次，代入計算上千次最慢
2. 還是必需對joint distribution of the risk factors做假設

- 最難之處：同時產生多個random variables的抽樣，且需符合其variance-covariance之設定，通常要利用 the technique of the covariance matrix decomposition
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• 目標，從  $\begin{pmatrix} r_x \\ r_y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}\right)$  抽樣出  $\tilde{r}_x, \tilde{r}_y$

(I)  $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$  抽樣出  $\tilde{Z}_1, \tilde{Z}_2$

(II) 
$$\begin{cases} \tilde{r}_x = \sigma_x \tilde{Z}_1 \\ \tilde{r}_y = \sigma_y (\tilde{Z}_1 \cdot \rho + \tilde{Z}_2 \sqrt{1 - \rho^2}) \end{cases}$$

◎但當超過兩個變數以上就不行了，要改用：

$$\begin{cases} \text{Cholesky Decomposition} \\ \text{Eigenvalue Decomposition} \end{cases}$$

※ In Excel, “=NORMSINV ( RAND( ) )” 可以模擬出標準常態分配  $\tilde{Z}$

# Cholesky Decomposition

Covariance matrix  $C = A^T A$ , 其中假設  $A = \begin{bmatrix} \alpha & \beta \\ 0 & \phi \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a = \sigma_x^2 & b = \rho\sigma_x\sigma_y \\ b = \rho\sigma_x\sigma_y & c = \sigma_y^2 \end{bmatrix} = \begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 + \phi^2 \end{bmatrix}$$

$$\text{解可得： } \alpha = \sqrt{a}, \quad \beta = \frac{b}{\sqrt{a}}, \quad \phi = \sqrt{c - \frac{b^2}{a}}$$

$$\begin{bmatrix} \tilde{r}_x & \tilde{r}_y \end{bmatrix}_{1 \times 2} = \begin{bmatrix} \tilde{z}_1 & \tilde{z}_2 \end{bmatrix}_{1 \times 2} \begin{bmatrix} A \end{bmatrix}_{2 \times 2} \text{ (可延伸到 } n \times n \text{)}$$

# Eigenvalue Decomposition

$$C = E^T \cdot \Lambda \cdot E$$

$$\text{where } \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \quad E^T E = I$$

$$= E^T \cdot \Lambda^{\frac{1}{2}} \cdot \Lambda^{\frac{1}{2}} \cdot E$$

\*If  $\lambda_i$  最大，表示risk factor  $i$  最重要

$$\text{where } \Lambda^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_N} \end{bmatrix}$$

$$= B^T B, \text{ where } B = \Lambda^{\frac{1}{2}} \cdot E$$

$$[\tilde{r}_1 \ \tilde{r}_2 \ \cdots \ \tilde{r}_N]_{1 \times N} = [\tilde{Z}_1 \ \tilde{Z}_2 \ \cdots \ \tilde{Z}_N]_{1 \times N} \cdot [B]_{N \times N}$$

- Why the Cholesky and Eigenvalue decompositions work?

The variance-covariance matrix of  $[r_1 \ r_2 \ \cdots \ r_N]$  is

$$E \begin{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \begin{bmatrix} r_1 & r_2 & \cdots & r_N \end{bmatrix} \end{bmatrix} = E \begin{bmatrix} A^T \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} \begin{bmatrix} z_1 & z_2 & \cdots & z_N \end{bmatrix} A \end{bmatrix}$$

$$= A^T E \begin{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} \begin{bmatrix} z_1 & z_2 & \cdots & z_N \end{bmatrix} \end{bmatrix} A = A^T I A = A^T A = C$$



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- p.124,  $FX$  與  $r$ 之例子
  - p.125~126, yield curve之例子
    - \* 因為 $E$ 是由eigenvectors構成，而eigenvectors是互相獨立(由 $E^T E = I$ 也可得知)，所以 $B$ 中的每個row，都是互相獨立，所以可以 $B$ 可看成多組互相獨立變動之情境 (independent scenarios of movements of market factors)
    - \* For example, changes in the yield curve can be regarded as independent movements of shift, twist, flex, and wiggle

# Improve the speed of Monte Carlo Simulation

- Parallel Processing (用多台電腦一起跑)
- Stabilization of Results (減少 scenarios)
  - (i) 不用每天都create那麼多 scenarios，因為這樣無法分辨是因risk-factor變而變，還是因不同 scenarios而變，如此一來，需要模擬非常多次來“average out” the random fluctuations
  - (ii) 固定一組 normally distribution，change variance-covariance matrix and the weight of each security daily in the portfolio (如何固定呢？一種是產生一次，存起來以後用，另一種是再每次模擬時，都使用相同的seed)
    - \* 或是可以一部分用舊的，另一部分從distribution中重新抽出

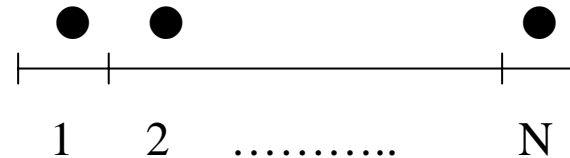
- Variance-reduction technique

- Antithetic sampling  $\tilde{Z}_1 \cdots \tilde{Z}_{\frac{N}{2}}, \tilde{Z}_{\frac{N}{2}+1}, \dots, \tilde{Z}_N$

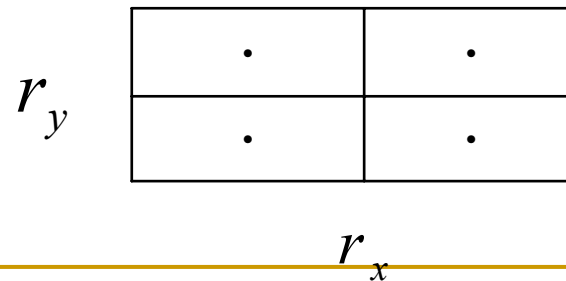
$$\begin{array}{ccc} & & \\ & & \\ & & \\ & & \\ & \Downarrow & \Downarrow \\ & - \tilde{Z}_1 & - \tilde{Z}_{\frac{N}{2}} \end{array}$$

- Stratification

(p.129, one dimension 切四格)



(p.128 與 p.129 Figure 6-6, two dimensions 切四格)



## ■ Importance sampling

focus on an important region，例如若要看crisis in the tail of the distribution，則加大 $\sigma$ 再模擬，不過記得要調整回原來真實的機率（現在一般不這樣做，一般都是change measure，將mean移到tail，抽樣後，再做change measure，將mean移回來）

## ■ Latin Hypercube

跟Stratification一樣，不過並非每格恰好一個sample，而是每格最多取一個

\* 當risk factor的數目增加，上述的方法很難產生well-balanced pseudorandom numbers，重點還是要減少risk factor，或是 concentrate on the more important risk factors

- Approximate Pricing (用比較不精準的評價模型來近似估計，例如，用BS model來估算American option之價值)

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# Attributes and limitations shared by all three methods

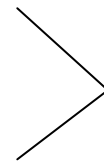
- 比較這三種算 VaR 的方法
    - p.104, Figure 6.1
    - Common attributes:
      1. Use Market-Risk factor to price thousands of different securities held by the bank (例如,  $r_t$ , FX, Equity Index, Forward Prices, Implied Volatility)
      2. Use the distribution of historical price changes (large amounts of information (長天期的歷史資料) vs. fresh information (最新的歷史資料))
    - Common limitations and disadvantages:
      1. Assume that past relationships between the risk factors will be repeated
      2. Each approach uses binning to put cash flows into a finite number of buckets
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## Binning:

(cash flow分類之方法很多，課本上是舉不同之到期日，相近的可成一group)

例：以三個月為最小單位，五個月後會發生之 cash flow，分配到三個月與六個月之cash flow中欲 match:

- ❶ cash-flow amount\*
- ❷ present value
- ❸ duration\*
- ❹ Stand-alone VaR



通常只取這兩個，因為只是將一分為二，只有二個變數 $C_3$ 與 $C_6$

$$C_5 = C_3 + C_6$$

$$D_5 = D_3 + D_6$$

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## Some Caveats for using VaR

- VaR is not the worst loss
- VaR does not describe the losses in the left tail
  - With the same value of VaR, there may be very different maximal losses
  - Therefore, the VaR of a portfolio may be greater than the sum of the VaRs for its components (VaR fails to qualify as a subadditive risk measure)
- VaR is measured with some error
  - Sampling period error
  - Simplification error
  - Higher confidence levels cause less precise VaR

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## Properties for Risk Measures

- Artzner, Delbaen, Eber, Heath (1999) list four desirable properties of risk measures for capital adequacy purposes: ( $X$  stands for random returns)
    - Monotonicity: if  $X_1 \leq X_2$ ,  $\rho(X_1) \geq \rho(X_2)$
    - Translation Invariance:  $\rho(X + k) = \rho(X) - k$
    - Homogeneity:  $\rho(bX) = b\rho(X)$
    - Subadditivity:  $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$
  - By definition, the VaR measure does not pass the subadditivity property, but the conditional VaR,  $E[-X|X \leq -VaR]$ , will satisfy all the above properties (In fact, only the parametric VaR qualifies the subadditivity property because the volatility of a portfolio is less than the sum of volatilities)
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- Consider a portfolio consisting of three corporate bonds, A, B, and C, with face value of \$100,000 and default probability of 0.5% individually
  - By definition, the VaR for each bond is \$0, and the summation of the VaR for each bond is \$0 as well
  - The VaR for this bond portfolio is \$100,000

State	Bonds	Probability	Payoff
No default		$0.995 \times 0.995 \times 0.995 = 0.9851$	\$0
1 default	A, B, C	$3 \times 0.005 \times 0.995 \times 0.995 = 0.0149$	-\$100,000
2 default	AB, BC, CA	$3 \times 0.005 \times 0.005 \times 0.995 = 0.0000746$	-\$200,000
3 default	ABC	$0.005 \times 0.005 \times 0.005 = 0.0000001$	-\$300,000