Ch 7. Greek Letters, Hedging, and Trading

I. Option Greek Letters

II. Numerical Differentiation to Calculate Greek Letters

III. Dynamic (Inverted) Delta Hedge

IV. Selected Trading Strategies

- This chapter analyzes the characteristics of the Greek letters of options first. Next, the numerical differentiation methods to calculate Greek letters are discussed. Third, the dynamic delta hedge method, which is the most common hedging method for institutional option traders, is synthesized. Moreover, the inverted delta hedge method to arbitrage from undervalued convertible bonds in Taiwan is introduced. Finally, several selected trading strategies are explored.

I. Option Greek Letters

- Consider an asset paying a yield q and its price process under the risk neutral measure Q as follows.

$$dS_t/S_t = (r-q)dt + \sigma dZ.$$

- The corresponding Black and Scholes formulas for call and put options on the asset are

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2),$$
$$p = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1),$$

where

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\ln(\frac{S_0}{K}) + (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

- Delta $\Delta \equiv \frac{\partial c}{\partial S_0}$
  - $\odot$ For calls: $\Delta = e^{-qT}N(d_1)$.
  - $\odot$ For puts: $\Delta = e^{-qT}[N(d_1) - 1]$.

  Derivation of Δ of calls:

  First, we can derive

$$\frac{\partial d_1}{\partial S_0} = \frac{\partial d_2}{\partial S_0} = \frac{\frac{1}{S_0}}{\sigma\sqrt{T}} = \frac{1}{S_0\sigma\sqrt{T}}.$$

  Next, we proceed the differentiation with respect to S_0:

$$\frac{\partial c}{\partial S_0} = e^{-qT}N(d_1) + S_0 e^{-qT}\phi(d_1)\frac{1}{S_0\sigma\sqrt{T}} - Ke^{-rT}\phi(d_2)\frac{1}{S_0\sigma\sqrt{T}},$$

  where the sum of the last two terms are

$$\begin{aligned}
& S_0 e^{-qT}\phi(d_1)\frac{1}{S_0\sigma\sqrt{T}} - Ke^{-rT}\phi(d_2)\frac{1}{S_0\sigma\sqrt{T}} \\
= & \frac{e^{-qT}\phi(d_1) - \frac{K}{S_0}e^{-rT}\phi(d_1 - \sigma\sqrt{T})}{\sigma\sqrt{T}}
\end{aligned}$$

$$\left\| \begin{array}{l}
\text{for the numerator} \\
e^{-qT}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} - \frac{K}{S_0}e^{-rT}\frac{1}{\sqrt{2\pi}}e^{-\frac{(d_1-\sigma\sqrt{T})^2}{2}} \\
\quad (-\frac{(d_1-\sigma\sqrt{T})^2}{2} = -\frac{d_1^2}{2} + d_1\sigma\sqrt{T} - \frac{\sigma^2 T}{2}) \\
= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}(e^{-qT} - \frac{K}{S_0}e^{-rT + d_1\sigma\sqrt{T} - \frac{\sigma^2 T}{2}}) \\
\quad (d_1\sigma\sqrt{T} = \ln(\frac{S_0}{K}) + rT - qT + \frac{\sigma^2 T}{2}) \\
= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}(e^{-qT} - \frac{K}{S_0}e^{\ln(\frac{S_0}{K}) - qT}) \\
= 0
\end{array} \right.$$

$$= 0.$$

  $\Rightarrow \frac{\partial c}{\partial S_0} = e^{-qT}N(d_1)$.

- $\frac{\partial c}{\partial K} = S_0 e^{-qT}\phi(d_1)\frac{\partial d_1}{\partial K} - e^{-rT}N(d_2) - Ke^{-rT}\phi(d_2)\frac{\partial d_2}{\partial K} = -e^{-rT}N(d_2)$, where $\frac{\partial d_1}{\partial K} = \frac{\partial d_2}{\partial K} = \frac{-1}{K\sigma\sqrt{T}}$.

- Theta $\theta \equiv -\frac{\partial c}{\partial T}$ (measuring the time decay of the option value)
  - $\odot$ For calls: $\theta = -\frac{S_0\phi(d_1)\sigma e^{-qT}}{2\sqrt{T}} + qS_0N(d_1)e^{-qT} - rKe^{-rT}N(d_2)$.
  - $\odot$ For puts: $\theta = -\frac{S_0\phi(d_1)\sigma e^{-qT}}{2\sqrt{T}} - qS_0N(-d_1)e^{-qT} + rKe^{-rT}N(-d_2)$.

- Gamma $\Gamma \equiv \frac{\partial^2 c}{\partial S_0^2}$
  - $\odot$ For calls and puts: $\Gamma = \frac{\phi(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$.

- Vega $\upsilon \equiv \frac{\partial c}{\partial \sigma}$
  - ⊙ For calls and puts: $\upsilon = S_0 e^{-qT}\sqrt{T}\phi(d_1) = Ke^{-rT}\sqrt{T}\phi(d_2)$.

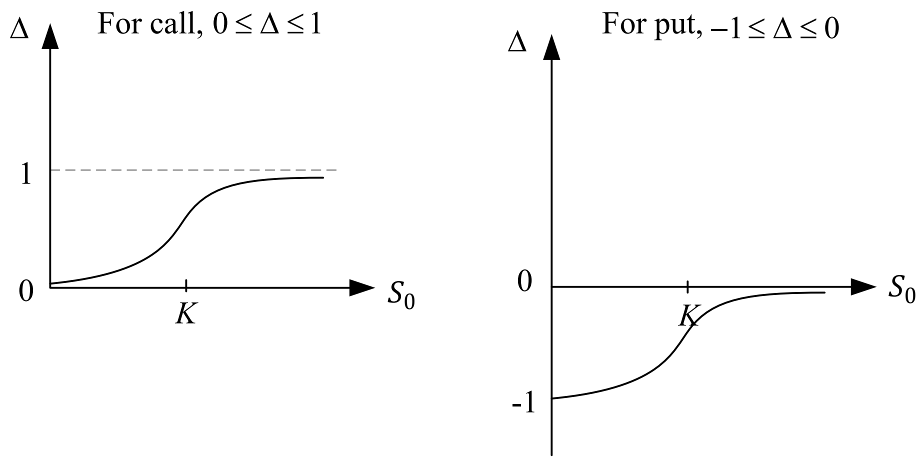
- Rho $\rho \equiv \frac{\partial c}{\partial r}$
  - ⊙ For calls: $\rho = KTe^{-rT}N(d_2)$.
  - ⊙ For puts: $\rho = -KTe^{-rT}N(-d_2)$.

- Characteristics of Greek letters

⊙ Δ:

(i) for calls, $0 \le \Delta \le 1$; for puts $-1 \le \Delta \le 0$.

Figure 7-1



(ii) When the time approaches the maturity T,

Δ(call) is 1 if the call is ITM and is 0 if the call is OTM.

Δ(put) is -1 if the put is ITM and is 0 if the call is OTM.

$*$ The value of Δ jumps almost between only two extreme values and thus varies discountinuously near maturity when the option is ATM.

$\odot$ Γ:

(i) For both calls and puts, the value of Γ, which measures the degree of curvature of the option value respect to S_0, is the same and always positive.

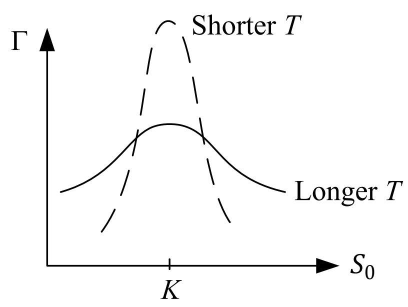
(ii) Since the value of Γ is positive, it benefits option holders.

(iii) The curve of Γ is similar to the probability density function of normal distributions because there is a term $\phi(d_1)$ in the formula of Γ on page 7-2.

(iv) The value of Γ attains its extreme when the option is at the money since the values Δ vary more intensely with respect to the change of S_0 when the option is at the money.

(v) The kurtosis of the curve of Γ because higher for shorter time to maturity T. The reason is due to the nearly discountinuous change of Δ, which implies an extremely high value of Γ, near maturity.

Figure 7-2



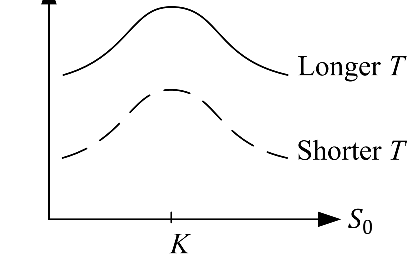
$\odot$ υ:

(i) For both calls and puts, their values of υ are the same and positive, which reflects that the values of options increases due to a higher degree of volatility of the underlying asset price.

(ii) The curve of υ is similar to the probability density function of normal distributions because there is a term $\phi(d_1)$ in the formula of υ on page 7-3.

(iii) When the time is close to the maturity date T, υ becomes smaller, which is due to that the period of time in which the volatility σ can affect the option value becomes shorter.

Figure 7-3

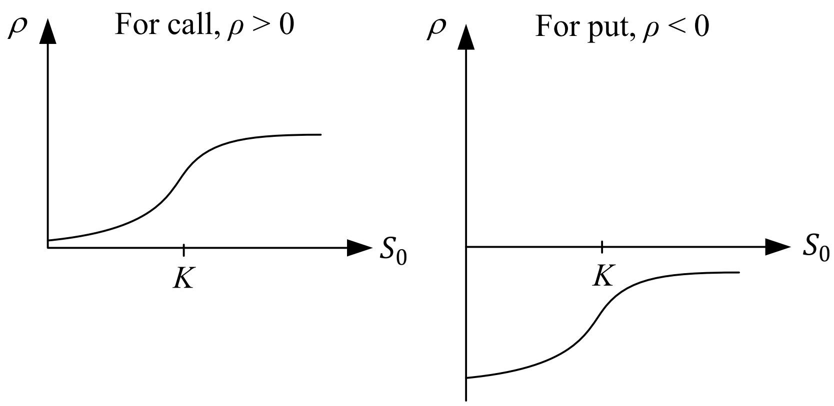


⊙ ρ:

(i) For calls, ρ is positive to reflect the positive impact of a higher growth rate r on the probability of being ITM.

(ii) For puts, ρ is negative to refelct the negative impact of a higher growth rate r on the probability of being ITM.

Figure 7-4

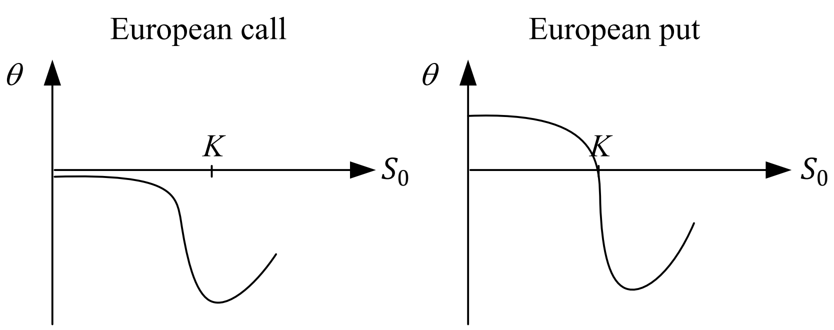


⊙ θ:

(i) θ can measure the speed of the option value decay with the passage of time. It is worth noting that unlike the stochastic S_0, σ, and r, the time is passing continuously and not a risk factor.

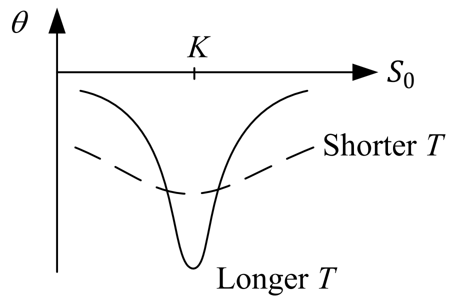
(ii) The value of θ is always negative for both American calls and puts. However, it is not always true for European puts because the dividend distritution could make the value of European put rise to cover the time decay of the put value.

Figure 7-5



(iii) For options that are at the money, the time decay of the option value is fastest (than other moneyness) and thus the corresponding θ is most negative. See Figure 7-6.

Figure 7-6



- The relationship among Δ, Γ, and θ:

  Based on the partial differential equation of any derivative f,
  $$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf,$$
  and the definitions of Δ, Γ, and θ, we can derive
  $$\theta + (r-q)S\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rf.$$
  If we know the value of the derivative f and any two values among Δ, Γ, and θ, then we can solve the value of the unknown Greek letter through the above equation.

  Moreover, suppose f represents a delta-neutral portfolio, i.e., its Δ equals 0. Then the relationship between the Γ and θ of the portfolio f can be expressed as
  $$\theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rf,$$
  which implies that for this delta-neutral portfolio, if the value of Γ is high, which is a desired feature for option holders, the speed of time decay is also fast, i.e., the value of θ is fairly negative and thus the option value declines quickly as time goes on.

II. Numerical Differentiation to Calculate Greek Letters

- One of the advantages of the Black-Scholes model is that the proposed option pricing formula is differentiable, which makes it simple to calculate the Greek letters. However, for other numerical option pricing methods, they need to resort to the numerical differentiation to calculate the Greek letters. It is worth noting that for American options, there is no analytic pricing formula and thus a numerical option pricing method combined with the numerical differentiation should be considered.

- The main idea of the numerical differentiation is to employ the finite difference as the approximation, i.e., if f is the option pricing function of any numerical method, then

$$\begin{aligned}\frac{\partial f}{\partial S_0} &\approx \frac{f(S_0+\Delta S)-f(S_0)}{\Delta S} \text{ (forward difference)},\\ &\approx \frac{f(S_0)-f(S_0-\Delta S_0)}{\Delta S} \text{ (backward difference)},\\ &\approx \frac{f(S_0+\Delta S)-f(S_0-\Delta S)}{2\Delta S} \text{ (central difference)}.\end{aligned}$$

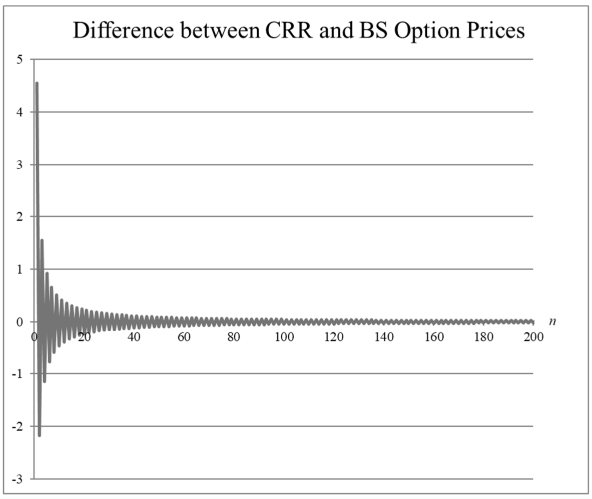
  $*$ Note that to compute the finite difference, it needs double time of the employed numerical option pricing model.

  $*$ In this chapter, we focus on two numerical option pricing models, the CRR binomial tree model and the Monte Carlo simulation.

- Numerical differentiation for the CRR binomial tree model.

  ⊙ Convergent behavior of the CRR binomial tree model: Given $S_0 = 100$, $K = 100$, $r = 0.05$, $q = 0.02$, $\sigma = 0.5$, and $T = 1$, and the Black-Scholes call value is 20.5465. For the CRR binomial tree model, the examined n ranges from 1 to 200. The differences between the results of the CRR binomial tree model and the Black-Scholes model are shown in Figure 7-7.

  Figure 7-7



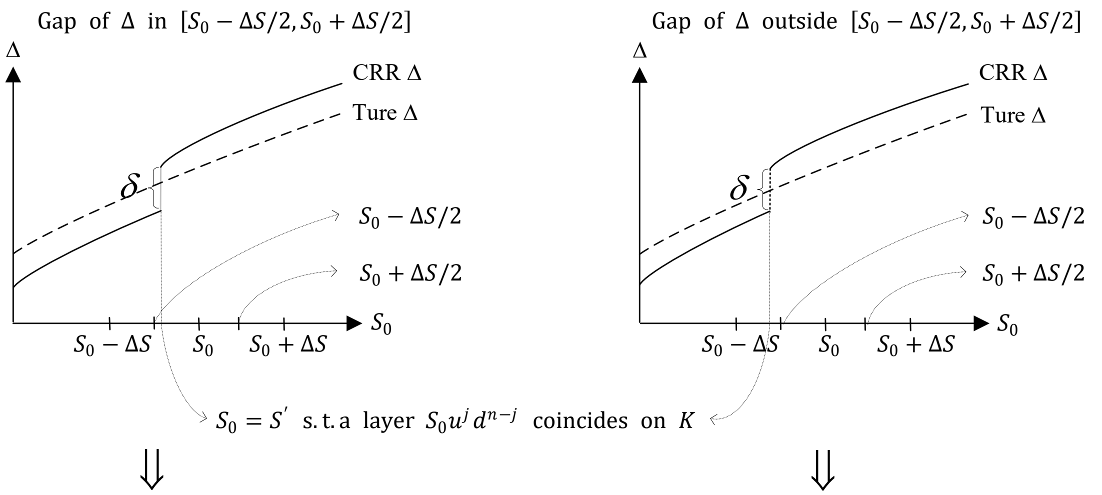
⊙ The reason for the oscillatory convergence behavior:
Suppose that when one of the stock price layers $S_0 u^j d^{n-j}$, $j = 0, 1, ..., n$, e.g., the layer k with $S_0 u^k d^{n-k}$, is close to or coincides on the strike price K, the error of the CRR binomial tree model is minimized.

Next, when the number of partitions n increases by 1, which changes the upward and downward multiplying factors u and d, the layer m could deviate from the strike price and it is very possible that there is no other layers that are quite close to or coincide on the strike price K. Therefore, the error of the CRR binomial tree model becomes larger.

In conclusion, as any stock price layer approaching and deviating from the strike price with the increase of n, the error of the CRR binomial tree decreases and increases, which causes the oscillatory convergence toward the theoretical value based on the Black-Scholes model.

⊙ Next, we fix the partition number of n and discuss about the choice of the ΔS when calculating the Greeks with the numerical differentiation. Based on the same reasoning mentioned above, the choice of the ΔS may shifts away a stock price layer, which is originally close to or coincides on the strike price K, from the strike price K. It is also possible to shift another stock price layer to become closer toward the strike price K. As a result, the oscillatory option values occurs and sometimes the errors caused from the oscillation increase with a decrease in ΔS. See Figure 7-8.

Figure 7-8



$$\text{CRR } \Gamma = \frac{\Delta(S_0 + \Delta S/2) - \Delta(S_0 - \Delta S/2)}{\Delta S}$$

$$\approx \text{True } \Gamma + \frac{\delta}{\Delta S}$$

$$\text{CRR } \Gamma = \frac{\Delta(S_0 + \Delta S/2) - \Delta(S_0 - \Delta S/2)}{\Delta S} \approx \text{True } \Gamma$$

$\odot$ By observing the above figure, we can conclude

(i) The oscillatory convergence causes the numerical differentiation method to over- or underestimated Δ by turns along the S_0 axis.

(ii) A smaller ΔS may not be better: The discontinuity of Δ could hurt the estimation of Γ based of the numerical differentiation seriously. Note that the error term $\delta/\Delta S$ increases with a smaller value of ΔS, which generates a counterintuitive result that it is not more accurate if you specify a smaller value of ΔS.

$*$ To remedy the estimation problem of Γ, a moderately large ΔS should be considered.

By enhancing the probability such that $\Delta(S_0 + \Delta S/2)$ and $\Delta(S_0 - \Delta S/2)$ are simultaneously over- or underestimated, then there is little error term associated with δ.

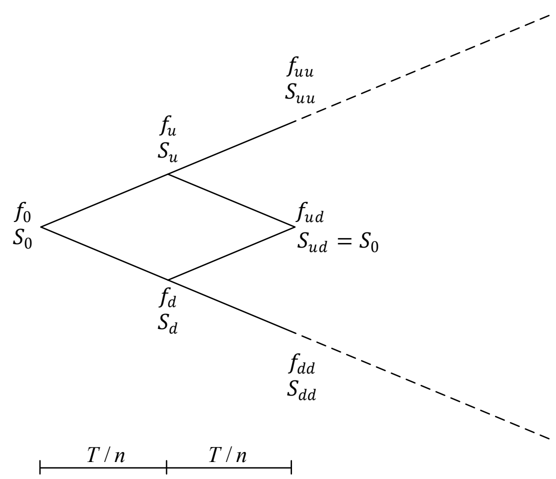
Even when one of $\Delta(S_0 + \Delta S/2)$ and $\Delta(S_0 - \Delta S/2)$ is overestimated and the other is underestimated, the error term $\delta/\Delta S$ is not significantly because firstly ΔS is large and secondly the difference between $\Delta(S_0 + \Delta S/2)$ and $\Delta(S_0 - \Delta S/2)$ is large enough to dominate the results of the numerical differentiation method.

- Another method to calculate Greek letters based on the CRR binomial tree model:

  (i) The above numerical differentiation method is with not only the accuracy problem but also the efficiency problem due to the necessity to calculate the binomial tree twice.

  (ii) To solve both these problem, another method is proposed based on one-time binomial tree calculation, as illustrated in Figure 7-9. However, this method can estimated only Δ, Γ, and θ.

Figure 7-9

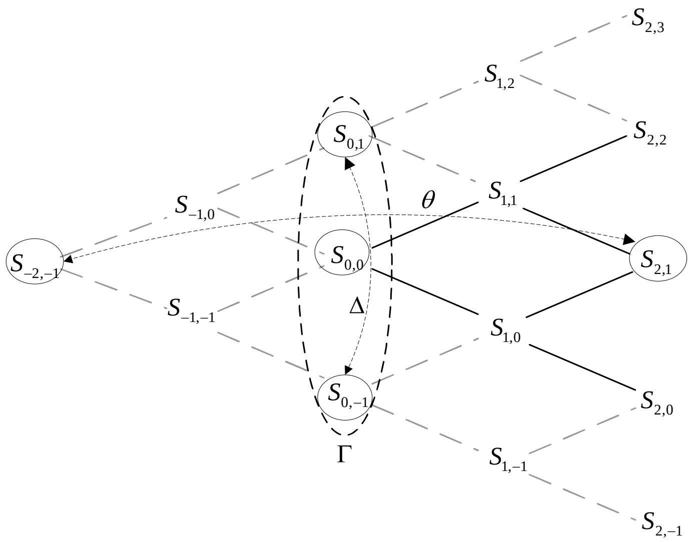


$$\Delta \approx \frac{f_u - f_d}{S_u - S_d} \qquad \Gamma \approx \frac{\frac{f_{uu}-f_{ud}}{S_{uu}-S_{ud}} - \frac{f_{ud}-f_{dd}}{S_{ud}-S_{dd}}}{\left(\frac{S_{uu}+S_{ud}}{2}\right) - \left(\frac{S_{ud}-S_{dd}}{2}\right)} \qquad \theta \approx \frac{f_{ud}-f}{2(T/n)}$$

⊙ The problems with this method: The estimations of Δ and Γ are actually Δ and Γ at T/n and $T/2n$, but not today. If n approaches infinity, the errors become negligible.

- The extended tree model proposed by Pelsser and Vorst (1994), which is an improvement of the above method. See Figure 7-10.

Figure 7-10



∗ The extended tree model is more accurate than the method in Figure 7-9. However, similar to the method in Figure 7-9, The extended tree model is applicable only for Δ, Γ, and θ.

- Numerical differentiation based on the Monte Carlo Simulation:

⊙ For the Monte Carlo simulation, the finite difference method together with the technique of common random variables can be employed to compute Greek letters. However, the huge computational burden of the Monte Carlo simulation deteriorates because the simulation needs to be performed twice in the finite difference method.

⊙ Here two one-time simulation methods to compute Greek letters are introduced. These two methods, the pathwise and likelihood methods, are proposed by Broadie and Glasserman (1996). In this section, only the European put is taken as example, and it is straightforward to apply these methods to the European call.

$\odot$ Pathwise method: all parameters affect S_T and in turn affect the option value, so to calculate $\partial f/\partial x$ for any parameter x, the calculation of $\frac{\partial f}{\partial S_T}\frac{S_T}{x}$ is considered.

Define $P = e^{-rT}(K - S_T)1_{\{K\geq S_T\}}$, and the option value today is $p = E[P]$.

$$\left\| \begin{array}{l} \frac{\partial P}{\partial S_T} = -e^{-rT}1_{\{K\geq S_T\}}. \\ \frac{\partial S_T}{\partial S_0} = \frac{S_T}{S_0} \text{ (because } S_T = S_0 e^{(r-q-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z}). \end{array} \right.$$

(i) $\Delta = E[\frac{\partial P}{\partial S_0}]$, where $\frac{\partial P}{\partial S_0} = -e^{-rT}\ 1_{\{K\geq S_T\}}\frac{S_T}{S_0}$.

(Note that to estimate Δ, we first simulate random samples of S_T and next approximate $E[\frac{\partial P}{\partial S_0}] = E[-e^{-rT}\ 1_{\{K\geq S_T\}}\frac{S_T}{S_0}]$ with the arithmetic average of $-e^{-rT}\ 1_{\{K\geq S_T\}}\frac{S_T}{S_0}$.)

(ii) $\Gamma = E[(\Delta(S_0+h) - \Delta(S_0))/h] = E[(-e^{-rT}\ (\frac{S_T}{S_0})\ (1_{\{K\geq S_T(S_0+h)\}} - 1_{\{K\geq S_T(S_0)\}}))/h]$

$= E[+e^{-rT}\ (\frac{S_T}{S_0})\ 1_{\{S_T(S_0+h)\geq K\geq S_T(S_0)\}}/h] = E[e^{-rT}(\frac{S_T}{S_0})\frac{G(S_T(S_0+h))-G(S_T(S_0))}{h}]$

(because $\frac{S_T(S_0+h)}{S_0+h} = \frac{S_T(S_0)}{S_0}$, and this ratio is independent of S_0)

$\overset{h\to 0, S_T\to K}{=}\ E[e^{-rT}\ (\frac{K}{S_0})^2\ g(K)]$,

where $G = \int dg, g(K) = \frac{1}{K\sigma\sqrt{T}}n(d(K))$, and $d(K) = \frac{\ln(K/S_0)-(r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$.

(iii) $\upsilon = E[\frac{\partial P}{\partial \sigma}]$, where $\frac{\partial P}{\partial \sigma} = \frac{\partial P}{\partial S_T}\cdot\frac{\partial S_T}{\partial \sigma} = e^{-rT}1_{\{K\geq S_T\}}S_0 e^{(r-q-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z}(-\sigma T + \sqrt{T}Z)$

$= e^{-rT}\ 1_{\{K\geq S_T\}}\ \frac{S_T}{\sigma}[\ln\frac{S_T}{S_0} - (r - q + \frac{1}{2}\sigma^2)T]$.

(iv) $\rho = E[\frac{\partial p}{\partial r}]$, where $\frac{\partial P}{\partial r} = -T\ e^{-rT}\ 1_{\{K\geq S_T\}}(K - S_T) - e^{-rT}1_{\{K\geq S_T\}}\frac{\partial S_T}{\partial r}$

$= -T\ e^{-rT}\ 1_{\{K\geq S_T\}}(K - S_T) - e^{-rT}1_{\{K\geq S_T\}}\frac{\partial S_T}{\partial r}S_T T$

$= -KTe^{-rt}1_{\{K\geq S_T\}}$.

(v) $\theta = E[-\frac{\partial p}{\partial T}]$, where $-\frac{\partial P}{\partial T} = re^{-rT}(K - S_T)1_{\{K\geq S_T\}}$

$+e^{-rT}\ 1_{\{K\geq S_T\}}\frac{S_T}{2T}[\ln(\frac{S_T}{S_0}) - (r - q - \frac{1}{2}\sigma^2)T]$.

$\odot$ Likelihood method: all parameters affect the probability density function of S_T and in turn affect the option value.

Define the option value as $p = \int_0^\infty e^{-rT} \max(K-x,0) g(x) dx$

$$\left\| \begin{array}{l} g(x) = \frac{1}{x\sigma\sqrt{T}} n(d(x)) \ (g(S_T) \text{ is the probability density function of } S_T), \\ n(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \\ d(x) = \frac{\ln\frac{x}{S_0} - (r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \end{array} \right.$$

(i) $\Delta = \frac{\partial p}{\partial S_0} = \int_0^\infty e^{-rT} \max(K-x,0) \frac{\partial g(x)}{\partial S_0} dx$

$$= \int_0^\infty e^{-rT} \max(K-x,0) \frac{\partial g(x)}{\partial S_0} \frac{1}{g(x)} g(x) dx$$

$$= E[e^{-rT} \max(K-S_T,0) \frac{\partial \ln g(S_T)}{\partial S_0}].$$

$$\left\| \begin{array}{l} \ln g(x) = -(\ln x + \ln\sigma + \ln\sqrt{T} + \ln\sqrt{2\pi}) - \frac{d(x)^2}{2} \\ \Rightarrow \frac{\partial \ln g(x)}{\partial S_0}|_{x=S_T} = -d(x)\frac{\partial d(x)}{\partial S_0}|_{x=S_T} = -\frac{\ln\frac{S_T}{S_0} - (r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \frac{-1}{S_0\sigma\sqrt{T}} \\ \qquad = \frac{\ln\frac{S_T}{S_0} - (r-q-\frac{1}{2}\sigma^2)T}{S_0\sigma^2 T} = \frac{d(S_T)}{S_0\sigma\sqrt{T}}. \end{array} \right.$$

(ii) $\upsilon = \frac{\partial p}{\partial\sigma} = E[e^{-rT} \max(K-S_T,0) \frac{\partial \ln g(S_T)}{\partial\sigma}].$

$$\left\| \begin{array}{l} \frac{\partial \ln g(x)}{\partial\sigma}|_{x=S_T} = -\frac{1}{\sigma} - d(S_T)\underbrace{\frac{\partial d(x)}{\partial\sigma}|_{x=S_T}}_{\Downarrow \; \frac{\ln\frac{S_0}{S_T} + (r-q+\frac{\sigma^2}{2})T}{\sigma^2\sqrt{T}}}. \end{array} \right.$$

(iii) $\Gamma = \frac{\partial^2 p}{\partial {S_0}^2} = \int_0^\infty e^{-rT} \max(K-x,0) \frac{\partial^2 g(x)}{\partial {S_0}^2} dx$

$$= \int_0^\infty e^{-rT} \max(K-x,0) \frac{\partial^2 g(x)}{\partial {S_0}^2} \frac{1}{g(x)} g(x) dx$$

$$= E[e^{-rT} \max(K-S_T,0) \frac{\partial^2 g(S_T)}{\partial {S_0}^2} \frac{1}{g(S_T)}].$$

$$\| \frac{\partial^2 g(x)}{\partial {S_0}^2}|_{x=S_T} \cdot \frac{1}{g(S_T)} = \frac{d(S_T)^2 - d(S_T)\sigma\sqrt{T}}{S_0^2\sigma^2 T}.$$

(iv) $\rho = \frac{\partial P}{\partial r} = \int_0^\infty \max(K-x,0)[-Te^{-rT} g(x) + e^{-rT} \frac{\partial g(x)}{\partial r} \frac{1}{g(x)} g(x)] dx$

$$= \int_0^\infty e^{-rT} \max(K-x,0)[-T + \frac{\partial g(x)}{\partial r} \frac{1}{g(x)}] g(x) dx$$

$$= E[e^{-rT} \max(K-S_T,0)[-T + \frac{\partial g(S_T)}{\partial r} \frac{1}{g(S_T)}]].$$

$$\| \frac{\partial g(x)}{\partial r}|_{x=S_T} \frac{1}{g(S_T)} = \frac{\partial \ln g(x)}{\partial r}|_{x=S_T} = -d(S_T)\frac{\partial d(x)}{\partial r}|_{x=S_T} = d(S_T)\frac{\sqrt{T}}{\sigma}.$$

(v) $\theta = -\frac{\partial P}{\partial T} = \int_0^\infty \max(K-x,0)[re^{-rT}g(x) - e^{-rT}\frac{\partial g(x)}{\partial T}\frac{1}{g(x)}g(x)]dx$

$= E[e^{-rT}\max(K-S_T,0)[r - \frac{\partial \ln g(S_T)}{\partial T}]].$

$$\frac{\partial \ln g(S_T)}{\partial T} = -\frac{1}{2T} - d(S_T)\underbrace{\frac{\partial d(x)}{\partial T}|_{x=S_T}}.$$

$$\Downarrow$$

$$\frac{-\ln\frac{S_T}{S_0} - (r-q-\frac{\sigma^2}{2})T}{2\sigma T^{\frac{3}{2}}}$$

$*$ To estimate each Greek letters, we simulate random sample of S_T, and next estimate $E[\omega(S_T)]$ with the arithematic average of $\omega((S_T)$, where $[\omega((S_T)$ could be any target functions in the formulas of different Greek letters.

III. Dynamic (Inverted) Delta Hedge

- Consider an institutional investor to issue a call option contract, in which $S_0 = 49$, $K = 50$, $r = 0.05$, $q = 0$, $\sigma = 0.2$, $T = 0.3846$ (20 weeks), and the underlying assets of this call option contract are 100,000 shares of stock. The Black-Scholes model generates the theoretical value of this call option contract to be \$240,000.

  $\odot$ Inspired from the derivation of the partial differentiation equation, longing Δ_t shares of S_t and shorting one share of C_t can form an instantaneous risk free portfolio, i.e.,

  $$\Delta_t S_t - C_t = B_t,$$

  where B_t is the balance of borrowing costs and lending interests. Rewrite the above equation to derive

  $$C_t = \Delta_t S_t - B_t.$$

  That means dynamically adjusting the position $\Delta_t S_t - B_t$ can replicate C_t for any t. More specifically, the trading strategy of this dynamic delta hedge method is as follows.

  $$\begin{cases} S_t \uparrow, \Delta_t \uparrow \Rightarrow \text{buying stock shares such that the total position is } 100{,}000\Delta_t \text{ shares} \\ S_t \downarrow, \Delta_t \downarrow \Rightarrow \text{selling stock shares such that the total position is } 100{,}000\Delta_t \text{ shares} \end{cases}$$

Table 7-1 The Call Option is ITM at Maturity (collected from Hull (2011))

Week	*Stock price*	*Delta*	*Shares purchased*	*Cost of shares purchased ($000)*	*Cumulative cost including interest ($000)*	*Interest cost ($000)*
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
3	50.25	0.596	19,600	984.9	2,966.6	2.9
4	51.75	0.693	9,700	502.0	3,471.5	3.3
5	53.12	0.774	8,100	430.3	3,905.1	3.8
6	53.00	0.771	(300)	(15.9)	3,893.0	3.7
7	51.87	0.706	(6,500)	(337.2)	3,559.5	3.4
8	51.38	0.674	(3,200)	(164.4)	3,398.5	3.3
9	53.00	0.787	11,300	598.9	4,000.7	3.8
10	49.88	0.550	(23,700)	(1,182.2)	2,822.3	2.7
11	48.50	0.413	(13,700)	(664.4)	2,160.6	2.1
12	49.88	0.542	12,900	643.5	2,806.2	2.7
13	50.37	0.591	4,900	246.8	3,055.7	2.9
14	52.13	0.768	17,700	922.7	3,981.3	3.8
15	51.88	0.759	(900)	(46.7)	3,938.4	3.8
16	52.87	0.865	10,600	560.4	4,502.6	4.3
17	54.87	0.978	11,300	620.0	5,126.9	4.9
18	54.62	0.990	1,200	65.5	5,197.3	5.0
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0.0	5,263.3	

⊙ At maturity, the cost of the dynamic hedging strategy is $5,263,300, and the institutional investor owns 100,000 shares of stock. On the other hand, the call holder exercise this option to purchase 100,000 shares of stock at $\$50 \times 100,000$. As a consequence, the net cost of the institutional investor is $263,300, which is close to the call premium $240,000.

Table 7-2 The Call Option is OTM at Maturity (collected from Hull (2011))

Week	*Stock price*	*Delta*	*Shares purchased*	*Cost of shares purchased ($000)*	*Cumulative cost including interest ($000)*	*Interest cost ($000)*
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	49.75	0.568	4,600	228.9	2,789.2	2.7
2	52.00	0.705	13,700	712.4	3,504.3	3.4
3	50.00	0.579	(12,600)	(630.0)	2,877.7	2.8
4	48.38	0.459	(12,000)	(580.6)	2,299.9	2.2
5	48.25	0.443	(1,600)	(77.2)	2,224.9	2.1
6	48.75	0.475	3,200	156.0	2,383.0	2.3
7	49.63	0.540	6,500	322.6	2,707.9	2.6
8	48.25	0.420	(12,000)	(579.0)	2,131.5	2.1
9	48.25	0.410	(1,000)	(48.2)	2,085.4	2.0
10	51.12	0.658	24,800	1,267.8	3,355.2	3.2
11	51.50	0.692	3,400	175.1	3,533.5	3.4
12	49.88	0.542	(15,000)	(748.2)	2,788.7	2.7
13	49.88	0.538	(400)	(20.0)	2,771.4	2.7
14	48.75	0.400	(13,800)	(672.7)	2,101.4	2.0
15	47.50	0.236	(16,400)	(779.0)	1,324.4	1.3
16	48.00	0.261	2,500	120.0	1,445.7	1.4
17	46.25	0.062	(19,900)	(920.4)	526.7	0.5
18	48.13	0.183	12,100	582.4	1,109.6	1.1
19	46.63	0.007	(17,600)	(820.7)	290.0	0.3
20	48.12	0.000	(700)	(33.7)	256.6	

⊙ At maturity, the cost of the dynamic hedging strategy is $256,600, and the institutional investor does not own any share of stock. On the other hand, the call holder give up his right to exercise this option. As a consequence, $256,600 is the net cost of the institutional investor and is close to the call premium $240,000.

∗ When the rebalancing frequency increases, the payoff of the dynamic delta hedge is more stable and close to $240,000, which is the theoretical value based on the Black-Scholes model.

∗ In practice, the option issuer sells the option higher than the theoretical value, e.g., $300,000 in the above example. The reasons are as follows.

(i) The option issuer needs some profit to undertake this business.

(ii) Theoretically speaking, the hedging costs can approach the Black-Scholes option value only when

(ii.1) the rebalancing frequency approaches infinity, which is impossible in practice.

(ii.2) the volatility of stock returns remains constant and as expected during the subsequent option life, which could occur with little probability.

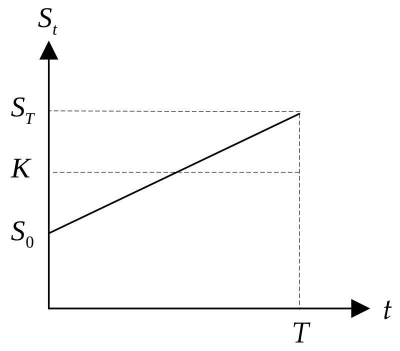
(iii) The transaction costs in the real world may incur more hedging costs than the theoretical amount.

As a result, it is common for option issuers to mark up the option premium. The general rule to decide the option premium is to use a volatility value that is higher than the estimated one by, for example, 20%.

⊙ Analysis of the hedging cost in different patterns of price movements:

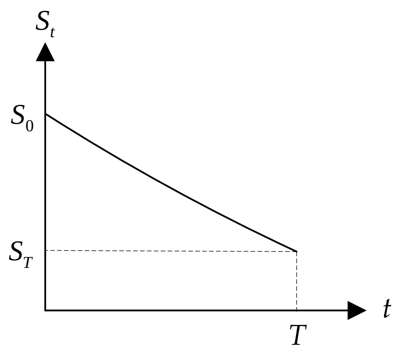
(i) Stock price rises continuously: Purchase $\Delta_0 S_0$ initially, and continue to buy stock shares until T. The net payoff of the dynamic delta hedge depends on the relative levels of the average purchasing price and the strike price, which is the selling price of stock shares for the option issuer at maturity.

Figure 7-11



(ii) Stock price declines continuously: Purchase $\Delta_0 S_0$ initially, and continue to sell those purchased stock shares until T. Since the purchasing price is higher than the selling price, the hedging cost accumulates continuously.

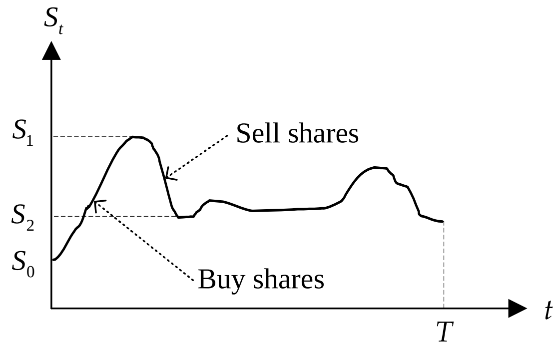
Figure 7-12



(iii) Normal pattern of the stock price movement: Purchase $\Delta_0 S_0$ initially, continue to purchase stock shares until S_t rises to reach S_1, sell stock shares until S_t declines to reach S_2, and so forth. It is apparent that the the some hedging costs occur in the $S_0 \to S_1 \to S_2$ round trip due to the trading strategy of purchasing at high and selling at low.

Hence, we can infer that if the stock price movement upward and downward by turns, these price fluctuations (or said price volatility) incur hedging costs.

Figure 7-13



$*$ If the time period is long enough and the price volatility is sufficiently high, then the hedging costs due to the price volatility will dominate the total hedging costs.

- Inverted delta hedging strategy: if you can find a relatively cheaper call option in the market, e.g., the call option sold at $200,000$ in the above example, then you can reverse your trading strategy to exploit this opportunity.

  $\odot$ That is, $(-1) \times [C_t = \Delta_t S_t - B_t]$, which is the trading strategy of the dynamic delta hedge, generate the trading strategy of the inverted delta hedge $-\Delta_t S_t + B_t$, which can replicate the position of $-C_t$. Since you can purchase a cheaper C_t and exploit $-\Delta_t S_t + B_t$ to generate the Black-Scholes option value, you can arbitrage from this opportunity almost certainly.

  (Note that in the dynamic delta hedge, the trading strategy incurs some hedging costs. On the other hand, in the inverted delta hedge, its trading strategy generates some trading profits, i.e., the strategy $-\Delta_t S_t + B_t = -C_t$ can generates the payoff equal to the amount of selling a call option at the theoretical Black-Scholes option value.)

  $\odot$ The trading strategy of this inverted delta hedging method is as follows.

  $$\begin{cases} S_t \uparrow, \Delta_t \uparrow \Rightarrow \text{selling stock shares such that the total position is } -100,000\Delta_t \text{ shares} \\ S_t \downarrow, \Delta_t \downarrow \Rightarrow \text{purchasing stock shares such that the total position is } -100,000\Delta_t \text{ shares} \end{cases}$$

⊙ The result of the above trading strategy:

(i) Short sell $\Delta_0 S_0$ shares of stock initially and deposit the selling proceed in the bank to earn the risk free rate

$$\text{(ii)} \begin{cases} S_t \uparrow, \Delta_t \uparrow \Rightarrow \text{Sell more shares at } S_t, \text{and deposit more money in the bank} \\ \\ S_t \downarrow, \Delta_t \downarrow \Rightarrow \text{Withdraw money from the bank, and use it to purchase stock shares at } S_t \end{cases}$$

⇒ The investor accumulates profits by continuously selling at high and buying back at low.

∗ If the price moves upward and downward alternately for many rounds due to its high volatility, the cumulative profits increase.

∗ If the price volatility is higher than the estimated one used in the Black-Scholes model, the investor adopting the inverted delta hedging strategy can earn the payoff in excess of the Black-Scholes option value.

$$\text{(iii)} \begin{cases} \text{If } S_T \geq K \Rightarrow \text{the final position at maturity is to short sell 100,000 shares} \\ \qquad\qquad\quad \Rightarrow \text{purchase 100,000 shares at } K \text{ through exercising the undervalued} \\ \qquad\qquad\qquad \text{call, and return these shares} \\ \qquad\qquad\quad \Rightarrow \text{the remaining balance in the bank account is what you can earn} \\ \\ \text{If } S_T < K \Rightarrow \text{all short selling shares are purchased back before maturity} \\ \qquad\qquad\quad \Rightarrow \text{the remaining balance in the bank account is what you can earn} \end{cases}$$

⊙ In Taiwan, the convertible bond (CB) can be regarded as a source of cheaper call options. Longing CBs and performing the inverted delta hedge can achieve the above arbitrage strategy.

IV. Selected Trading Strategies

- Interval trading (區間操作) for volatile stocks: purchase and sell shares proportionately when the stock price falls and rises in the pre-specified price interval.

  Table 7-3 Hypothetical Example

Stock price	Holding shares
70	0
65	25
60	50
55	75
50	100
45	125
40	150
35	175
30	200

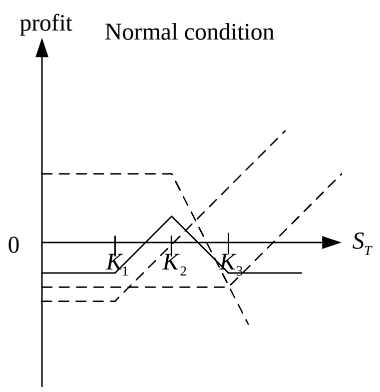
  ∗ Estimating the expected stock returns is far difficult than estimating the stock return volatility.

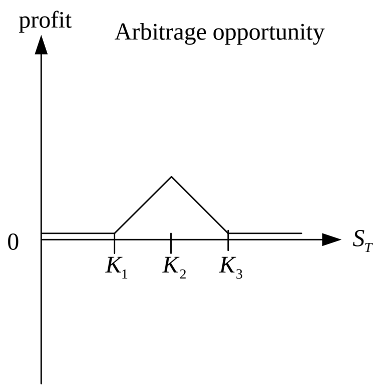
  ∗ Accumulate profits through continuously performing the strategy of buying at low and selling as high.

  ∗ If the stock price movements break the interval, the trading strategy pauses until the stock price moves back into the interval.

- Butterfly (蝶狀價差) strategy for calm stocks: buy one share of call(K_1) and one share of call(K_3) and sell two shares of call(K_2), where K_2 is the middle point of K_1 and K_3. It is also possible to employ put options to construct the butterfly strategy. If the net payoff after the deduction of the transaction costs is always positive regardless of the stock price, the butterfly strategy can result in an arbitrage opportunity.

  Figure 7-14





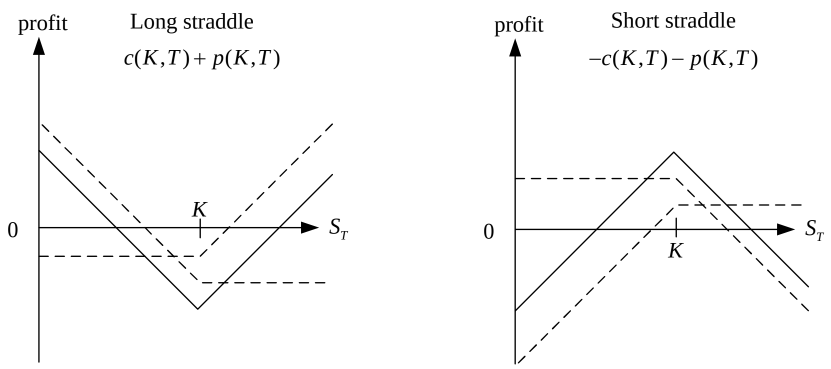
∗ When the option markets become more efficient, it is rarer to find this arbitrage opportunity.

∗ An asymmetric butterfly strategy is proposed as follow. For example, if $K_1 = 5800$, $K_2 = 6000$, $K_3 = 6100$, an asymmetric butterfly can be formed through buying one share of call($K_1 = 5800$) and two shares of call($K_3 = 6100$) and selling 3 shares of call($K_2 = 6000$).

- Straddle (跨式部位) and Strangle (勒式部位)

  ⊙ Long (short) straddle is an investment strategy involving the purchase (sale) of each share of the call and put option with the same strike price and time to maturity. The chosen stirke price is usually close to the current stock price.

Figure 7-15

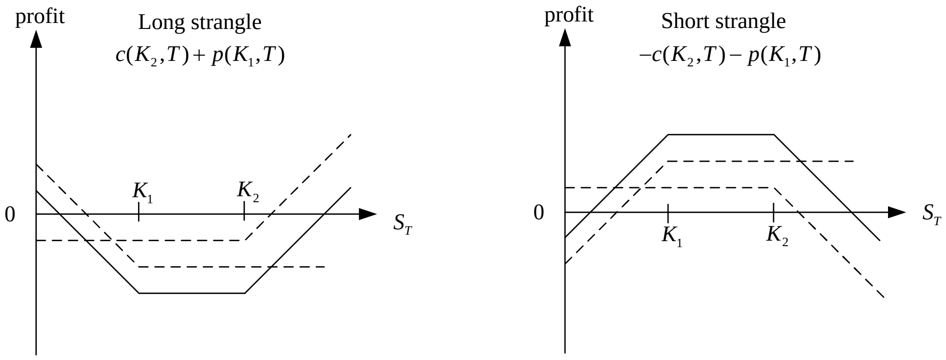


∗ Long straddle makes profits if the stock price deviates from the strike price moderately. On the contrary, short straddle makes profits if the stock price fluctuation is not far from the strike price.

∗ Institutional investors use the trading strategy of long straddle for volatile stocks frequently.

⊙ Long (short) strangle is an investment strategy involving the purchase (sale) of each share of the call and put option with the same time to maturity but different strike prices. Note that the strike price of the call is higher than the strike price of the put, and it is usual that the current stock price is between this two strike prices.

Figure 7-16



∗ Long strangle makes profits if the stock price deviates from the two strike price significantly. On the contrary, short strangle makes profits if the stock price fluctuation is roughly between the two strike prices of the call and put options.

∗ Institutional investors use a lot the trading strategy of short strangle to bet that the maximum stock price movements do not exceed the range of $[K_1, K_2]$.

- Arbitrage between American deposit receipt (ADR) and the stock share in Taiwan

  ⊙ Taking the ADR of TSMC as example: Suppose one shares of ADR of TSMC can exchange for 5 stock shares of TSMC, and one share of ADR is worth US\$8, one share of stock of TSMC is worth NT\$40, and the exchange rate is US\$1 = NT\$30.

  ⊙ Therefore, the current premium ratio equals $\frac{240-200}{200} = 0.25 = 25\%$.

  ⊙ The intuition behind this strategy:

  · If the premium ratio is high, the price of ADR compared with the share price of TSMC is relatively high, and thus longing undervalued TSMC shares and shorting overvalued ADR is profitable if the premium ratio moves back to the normal level.

  · If the premium ratio is low, the price of ADR compared with the share price of TSMC is relatively low, and thus shorting overvalued TSMC and longing undervalued ADR is profitable if the premium ratio moves back to the normal level.

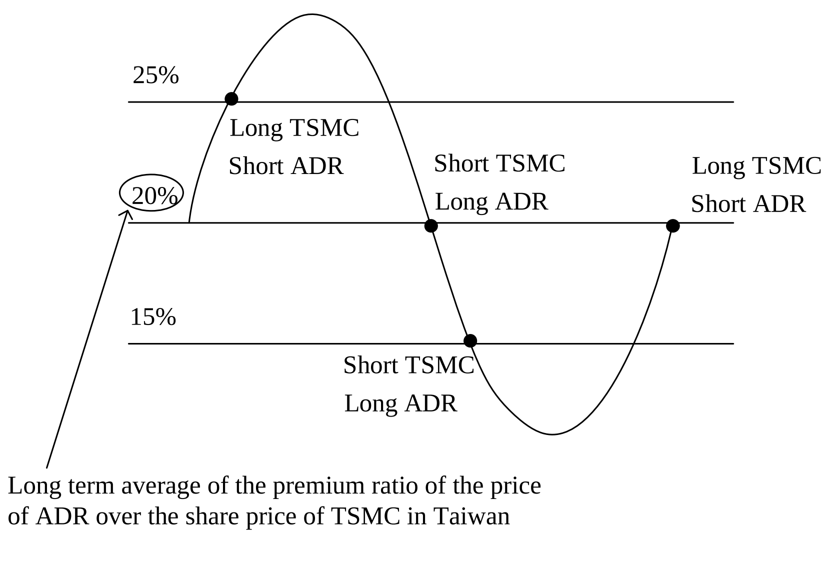
  ⊙ The detailed trading strategy is as follows.

  (i) Estimated the long term average of the premium ratio, which is assumed to be 20%.

  (ii) Define the upper and lower bounds of the premium ratio. In the following figure, the upper and lower bounds are assumed to be 25% and 15%, respectively.

  (iii) If the premium ratio penetrates 25% from below,
  buy 5 shares of TSMC and sell 1 share of ADR, and the position is closed when the premium ratio moves back to 20%.
  If the premium ratio penetrates 15% from above,
  short 5 shares of TSMC and buy 1 share of ADR, and the position is closed when the premium ratio moves back to 20%.

Figure 7-17



⊙ The tradoff to decide the upper and lower bounds of the premium ratio.

· If the band between the upper and lower bounds is too wide, then the frequency to undertake the strategy is too few and thus less profits are earned.

· If the band is too narrow, then it is difficult to cover the transaction costs.

⊙ The main concern of this strategy is the reliability of the estimation of the long term average premium ratio. If the estimation is wrong, it is possible to suffer a lot of losses based on this strategy.

⊙ This strategy is preferred by some foreign financial institutions in Taiwan when the market is preditable in the near future.