Interest Rate Term Structure

Zero Rates

A zero rate (or spot rate), for maturity T is the rate of interest earned on an investment that provides a payoff only at time T

Example

Maturity (years)	Zero Rate (%)
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

Bond Pricing

- To calculate the cash price of a bond we discount each cash flow at the appropriate zero rate
- In our example, the theoretical price of a two-year bond providing a 6% coupon semiannually is

$$3e^{-0.05\times 0.5} + 3e^{-0.058\times 1.0} + 3e^{-0.064\times 1.5}$$
$$+ 103e^{-0.068\times 2.0} = 98.39$$

Bond Yield

- The bond yield is the discount rate that makes the present value of the cash flows on the bond equal to the market price of the bond
- Suppose that the market price of the bond in our example equals its theoretical price of 98.39
- The bond yield is given by solving

$$3e^{-y\times 0.5} + 3e^{-y\times 1.0} + 3e^{-y\times 1.5} + 103e^{-y\times 2.0} = 98.39$$

to get y=0.0676 or 6.76%.

Par Yield

- The par yield for a certain maturity is the coupon rate that causes the bond price to equal its face value.
- In our example we solve

$$\frac{c}{2}e^{-0.05\times 0.5}+\frac{c}{2}e^{-0.058\times 1.0}+\frac{c}{2}e^{-0.064\times 1.5}$$

$$+\left(100+\frac{c}{2}\right)e^{-0.068\times 2.0}=100$$

to get $c = 6.87$ (with s.a. compounding)

Par Yield
(continued)

In general if m is the number of coupon payments per year, d is the present value of \$1 received at maturity and A is the present value of an annuity of \$1 on each coupon date

$$100 = A\frac{c}{m} + 100d$$

$$c = \frac{(100 - 100d)m}{A}$$

Sample Data for Determining the Zero Curve

Bond Principal (dollars)	Time to Maturity (years)	Annual Coupon (dollars)	Bond Price (dollars)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

The Bootstrap Method

- An amount 2.5 can be earned on 97.5 during 3 months.
- The 3-month rate is 4 times 2.5/97.5 or 10.256% with quarterly compounding
- This is 10.127% with continuous compounding
- Similarly the 6 month and 1 year rates are 10.469% and 10.536% with continuous compounding

The Bootstrap Method
(continued)

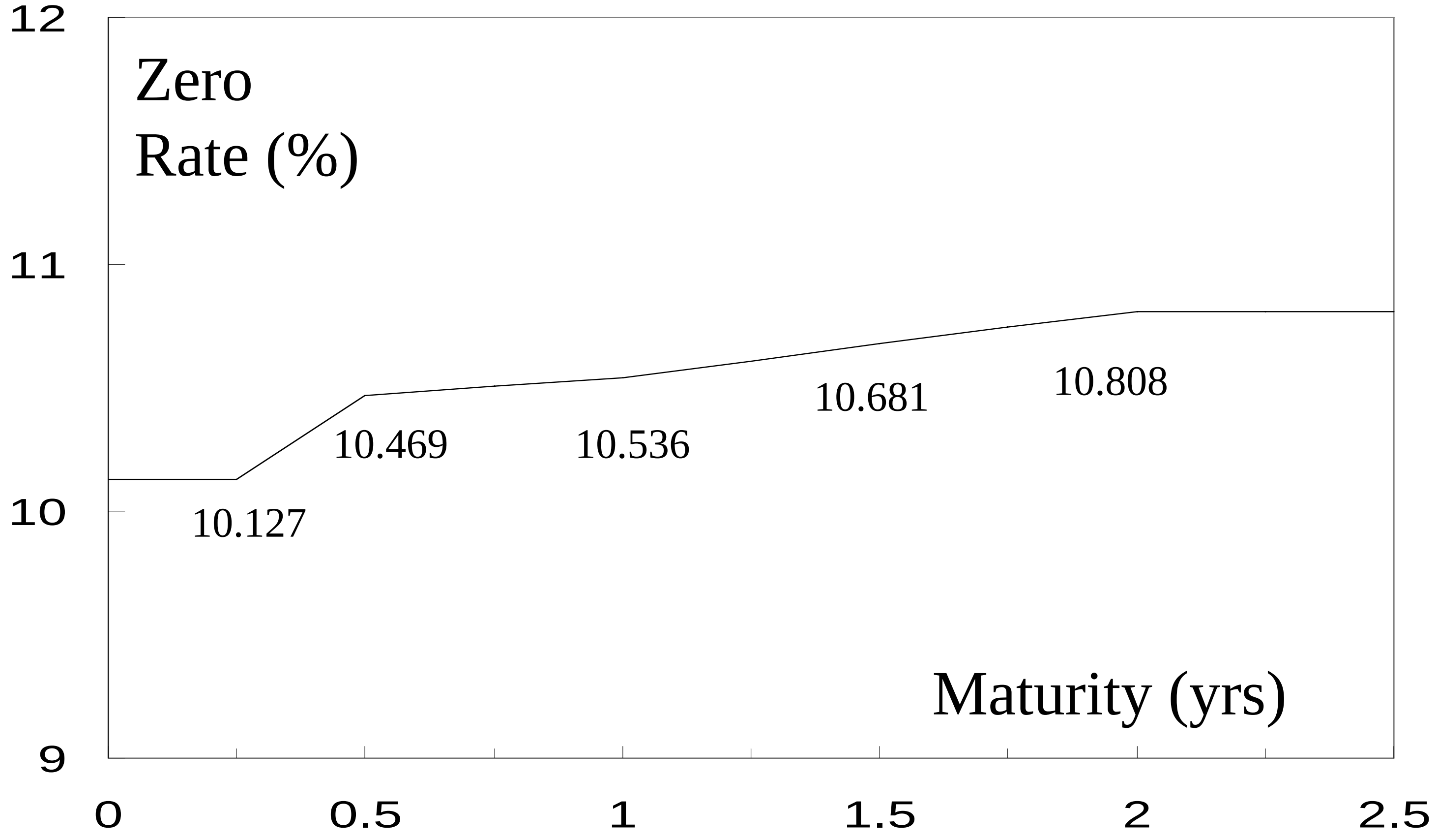
- To calculate the 1.5 year rate we solve

$$4e^{-0.10469\times 0.5}+4e^{-0.10536\times 1.0}+104e^{-R\times 1.5}=96$$

to get R = 0.10681 or 10.681%

- Similarly the two-year rate is 10.808%

Zero Curve Calculated from the Data



Forward Rates

The forward rate is the future zero rate implied by today's term structure of interest rates

Calculation of Forward Rates

Year (n)	Zero Rate for an n-year Investment (% per annum)	Forward Rate for n th Year (% per annum)
1	10.0	
2	10.5	11.0
3	10.8	11.4
4	11.0	11.6
5	11.1	11.5

Formula for Forward Rates

- Suppose that the zero rates for time periods T_1 and T_2 are R_1 and R_2 with both rates continuously compounded.
- The forward rate for the period between times T_1 and T_2 is

$$\frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

Upward v.s. Downward Sloping Yield Curve

- For an upward sloping yield curve:

  Fwd Rate > Zero Rate

- For a downward sloping yield curve

  Zero Rate > Fwd Rate

Theories of the Term Structure

- Expectations Theory: forward rates equal expected future zero rates
- Market Segmentation: short, medium and long rates determined independently of each other
- Liquidity Preference Theory: forward rates higher than expected future zero rates (Lenders prefer to lend for short period, and borrowers prefer to borrow for long period. As a result, financial institutions raise the long-term interest rates relative to expected future zero rates.)

Duration

- Duration of a bond that provides cash flow c_i at time t_i is

$$B = \sum_{t_i} c_{t_i} e^{-yt_i}$$

$$D = \sum_{t_i} t_i \left[\frac{c_{t_i} e^{-yt_i}}{B} \right]$$

- This leads to

$$-\frac{dB \ / \ B}{dy} = D$$

Duration

(Continued)

$$B = \sum_{t_i} c_{t_i} e^{-yt_i}, \quad D = \sum_{t_i} t_i \frac{c_{t_i} e^{-yt_i}}{B}$$

$$-\frac{dB\ /\ B}{dy} = \frac{1}{B}\left(-\sum_{t_i} c_{t_i} e^{-yt_i}(-t_i)\right)$$

$$= \frac{1}{B}\left(\sum_{t_i} t_i c_{t_i} e^{-yt_i}\right)$$

$$= \sum_{t_i} t_i \frac{c_{t_i} e^{-yt_i}}{B}$$

$$= D$$

Duration
(Continued)

- When the yield y is expressed with compounding m times per year

$$-\frac{dB/B}{dy} = \frac{1}{1+y/m} D$$

- The expression

$$D^* = \frac{1}{1+y/m} D$$

is referred to as the “modified duration”

Duration

(Continued)

$$B = \sum_{t_i} \frac{c_{t_i}}{(1+y/m)^{m \cdot t_i}}, \quad D = \sum_{t_i} t_i \frac{c_{t_i}/(1+y/m)^{m \cdot t_i}}{B}$$

$$-\frac{dB/B}{dy} = \frac{1}{B}\left(-\sum_{t_i} \frac{c_{t_i}}{(1+y/m)^{m \cdot t_i + 1}}(-m \cdot t_i) \cdot (\frac{1}{m})\right)$$

$$= \frac{1}{B}\left(\sum_{t_i} t_i \frac{c_{t_i}}{(1+y/m)^{m \cdot t_i + 1}}\right)$$

$$= \frac{1}{1+y/m}\left(\sum_{t_i} t_i \frac{c_{t_i}/(1+y/m)^{m \cdot t_i}}{B}\right)$$

$$= \frac{1}{1+y/m} D$$

$$D^* = -\frac{dB/B}{dy} = \frac{1}{1+y/m} D$$

Duration Matching

- This involves hedging against interest rate risk by matching the durations of assets and liabilities
- It provides protection against small parallel shifts in the zero curve