The eBay Market as Sequential Second Price Auctions—Theory and Experiments

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Abstract

We consider the sequential second price auction in which the two highest bids are announced after each stage with unit demand bidders of affiliated private values. We find an efficient symmetric sequential equilibrium in monotonic strategies where bidders bid the expected winning price of the next stage, conditional on being the tied winner in this stage. Such an equilibrium survives in an eBay auction setting, in which bidders are allowed to submit multiple proxy (second-price) bids in a given fixed time period, when all bidders bid only at the last minute, or snipe. We then conduct controlled (laboratory) experiments with the special case of three items sold to bidders with independent private values, using either a sealed-bid second price auction or an eBay auction format. We show the timing of bidding in the eBay auction demonstrate "last minute bidding (sniping)" as reported by Roth and Ockenfels (2002), and early "squatting" coined by Ely and Hossain (2006). Moreover, both formats achieve high efficiency levels and have estimated bid functions seemly as predicted. However, prices decline, but revenue is higher than predicted in the early stages. Individual bids reveal that a significant portion of subjects either bid their valuations or zero in early auctions, even though they bid their valuations in the last one. To explain this, we construct a steps-of-thinking (cognitive hierarchy) model anchored on L0 bidders who always bid their valuations. The combination of various types of bidders explain the data better than that of standard equilibrium theory and shed light on the "declining price anomaly" documented by Ashenfelter (1989) and previously attributed to extreme risk aversion by McAfee and Vincent (1993).

Keywords: Repeated auctions, Last minute bidding, Revenue equivalence

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1 Introduction

Internet auctions have been a growing phenomenon in the last few years. Even after most dot-com companies tumbled during the internet bubble burst, online auction houses such as eBay still prevail and make huge profits. However, the auction rules employed by eBay are very different from rules we usually see in traditional auction houses and auction theory. As summarized in Wang (2006), the eBay auction is characterized by an out-cry auction format (instead of a button auction in auction theory) with proxy-bidding (instead of paying one's bid in traditional auction houses) and a fixed time ending rule (instead of the going-going-gone rule utilized by traditional auction houses, or the drop-off rule used to mimic it).

Regarding the eBay auction rule, Bajari and Hortacsu (2000) note that if all bidders only bid at the last minute, eBay auctions are as if a second-price auction (due to proxy-bidding). This draws attention to the relation between the eBay auction and the second price auction. Moreover, Roth and Ockenfels (2002) note that due to the different ending rules, the timing of bids submitted in eBay (fixed-time ending) auctions and Amazon (going-going-gone) auctions differ. In fact, bidders attempt to “”snipe” and bid exactly at the last minute of eBay auctions, but seldom do so in Amazon auctions. However, standard auction theory with a private value setting cannot explain this kind of last minute bidding behavior since the proxy bidding system of online auctions allow bidders to submit their maximum willingness-to-pay at any time, and hence, it is a puzzle why bidder do not simply submit their maximum bid as soon as they see the auction. For example, Wang (2006) discuss how last minute bidding is incompatible with the eBay auction rule under the private value setting when only one item sold.

To solve this “last minute puzzle,” Ockenfels and Roth (2006) suggest bidders bid at the last minute to collude over the possibilities of lost bids in the last minute.[1] Nevertheless, Ariely, Ocken-

[1]Note that we focus on private values only, since there is no real “puzzle” if common values are introduced. In fact, Ockenfels and Roth (2006), as well as many others, show that last minute bidding can be easily explained under a common value setting, since informed bidders would have incentives to withhold private information and bid only at the last minute.

fels and Roth (2006) show that bidders attempt to bid at the last minute even without the possibility of lost bids, suggesting other explanations such as avoiding bidding wars against incremental bidders, also discussed in Ockenfels and Roth (2006) or repeated sales proposed by Wang (2006), are more plausible.

Furthermore, since many similar or even identical items are sold within short periods of time, the eBay auction house seems more like a "market," instead of a collection of individual auctions unrelated to each other. In fact, the "About eBay" webpage explicitly states "eBay offers an online platform where millions of items are traded each day."[2] However, unlike financial markets where large transactions are executed simultaneously, eBay auctions typically have one unit for sale, and each have a different (fixed) ending time. Therefore, a sequential second-price auction model would better approximate this empirical phenomenon.[3]

Sequential auctions are not new. Weber (1983) and Milgrom and Weber (2000)[4] extend the affiliation model of Milgrom and Weber (1982) to identical auctions done one after another. Weber (1983) shows that with independent private values, the expected price of each stage are the same for both the sequential first-price and second-price auctions, since the effect of unit demand winners leaving the auction and that of bidders shading their bids more in early auctions (anticipating more opportunities to win the item) cancel out. With affiliated values, Milgrom and Weber (2000) show that the sequential first-price auction (with price announcements) would have ascending expected prices due to information revelation through the winning prices. Milgrom and Weber (2000) also conjecture an equilibrium for the sequential second-price auction (without price announcements), but admit that "the proof(s) refused to come together."[5]

On the other hand, empirical evidence suggests that price sequences actually decline. For example, Ashenfelter (1989) discusses the "afternoon effect" in wine auctions. More recently, van

[2] See http://pages.ebay.com/aboutebay.html

[3] This is contrary to Peters (2000?) TBA, which models eBay auctions as multi-unit simultaneous auctions.

[4] This paper was written in early 1982, but remained unpublished until being included in *The Economic Theory of Auctions* (2000), edited by P. Klemperer.

[5] See their foreword written in April, 1999.

den Berg, *et al.* (2001) showed declining prices in the Dutch flower auction with many stages of identical Dutch auctions.[6] However, in experimental settings, Neugebauer and Pezanis-Christou (2001) show that with independent private values, sequential first-price auctions had average prices not significantly different from each other, somewhat confirming Weber (1983)'s prediction. Facing this empirical literature, McAfee and Vincent (1993) show how increasing relative risk aversion can explain such declining price anomaly in the case of two auctions, but did not discuss what would happen if more than two stages of auctions occur. More recently, Mezzetti, Pekec and Tsetlin (2004) consider two stages of multiple-unit sales and show how prices may either ascend or decline in different situations.

In this paper, we address the relation between eBay auctions and sequential auctions, both theoretically and experimentally. In the theory front, we pursue the symmetric sequential (or perfect Bayesian) equilibrium for the sequential second-price auction, restricting attention to (strictly) increasing strategies and affiliated private values.[7] However, unlike Milgrom and Weber (2000), which purposely ignore the case where previous bids are revealed, we consider the case where both the highest and the second-highest bids are announced. Since higher valuation bidders bid higher, the highest loser of one stage is the (designated) winner of the next stage.[8]

We show that bidders bid the expected price of the next stage, conditional on being tied with the winner in the current stage and on past winner's information, while adjusting for the fact that her current bid might affect the price of the next stage. Interestingly, when calculating the expected price, bidders do *not* take into consideration the available information about the highest loser in the

[6] Some empirical studies do show ascending prices, but not much. See van den Berg, *et al.* (2001) for further discussion of more recent work.

[7] Mezzetti, Pekec and Tsetlin (2004) also attempt the unfinished quest of Milgrom and Weber (2000), and consider the more general affiliated values (which incorporates common values) and allow for multiple-units to be sold in each stage. However, they could not extend the general results to three stages or more, and hence, focused on only two stages. In contrast, due to empirical concerns of internet auctions, we consider only affiliated private values, and one unit sold at a time, but provide results for any finite number of stages.

[8] Milgrom and Weber (2000) think price announcements would hamper the symmetry of the game after the first stage since one remaining bidder's information is revealed through the price announcement. We actually view this as an interesting feature of the model.

previous stage, even though they *do* consider all past winner's information.[9] Moreover, since other bidders ignore this obvious information, a bidder actually need not adjust for the effect of her bid on future prices, since this information is not used anyway. Hence, we can construct an efficient equilibrium where bidders bid exactly the expected price of the next stage and ignore the effect of her bid on future prices.

Since the equilibrium does not depend on the second-highest bids, we may easily construct the same equilibrium in the setting where only the highest bids are observed.[10] Moreover, this equilibrium also survives in an eBay auction setting where there are multiple rounds of bidding before the fixed ending time, provided that subjects hold out-of-equilibrium beliefs that seeing a change in the standing price indicates a valuation higher than this standing price.

Based on the theory, we conduct controlled (laboratory) experiments to compare the two auction formats. In particular, we auction three items sequentially to unit demand bidders with independent private value using either the eBay auction format or the sealed-bid second price auction format. In the eBay bidding stage of sixty seconds, we find the timing of bids replicate the last minute bidding behavior reported by Roth and Ockenfels (2002), having 17% to 29% of the bids come in at the last two seconds. We also find some evidence of early "squatting" behavior coined by Ely and Hossain (2006), since approximately 20% of the bids come in during a ten second window (58 to 48 seconds left) at the beginning of each stage. Moreover, we find both formats having efficiency levels of above 90%, and significantly higher revenue in the first two stages (than both the third stage and what theory predicts), which is in line with the "declining price anamoly" discussed in Ashenfelter (1989) and McAfee and Vincent (1993). Thirdly, the estimated linear bid function is close to what theory predicts, having bidders shading their bids more in earlier rounds ($1/2$ and $2/3$ as opposed to 1). However, individual bids show a different story, showing

[9]Note that this would not hurt the bidders since in second-price auctions, bidders are indifferent about what they bid, given the same winning probability. In this particular case, since the previous highest-loser's valuation is inferred from previous bids, the winning probability is either 0 or 1.

[10]In other words, if in each stage the auctioneer only announces the highest winner and her bid, but not the actual price she pays, the symmetric equilibrium would be the same as that when the actual price is also announced.

concentrations on either bidding one's valuation or bidding zero in the first two auctions.

To explain the experimental data, we construct a steps-of-thinking (cognitive hierarchy) model anchored on L0 type bidders who bid their valuation in every stage, either due to naivete or perceiving the current stage as the last one (for herself). Facing L0 type bidders, L1 type bidders "dodge" and bid zero in the first two auctions. On the other hand, L2 type bidders face a combinaion of L0 and L1 type bidders and best respond to the mixture of bids equal to one's valuation or zero. Combined with Equilibrium type bidders who follow the (fully rational) equilibrium strategy, we can drastically improve fit to the experimental bidding data in the first two auctions, obtain 91-93% efficiency levels, and simulate declining prices. Thus, using the cognitive hierarchy model, we provide an (at least partial) explanation of the experimental data, and suggest that declining prices in sequential auctions can be explained by limits of rationality, instead of (extreme) risk aversion previously suggested by McAfee and Vincent (1993).

The rest of the paper is structured as follows. Section 2 describes the theory for sequential auctions, section 3 describes the experiment and presents experimental findings as well as the "puzzles," section 4 provides a steps-of-thinking (cognitive hierarchy) model that explains the puzzle, and section 5 concludes.

2 Theory

2.1 Basic Settings

We consider multiple bidders and multiple (passive) sellers, each with unit demand or unit supply. The valuations of the sellers are normalized to zero, while the bidders' valuations are private but affiliated. We consider the symmetric case where the valuation distribution is ex ante symmetric, and hence, focus on symmetric equilibrium. This is the standard affiliated private value model proposed by Milgrom and Weber (2000).[11] To be precise, we assume

Assumption 1. *N bidders and M sellers. $N > M$, all are risk neutral and no discounting. Sellers each have one item. Sellers' valuation is $v_0 = 0$. Bidders want only one item. Valuations (types) V_i are private and affiliated (APV) with support $[\underline{v}, \overline{v}]$, $\mathbf{V} = (V_1, \cdots, V_N)$ has symmetric joint pdf $f(v_1, \cdots, v_N)$, and order statistics are*

$$V_{(1)} \geq V_{(2)} \geq \cdots \geq V_{(N)}.$$

For any V_i, we define the order statistics of the other $n-1$ bidders as

$$Y_1 \geq Y_2 \geq \cdot \geq Y_{n-1}.$$

The equilibrium concept we consider is the symmetric sequential (perfect Bayesian) equilibrium with strictly increasing strategies.[12]

We consider the sequential second-price auction, in which each item is auctioned off one at a time, using the same sealed-bid second-price auction rule each time.

Assumption 2. (Sequential Second-Price Auction)

1. *In each auction stage,*[13] *bidders simultaneously submit sealed bids. We denote bidder i's bid at stage m as b_{mi}.*

2. *The bidder who submitted the highest bid wins the item and pays the second-highest bid. Hence, in stage m, bidder $j = \arg\max_i \{b_{mi}\}$ is the winner, and the winning price is $p_m = \max_i \{b_{mi}\} \setminus \{b_{mj}\}$.*

[11] However, we do not consider the most general affiliated value which include common values.

[12] It is known that even with only one auction, there are asymmetric equilibria where one bidder bids very high and others bid zero. Restricting to symmetric equilibrium rules out these asymmetric equilibria.

[13] We use auction stage, stage and auction interchangeably to indicate each stage of sale where one item is sold.

3. *At the end of each stage, the two highest bids, the winning price and the winner's "maximum" are publicly announced. Then, a next stage of auction is run with the same rules. Therefore, the bidding history is* $H_1 = (b_{1(1)}, b_{1(2)})$, $H_2 = [H_1, (b_{2(1)}, b_{2(2)})]$, $H_3 = [H_2, (b_{3(1)}, b_{3(2)})]$, *and so on. Also, the winning prices are* $p_1 = b_{1(2)}$, $p_2 = b_{2(2)}$, $\cdots$, $p_M = b_{M(2)}$.

Note that the last stage is just like the standard (one-item) sealed-bid second-price auction where bidding one's valuation is a (weakly) dominant strategy. Moreover, in a strictly increasing equilibrium, the bidder with the highest valuation $v_{(1)}$ wins the first auction, the bidder with the second-highest valuation $v_{(2)}$ wins the second auction, and so on. Since bids are revealed, other bidders can infer the valuations of the first and second-highest remaining bidders. In particular, for bid functions $\beta_1(v)$, $\beta_2(v; H_1)$, $\cdots$, $\beta_{M-1}(v; H_{M-1})$, and $\beta_M(v; H_M)(= v)$, all bidders infer

$$
\begin{aligned}
v_{(1)} &= \beta_1^{-1}(b_{1(1)}) = z_1 && \text{after stage 1} \\
v_{(2)} &= \beta_1^{-1}(b_{1(2)}) = z_2 && \text{after stage 1} \\
v_{(3)} &= \beta_2^{-1}(b_{2(2)}, H_1) = z_3 && \text{after stage 2} \\
&\cdots \\
v_{(m+1)} &= \beta_m^{-1}(b_{m(2)}, H_m) = z_{m+1} && \text{after stage } m \\
&\cdots \\
v_{(M)} &= \beta_{M-1}^{-1}(b_{M-1,(2)}, H_{M-1}) = z_M && \text{after stage } (M-1)
\end{aligned}
$$

since β_1^{-1}, β_2^{-1}, $\cdot$, β_{M-1}^{-1} all exist by the increasing assumption. Hence, in equilibrium, the bid functions depend on the winners' valuations, and can be re-written as $\beta_1(v)$, $\beta_2(v; z_1, z_2)$, $\cdots$, $\beta_{(M-1)}(v; z_1, z_2, \cdots, z_{M-1})$, and $\beta_M(v; z_1, z_2, \cdots, z_M) = v$. Moreover, at stage m, if a bidder has valuation $v < z_m$, she *knows* that she will not win that stage, unless she bids above $\beta_m(z_m; z_1, \cdots, z_m)$.

To summarize, we have

Lemma 1. (Valuation Inference)

A strictly increasing equilibrium $(\beta_1, \beta_2, \beta_3, \cdot, \beta_M)$ *requires*

1. *The last stage has a dominant strategy of* $\beta_M(v) = v$.

2. *Beliefs: In the second to last stages, the (designated) winner's valuation (and identity) is known* before *the bidding starts. Hence, other bidders* know *they have no chance to win (unless they bid above the "inferred" highest bid.)*

Therefore, we denote the bid functions $\beta_1(v)$, $\beta_2(v; z_1, z_2)$, $\cdots$, $\beta_M(v; z_1, \cdots, z_M)$ *where* v *is one's own valuation, and* z_m *are the inferred valuation of* $v_{(m)}$.

Note that information is inferred due to bid revelation. Hence, one can possibly construct symmetric equilibria where bidders "coordinate" on this revelation, and hence, are "asymmetric." For example, we might be able to construct a symmetric equilibrium such that bidders all bid zero in the first stage, and after the first stage, whoever's bid is chosen to be revealed (but lost in tie-breaking) would bid $\overline{v}$ in the next stage, while others continue to bid zero. This is similar to the asymmetric equilibria in the standard single item second price auction where one bidders bids very high and all others bid zero, but is indeed symmetric in the sense that bidders do use the same bidding strategy. However, we focus on the more interesting case where the bidding strategy does not hinge on such possibilities.[14]

2.2 Method of Proof (M=3)

When $M = 2$, the second (and last) auction has a dominant strategy of bidding one's valuation, and hence, information obtained from the first stage does not matter. Therefore, to see how information from early auctions can influence the result of later ones, and how such influence can feedback into

[14]Note that these "asymmetric" equilibria hinge on the fact that all bidders correctly infer the valuation and/or identity of the designated winner, and hence, might not be robust.

the bid functions of early auctions, we need at least three auction. To illustrate these issues and demonstrate the method of the general proof, we first focus on the case where $M = 3$.

We consider the problem of choosing to act as if one had valuation x_1, x_2 and x_3 in stage 1, 2, and 3, respectively, maximizing the bidder's expected payoff, given one's (true) valuation v:

$$\max_{x_1,x_2,x_3} u(v,x_1,x_2,x_3) = \underbrace{W_1(v,x_1)}_{\text{win 1st}} + \underbrace{W_2(v,x_1,x_2)}_{\text{win 2nd}} + \underbrace{W_3(v,x_1,x_2,x_3)}_{\text{win 3rd}}$$

where W_1, W_2 and W_3 are the expected utility of winning the first, second and third auction, respectively.

Given the following conditional pdf

$$Y_1\Big|_V \sim f_1(y_1|v) \qquad Y_1,Y_2\Big|_V \sim f_2(y_1,y_2|v) \qquad Y_1,Y_2,Y_3\Big|_V \sim f_3(y_1,y_2,y_3|v),$$

the expected utility of bidder v acting as x_1, x_2, x_3, winning the first, second and third auction are:

$$\begin{aligned}
W_1 &= \int_{\underline{v}}^{x_1} \Big[v - \beta_1(y_1)\Big] f_1(y_1|v)dy_1 \\
W_2 &= \int_{x_1}^{\overline{v}}\int_{\underline{v}}^{\min\{x_2,y_1\}} \Big[v - \beta_2(y_2;y_1,x_1)\Big] f_{12}(y_1,y_2|v)dy_2dy_1 \\
W_3 &= \int_{x_1}^{\overline{v}}\int_{x_2}^{\max\{x_2,y_1\}}\int_{\underline{v}}^{\min\{x_3,y_2\}} \big[v - \beta_3(y_3;y_1,y_2,x_2)\big] f_{123}(y_1,y_2,y_3|v)dy_3dy_2dy_1
\end{aligned}$$

To find the sequential (perfect Bayesian) equilibrium, we have to

1. Find $\beta_3(v;z_1,z_2,z_3)$ such that given any x_1 and x_2, choosing $x_3 = v$ is optimal in the third stage.

2. Find $\beta_2(v;z_1,z_2)$ such that given any x_1, and knowing one will choose $x_3^* = v$ in the third stage, choosing $x_2 = v$ is optimal in the second stage.

3. Find $\beta_1(v)$ such that knowing one will choose $x_2^* = x_3^* = v$ in the future stages, choosing $x_1 = v$ is optimal in the first stage.

2.2.1 Choice of x_3

We now show that $\beta_3(v; z_1, z_2, z_3) = v$, ignoring all other information (from valuation inference), is indeed optimal.

The first order condition of x_3 requires, at $x_3 = v$,

$$\frac{\partial u}{\partial x_3} = \frac{\partial W_3}{\partial x_3}(v, x_1, x_2, x_3) = 0$$

Since for $\beta_3(v; z_1, z_2, z_3) = v$,

$$\begin{aligned}
\frac{\partial W_3}{\partial x_3}(v, x_1, x_2, x_3) &= \int_{x_1}^{\overline{v}}\int_{\max\{x_2,x_3\}}^{\max\{x_2,y_1\}} \big[v - \beta_3(x_3; y_1, y_2, x_2)\big] f_{123}(y_1, y_2, x_3|v) dy_2 dy_1 \\
&= \big(v - x_3\big) \cdot \int_{x_1}^{\overline{v}}\int_{\max\{x_2,x_3\}}^{\max\{x_2,y_1\}} f_{123}(y_1, y_2, x_3|v) dy_2 dy_1 \\
&= 0
\end{aligned}$$

at $x_3^* = v$, $\beta_3(v; z_1, z_2, z_3) = v$ does indeed satisfy the first order condition.

For sufficiency, note that for $x_3 < v$,

$$\frac{\partial W_3}{\partial x_3}(v, x_1, x_2, x_3) = \big(v - x_3\big) \cdot \int_{x_1}^{\overline{v}}\int_{\max\{x_2,x_3\}}^{\max\{x_2,y_1\}} f_{123}(y_1, y_2, x_3|v) dy_2 dy_1 \geq 0 \text{[15]}$$

and for $x_3 > v$,

$$\frac{\partial W_3}{\partial x_3}(v, x_1, x_2, x_3) = \big(v - x_3\big) \cdot \int_{x_1}^{\overline{v}}\int_{\max\{x_2,x_3\}}^{\max\{x_2,y_1\}} f_{123}(y_1, y_2, x_3|v) dy_2 dy_1 \leq 0$$

[15] Equality might hold when the integration is zero, such as when $x_1 = \overline{v}$.

Hence, we have shown that

$$\frac{\partial W_3}{\partial x_3}(v, x_1, x_2, x_3) \geq 0 \text{ if } x_3 < v$$
$$\frac{\partial W_3}{\partial x_3}(v, x_1, x_2, x_3) \leq 0 \text{ if } x_3 > v$$

and hence, $x_3^* = v$ is indeed a global maximum.[16] Therefore, it is optimal to choose $x_3 = v$.

Thus, we have the following lemma:

Lemma 2. *For $\beta_3(v) = v$, $x_3 = v$ is indeed a best response, regardless of what x_1, x_2 was chosen in the first two stages.*

In other words, bidding one's valuation is indeed a (weakly) dominant strategy in the last stage.

2.2.2 Choice of x_2 Given $x_3^* = v$

We now turn to stage 2 and consider the choice of x_2 given any x_1 and knowing that one will follow $x_3^* = v$ in stage 3.

We show that bidders tradeoff the possibility of winning the second and third auction, and hence, bid in the second auction stage the conditional expectation of the price of the third auction. Such expectation is taken using some (but not all) of the valuations inferred.

Lemma 3. *For all x_1, given $\beta_3(v; z_1, z_2, z_3) = v$ and knowing one will choose $x_3^* = v$ in stage 3, for the following bid function of stage 2, it is optimal to choose $x_2^* = v$.*

$$\begin{aligned}\beta_2(v; z_1) &= E[Y_3 | Y_1 = z_1, Y_2 = v, V_i = v] = E[V_{(4)} | V_{(1)} = z_1, V_{(2)} = v, V_{(3)} = v] \\ &= \frac{\int_{\underline{v}}^{v} y_3 f_{123}(z_1, v, y_3 | v) dy_3}{f_{12}(z_1, v | v)}\end{aligned}$$

where z_1 is observed and used, but z_2 is ignored (or not observed).

[16] One might be indifferent between several x_3's, but $x_3^* = v$ is still one of these global maxima.

On the other hand, if $\beta_2(v; z_1, z_2) = \beta_2(v)$, or the inferred valuations z_1 and z_2 are both ignored[17]*, then for all given x_1, the first order condition requires*

$$\begin{aligned}
\beta_2(v) &= E[Y_3|Y_1 > \max\{x_1, v\}, Y_2 = v, V_i = v] \\
&= E[V_{(4)}|V_{(1)} > \max\{x_1, v\}, V_{(2)} = v, V_{(3)} = v] \\
&= \frac{\int_{\max\{x_1,v\}}^{\overline{v}} \int_{\underline{v}}^{v} y_3 f_{123}(y_1, v, y_3|v) dy_3 dy_1}{\int_{\max\{x_1,v\}}^{\overline{v}} f_{12}(y_1, v|v) dy_1}
\end{aligned}$$

Such $\beta_2(v)$ is well-defined only if $E[V_{(4)}|V_{(1)} > \max\{x_1, v\}, V_{(2)} = v, V_{(3)} = v]$ is the same for all x_1.[18]

Proof. See Appendix. □

Note that $\beta_2(v) = E[Y_3|(Y_1 > v), Y_2 = v, V_i = v] = E[V_{(4)}|V_{(2)} = v, V_{(3)} = v]$ (not using z_1, z_2) is what Milgrom and Weber (2000) conjectured for their sequential second price auction *without price announcements*. However, since we assume bids are revealed, affiliation requires

$$E[Y_3|Y_1 > x_1', Y_2 = v, V_i = v] \geq E[Y_3|Y_1 > x_1, Y_2 = v, V_i = v] \geq E[Y_3|Y_1 > v, Y_2 = v, V_i = v]$$

for $x_1' > x_1 > v$, and thus, $\beta_2(v)$ might not be well-defined after histories where $x_1 > v$. However, such consideration is beyond the scope of this paper.

2.2.3 Choice of x_1 Given $x_2^* = x_3^* = v$

Given $\beta_2(v; z_1, z_2)$, $\beta(v; z_1, z_2, z_3)$ and $x_2^* = x_3^* = v$, the first order condition of x_1 requires

$$\frac{\partial u}{\partial x_1} = \frac{\partial W_1}{\partial x_1}(v, x_1)\frac{\partial W_2}{\partial x_1}(v, x_1, v) + \frac{\partial W_3}{\partial x_1}(v, x_1, v, v) = 0$$

[17] Or, not observed, as in Milgrom and Weber (2000).

[18] For example, when bidder valuations are independent.

at $x_1 = v$. We are to show, in the following lemma, bidders again tradeoff winning the first and second auction, leaving the third auction out since it is already incorporated into the second auction, and bid the conditional expectation of the prices of the second auction, minus the announcement effect revealing his valuation as the highest loser in the first auction would caused in the second auction. Furthermore, when z_2 is ignored in the second auction, as in Lemma 3, the announcement effect vanishes, and the conditional expectation suffices.

Lemma 4. *Given β_2, β_3 and knowing one will choose $x_2^* = x_3^* = v$ in future stages, the first order condition requires $\beta_1(v)$ to satisfy,*

$$\beta_1(v) = \frac{1}{f_1(v|v)} \left[\int_{\underline{v}}^{v} \beta_2(y_2; v, v) f_{12}(v, y_2|v) dy_2 \right. \\ \left. - \int_{v}^{\overline{v}} \int_{\underline{v}}^{v} \frac{\partial \beta_2}{\partial z_2}(y_2; y_1, v) f_{12}(y_1, y_2|v) dy_2 dy_1 \right]$$

In particular, when $\beta_2(v; z_1, z_2) = \beta_2(v; z_1) = E\big[Y_3 \big| Y_1 = z_1, Y_2 = v, V_i = v\big]$, the bid function in stage 1 is

$$\begin{aligned}\beta_1(v) &= E_{Y_2}\Big[\beta_2(Y_2; v, v)\Big| Y_1 = v, V_i = v\Big] \\ &= E_{Y_2}\Big[E\big[\tilde{Y}_3 \big| \tilde{Y}_1 = v, \tilde{Y}_2 = Y_2, \tilde{V}_i = Y_2\big]\Big| Y_1 = v, V_i = v\Big] \\ &= \int_{\underline{v}}^{v}\int_{\underline{v}}^{y_2} y_3 \underbrace{\left(\frac{f_{123}(v, y_2, y_3|y_2)}{f_{12}(v, y_2|y_2)}\right)}_{f_{3|12}(y_3|y_2, v, y_2)} dy_3 \underbrace{\left(\frac{f_{12}(v, y_2|v)}{f_1(v|v)}\right)}_{f_{2|1}(y_2|v,v)} dy_2\end{aligned}$$

Given this $\beta_1(v)$, and knowing one will choose $x_2^* = x_3^* = v$ in future stages, choosing $x_1 = v$ is optimal.

Proof. See Appendix. $\square$

Now we are ready to state and prove the equilibrium for $M = 3$.

Theorem 1. *The following bidding strategy*

$$
\begin{aligned}
\beta_1(v) &= E\Big[\beta_2(V_{(3)}; v, v)\Big|V_{(1)} = v, V_{(2)} = v\Big] \\
\beta_2(v; z_1, z_2) &= E\big[V_{(4)}\big|V_{(1)} = z_1, V_{(2)} = v, V_{(3)} = v\big] && = \beta_2(v; z_1) \\
\beta_3(v; z_1, z_2, z_3) &= v && = \beta_3(v)
\end{aligned}
$$

Consists an (symmetric, strictly increasing, sequential) equilibrium. Moreover, the expected price sequence, $p_1 = \beta_1(v_{(2)})$, $p_2 = \beta_2(v_{(3)}; v_{(1)})$, *and* $p_3 = \beta_3(v_{(4)}) = v_{(4)}$ *ascends*[19]

$$
E\left[p_1\right] \leq E\left[p_2\right] \leq E\left[p_3\right] = E\left[V_{(4)}\right]
$$

Proof. From lemma 4, 3, 2, we see that given the following symmetric, strictly increasing bid functions

$$
\begin{aligned}
\beta_1(v) &= E\Big[\beta_2(V_{(3)}; v, v)\Big|V_{(1)} = v, V_{(2)} = v\Big] \\
\beta_2(v; z_1, z_2) &= E\big[V_{(4)}\big|V_{(1)} = z_1, V_{(2)} = v, V_{(3)} = v\big] && = \beta_2(v; z_1) \\
\beta_3(v; z_1, z_2, z_3) &= v && = \beta_3(v),
\end{aligned}
$$

it is optimal to choose $x_3 = v$ for any x_1 and x_2. It is also optimal to choose $x_2 = v$ for any x_1 and knowing $x_3^* = v$. Last but not the least, it is optimal to choose $x_1 = v$ knowing $x_2^* = x_3^* = v$. Hence, this is indeed an equilibrium.

[19]Contrary to what some empirical evidence, such as Ashefelter (1989), show.

Moreover, the expected prices satisfy

$$
\begin{aligned}
E\left[p_1\right] &= E\left[\beta_1(V_{(2)})\right] \\
&= E\left[E\left[\beta_2(\tilde{V}_{(3)}; V_{(2)}, (V_{(2)}))\middle|\tilde{V}_{(1)} = V_{(2)}, (\tilde{V}_{(2)} = V_{(2)})\right]\right] \\
&\leq E\left[E\left[\beta_2(\tilde{V}_{(3)}; V_{(1)}, (V_{(2)}))\middle|\tilde{V}_{(1)} = V_{(1)}, (\tilde{V}_{(2)} = V_{(2)})\right]\right] \\
&= E\left[\beta_2(V_{(3)}; V_{(1)}, (V_{(2)}))\right] = E\left[p_2\right] \\
&= E\left[E\left[\tilde{V}_{(4)}\middle|\tilde{V}_{(1)} = V_{(1)}, \tilde{V}_{(2)} = V_{(3)}, \tilde{V}_{(3)} = V_{(3)}\right]\right] \\
&\leq E\left[E\left[\tilde{V}_{(4)}\middle|\tilde{V}_{(1)} = V_{(1)}, \tilde{V}_{(2)} = V_{(2)}, \tilde{V}_{(3)} = V_{(3)}\right]\right] \\
&= E\left[V_{(4)}\right] = E\left[p_3\right]
\end{aligned}
$$

$\square$

The result here is very close to what Milgrom and Weber (2000) conjectured as the solution to the sequential second price auction. However, their model was a more general affiliated value setting without any price announcements or only announcing the second highest bid (winning price).

2.3 The General Case

We first state the general result regarding $M \geq 3$.

For any $V_i = V$, let $Y_1, \cdots, Y_{M-1}$ be the order statistics of other bidders' valuations,[20] and given $V = v$, the distributions of these order statistics are

$$
Y_1, Y_2, \cdots, Y_m \Big|_V \sim f_{12\cdots m}(y_1, y_2, \cdots, y_m | v)
$$

for all $m = 1, 2, \cdots, M-1$. Then, we have

[20]Since $V_1, \cdots, V_N$ are symmetric, we can use any V_i without loss of generality.

Theorem 2. *The following iteratively defined bidding strategy,*

$$\begin{aligned}
&\beta_m(v; z_1, \cdots, z_m) \\
&= \int_{\underline{v}}^{v} \beta_{m+1}(y_2; z_1, \cdots, z_{m-1}, v, v) \frac{f_{12\cdots(m+1)}(z_1, \cdots, z_{m-1}, v, y_{m+1}|v)}{f_{12\cdots(m)}(z_1, \cdots, z_{m-1}, v|v)} dy_{m+1} \\
&= E\Big[\beta_{m+1}(Y_{m+1}; z_1, \cdots, z_{m-1}, v, v)\Big| Y_1 = z_1, \cdots, Y_{m-1} = z_{m-1}, Y_m = v, V_i = v\Big] \\
&= \beta_m(v; z_1, \cdots, z_{m-1}),
\end{aligned}$$

for $m = 1, 2, \cdots, (M-1)$, which is independent of z_m, starting from the last period:

$$\beta_M(v; z_1, \cdots, z_M) = v = \beta_M(v)$$

consists an (symmetric, strictly increasing, sequential) equilibrium for the sequential second price auctions when the highest and second-highest bids are announced after each stage.

Moreover, the equilibrium outcome is efficient,[21] *while the expected price sequence, $E[\beta_1(V_{(2)})]$, $E[\beta_2(V_{(3)}; V_{(1)})]$, $\cdots$, $E[\beta(V_{(M+1)})] = E[V_{(M+1)}]$ satisfies*

$$E\left[\beta_1(V_{(2)})\right] \leq E\left[\beta_2(V_{(3)}; V_{(1)})\right] \leq \cdots \leq E\left[\beta(V_{(M+1)})\right] = E\left[V_{(M+1)}\right]$$

For subgame perfection, we do have to check whether one would want to deviate in later stages, given all possible actions in the previous stages. The next lemma provides a handle on the typical subgame of stage k, after the bidder has behaved as x_1, x_2, ..., x_{k-1}, in stage 1, 2, ..., $k-1$, respectively, showing that bidders would bid as $x_k^* = v$, knowing by subgame perfection that they would bid as $x_{k+1}^* = x_{k+2}^* = \cdots = x_M^* = v$ later.

Lemma 5. *In the symmetric equilibrium proposed in Theorem 2, in stage k, $k = 1, 2, \cdots, (M-$*

[21]Note that the M highest valuation bidders each win an item since bid functions are increasing.

1),[22] *consider the subgame after the bidding history in which one bid as if they were* x_1 *in stage* 1, x_2 *in stage* 2, ... , *and* x_{k-1} *in stage* $(k-1)$. *Then, a bidder with valuation* v *would choose* $x_k = v$, *knowing that by subgame perfection, she will choose* $x^*_{k+1} = x^*_{k+2} = \cdots = x^*_M = v$ *in later stages.*

Proof. See Appendix. □

Now we are ready to prove the general theorem.

Proof. **(Proof of Theorem 2)**
First note that the last period is the standard second price (private value) auction where bidding one's valuation is a weakly dominant strategy. Hence, $\beta_M(v) = v$ is indeed optimal.

In fact,

$$\begin{aligned}
\frac{\partial u}{\partial x_M} &= \int_{x_1}^{\overline{v}}\int_{x_2}^{\max\{x_2,y_1\}} \cdots \int_{\max\{x_{M-1},x_M\}}^{\max\{x_{M-1},y_{M-2}\}} \Big\{ \Big[v - \beta_M(x_M; y_1, \cdots, y_{M-1}, x_{M-1}) \Big] \\
&\qquad\qquad \cdot f_{1\cdots M}(y_1, \cdots, y_{M-1}, x_M | v) \Big\} dy_{M-1} \cdots dy_1 \\
&= \big(v - x_M\big) \int_{x_1}^{\overline{v}} \cdots \int_{\max\{x_{M-1},x_M\}}^{\max\{x_{M-1},y_{M-2}\}} f_{1\cdots M}(y_1, \cdots, y_{M-1}, x_M | v) dy_{M-1} \cdots dy_1 \\
&= 0
\end{aligned}$$

at $x^*_M = v$, and $\beta_M(v; z_1, \cdots, z_M) = v$ does indeed satisfy the first order condition.

For sufficiency, note that

$$\frac{\partial u}{\partial x_M}(v, x_1, \cdots, x_M) \geq 0 \text{ if } x_M < v$$
$$\frac{\partial u}{\partial x_M}(v, x_1, \cdots, x_M) \leq 0 \text{ if } x_M > v$$

[22]If $k = 1$, we have the trivial case where there is no history, and hence, all integration regarding y_j where $j \leq k-1$ stated below are trivial. If $k = (M-1)$, we are at the next to last stage, and all integration regarding y_i where $i \geq k+2$ are trivial.

Hence, $x_M^* = v$ is indeed a global maximum.[23] Therefore, it is optimal to choose $x_M = v$.

Now consider the subgame of stage $k = (M-1), (M-2), \cdots, 1$. Lemma 5 shows that with the stated shows that with the stated bid function, $x_k = v$ is indeed optimal, given any history where one bids as if she were x_1, x_2, $\cdots$, x_{k-1} in stages 1, 2, $\cdots$, $(k-1)$, respectively, and knowing that she will choose $x_{k+1}^* = \cdots = x_M^* = v$.

Thus, we have shown that the proposed bid functions consist an equilibrium.

Moreover, since β_m is independent of z_m, we have

$$\begin{aligned}
&E\left[\beta_m\Big(V_{(m+1)}; V_{(1)}, \cdots, V_{(m-1)}\Big)\right] \\
&= E\left\{E\Bigg[\beta_{m+1}\Big(\tilde{V}_{m+2}; V_{(1)}, \cdots, V_{(m-1)}, V_{(m+1)}\Big)\right. \\
&\qquad\qquad\qquad \left.\Bigg|\tilde{V}_1 = V_{(1)}, \cdots, \tilde{V}_m = V_{(m+1)}, \tilde{V}_{(m+1)} = V_{(m+1)}\Bigg]\right\} \\
&\le E\left\{E\Bigg[\beta_{m+1}\Big(\tilde{V}_{m+2}; V_{(1)}, \cdots, V_{(m-1)}, V_{(m)}\Big)\right. \\
&\qquad\qquad\qquad \left.\Bigg|\tilde{V}_1 = V_{(1)}, \cdots, \tilde{V}_m = V_{(m)}, \tilde{V}_{(m+1)} = V_{(m+1)}\Bigg]\right\} \\
&= E\left[\beta_{m+1}\Big(V_{(m+2)}; V_{(1)}, \cdots, V_{(m-1)}, V_{(m)}\Big)\right],
\end{aligned}$$

for $m = 1, 2, \cdots, (M-1)$, and hence, the expected price sequence (weakly) ascends. □

Therefore, the three auction example is not just a special case, but can be generalized to any finite number of auctions. The intuition is the following. The last stage is the standard auction where bidding one's valuation is a (weakly) dominant strategy. For all the early stages, as illustrated in the three auction example, a bidder is weighing between winning this auction and winning the next one, as well as considering the effect of the price announcement in this auction on the next one. Hence, she will bid the expected winning price of the next auction, condition on herself being the tied winner of this auction, minus the adjustment for price announcement effects. Therefore,

[23] One might be indifferent between several x_M's, but $x_M^* = v$ is still one of these global maxima.

knowing other bidders do not utilize the price announcement of the immediate previous period, one would bid the conditional expected price of the next auction. Moreover, further auctions beyond the immediate next one do not directly influence the choice in the current auction since bids below the second-highest bid are not announced. However, they do indirectly determine the bids through the iterative nature of the bid functions.[24]

Note that this equilibrium is efficient since bid functions are increasing. Therefore, this equilibrium with unused information is interesting since it provides an example where there are "too much" information, and agents need to drop some in equilibrium to obtain efficiency, contrary to the usual case where information aggregation is the key to reach the social optimum. Nonetheless, it is essential that even though one ignores the most critical information of their next opponent, the bid functions do depend on the information of past winners. In fact, in the case of three auctions, we show the bid function might not be not well-defined if bidders ignore all information, save in very special cases, such as independence, where information does not matter at all.

Since the equilibrium bid functions do not depend on second-highest bids, we immediately have the corollary that the same equilibrium exists even if the second-highest bids are not announced.

Corolary 3. *The iteratively defined bidding strategy for the sequential second price auction when both the highest and second-highest bids are announced,*

$$\begin{aligned}
&\beta_m(v; z_1, \cdots, z_{m-1}) \\
&= \int_{\underline{v}}^{v} \beta_{m+1}(y_2; z_1, \cdots, z_{m-1}, v) \frac{f_{12\cdots(m+1)}(z_1, \cdots, z_{m-1}, v, y_{m+1}|v)}{f_{12\cdots(m)}(z_1, \cdots, z_{m-1}, v|v)} dy_{m+1} \\
&= E\Big[\beta_{m+1}(Y_{m+1}; z_1, \cdots, z_{m-1}, v)\Big| Y_1 = z_1, \cdots, Y_{m-1} = z_{m-1}, Y_m = v, V_i = v\Big] \\
&= \beta_m(v; z_1, \cdots, z_{m-1}),
\end{aligned}$$

[24]The winning price of the next auction depends on the bid function of the next auction, which further depends on the bid function of future auctions, etc.

for $m = 1, 2, \cdots, (M-1)$*, starting from the last period with*

$$\beta_M(v; z_1, \cdots, z_M) = v = \beta_M(v),$$

consists an (symmetric, strictly increasing, sequential) equilibrium for the sequential second price auction when only the highest bid is announced after each stage.

Also, the equilibrium outcome is efficient, while the expected price sequence, $E[\beta_1(V_{(2)})]$, $E[\beta_2(V_{(3)}; V_{(1)})]$, $\cdots$, $E[\beta(V_{(M+1)}) = V_{(M+1)}]$ *satisfies*

$$E\left[\beta_1(V_{(2)})\right] \leq E\left[\beta_2(V_{(3)}; V_{(1)})\right] \leq \cdots \leq E\left[\beta(V_{(M+1)})\right] = E\left[V_{(M+1)}\right]$$

In both cases, the equilibrium is efficient, due to increasing bid functions, and have ascending prices. However, the equilibrium for the sequential second price auction when both the highest and second-highest bids are announced is especially interesting since there are unused information in each stage, while the outcome is still efficient. Unlike the typical case where information aggregation is in the essence of achieving social optimality, here we have "too much" information, and hence, bidders have to drop some to achieve efficiency.[25] However, one should note that it is crucial that the information of previous winners *are* used.

2.4 eBay Auctions

We now see how the above theory extends to eBay auctions. The eBay auction format is an non-standard auction rule observed in the field. According to Wang (2006), an *ebay auction* is defined to be an *out-cry auction* with *proxy bidding* and a *fixed-time ending rule* with the following rules:[26]

[25]Note that the above equilibrium might no longer sustain if more bids are announced since $V_{(M+1)}$ would be revealed in stage $M-1$.

[26]In addition to the rules here, Wang (2006) deals with a minimum increment $s > 0$. However, here we abstract from this complication.

Assumption 3. (Out-Cry Auction with Proxy Bidding)

1. *Submitting Proxy Bids: Bidding starts from the reserve price $r = v_0 = 0$ at time $t = 0$. Bidders can submit a "proxy bid" any time $t \in [0, T]$. If no bid is submitted in $[0, T]$, the auction ends without a sale. Bidder i's* proxy bid *at time t is recorded as $b_i(t)$. However, this proxy bid is not revealed immediately. The (hidden) bidding record of proxy bids at time $\tilde{t}$ is $h_{\tilde{t}} = \Big\{b_i(t)\Big\}_{t\in[0,\tilde{t})}$*

2. *High Bidder and its Standing Bid: For any time $t \in [0, T]$, the* high bidder, *denoted by bidder k, is the bidder who submitted the highest proxy bid*

$$b_k(t_1) = \max_{b_i(\tilde{t})\in h_t} b_i(\tilde{t}), \quad t_1 = \arg\max_{\tilde{t}\in[0,t)} \Big\{b_i(\tilde{t})\Big|b_i(\tilde{t}) \in h_t\Big\}.$$

The second-highest bidder, denote by bidder j, is the bidder who submitted the second-highest proxy bid

$$b_j(t_2) = \max_{b_i(\tilde{t})\in h_t\setminus\{b_k(t_1)\}} b_i(\tilde{t}), \quad \textit{at time } t_2 = \arg\max_{\tilde{t}\in[0,t)} \Big\{b_i(\tilde{t})\Big|b_i(\tilde{t}) \in h_t \setminus \{b_k(t_1)\}\Big\}.$$

The standing bid *or "current price" S_t at any time $t \in [0, T]$ is $b_j(t_2)$, if there are two or more bidders who submitted their proxy bids. If there is only one active bidder, it is the reserve price r.*[27] *The standing bid S_t is common knowledge immediately after time t.*

3. *Ascending Proxy Bids: Later proxy bids have to be strictly higher than the bidder's last submitted proxy bid and the standing bid. In other words, if $b_i(\hat{t})$ is the last proxy bid submitted by bidder i, then its new proxy bid $b_i(t)$ must satisfy*

$$b_i(t) > b_i(\hat{t}), \quad \textit{and} \quad b_i(t) > S_t.$$

[27] If two bidders submitted the same highest proxy bid, the bidder who submitted their proxy bid earlier becomes the high bidder, and the standing bid equals to the tied highest proxy bid.

One might ask why are we interested in the case where the highest bid in the last minute is actually observed, while we seldom see the highest bids revealed in the field. For example, in eBay auctions, the winner's maximum bid is usually not revealed since the robot only bids up to the second-highest bid plus a minimum increment. However, it should be noted that since the highest loser in one stage is the winner in the next, we have two chances to infer past winner's valuation. One is from the highest bid when she wins, the other is the previous stage's winning price (second-highest bid) when she was the highest loser. Therefore, it is fine to not observe all highest bids (except for the first one where we have no "previous" stage to observe). Moreover, if we are concerned about empirical evidence, one should note that on eBay, the highest bids *are* sometimes revealed due to a minimum increment rule.[28] Thus, assuming that one sees the highest bid (only in the first stage) is not as implausible as one might first think.

Assumption 4. (Fixed-Time Ending Rule) *The auction ends at a fixed time T. The high bidder wins the item pays the standing bid S_T.*[29]

Moreover, when we consider more than one item sold, we have the same auction rule executed *repeatedly*, and have the following bid history:

Assumption 5. (History in the Repeated eBay Auction)

1. *Revelation of Last Minute Bids:*

   We assume that all last minute bids are accepted, but only the two highest ones are revealed.[30] *Hence, the last minute bidding record for stage m is $\{b_{m,1}, b_{m,2}\}$.*

[28] Specifically, when the hidden maximum of the winner is so close to the second-highest bid that adding a minimum increment exceeds it, then the hidden maximum is actually shown as the winning price (instead of the second-highest bid plus the minimum increment). Since ebay's minimum increment is not fixed, but proportional to the standing bid (current price), in the earliest auction with the most bidders, it is likely that the highest and second-highest bids are within a minimum increment.

[29] Note that due to the proxy bidding system, bidders may submit their reservation price at time $t = T$. This makes the "last minute" T *as if* a sealed bid second price auction.

[30] This makes the eBay auction comparable to the sealed-bid second price auction discussed above. Note that as shown in Wang (2006), if we have only two auctions and private values, bid revelation does not matter since there is a dominant strategy in the last auction.

2. *Bidding Record for Early Bids:*

   The bidding record of stage m, at time $t \in [0, T]$ is denoted by $\tilde{h}_{m,t} = \{S_{m,\tau}, i_{m,\tau}\}_{\tau \in [0,t]}$ consisting of the sequence of changes in standing bids, $S_{m,\tau} \in [r, \overline{v}]$, as well as the corresponding bidders who triggered the change, $i_{m,\tau} \in I_0 = \{0, 1, \cdots, N\}$.

3. *Bidding History:*

   The bidding record of an ended auction stage m is the combination of the early bids and last minute bids $\hat{h}_m = \left\{\tilde{h}_{m,T}, \{b_{m,1}, b_{m,2}\}\right\}$, and hence, the entire bidding history up to stage m, time t is defined by

$$h_{m,t} = \left\{\hat{h}_1, \cdots, \hat{h}_{m-1}, \tilde{h}_{m,t}\right\}$$

Given these settings, we are ready to state the corollary that extends the sequential sealed-bid second price auction equilibrium to the eBay auction, in which bidders only bid at the last minute since they hold (out-of-equilibrium) beliefs that others would correctly revise their conditional expectations upwards (when bidding later) if they bid early.

Corolary 4. *The iteratively defined bidding strategy for the sequential second price auction when both the highest and second-highest bids are announced,*

$$\begin{aligned}
&\beta_m(v; z_1, \cdots, z_{m-1}) \\
&= \int_{\underline{v}}^{v} \beta_{m+1}(y_2; z_1, \cdots, z_{m-1}, v) \frac{f_{12\cdots(m+1)}(z_1, \cdots, z_{m-1}, v, y_{m+1}|v)}{f_{12\cdots(m)}(z_1, \cdots, z_{m-1}, v|v)} dy_{m+1} \\
&= E\Big[\beta_{m+1}(Y_{m+1}; z_1, \cdots, z_{m-1}, v)\Big| Y_1 = z_1, \cdots, Y_{m-1} = z_{m-1}, Y_m = v, V_i = v\Big] \\
&= \beta_m(v; z_1, \cdots, z_{m-1}),
\end{aligned}$$

for $m = 1, 2, \cdots, (M-1)$, starting from the last period with

$$\beta_M(v; z_1, \cdots, z_M) = v = \beta_M(v),$$

consists an (symmetric, strictly increasing, sequential) equilibrium for the eBay auction when bidders only bid at the last minute, holding out of equilibrium beliefs that at any stage,

1. *If at some time t, the standing price S_t changes to $S_0 = a > \underline{v}$, triggered by bidder j, then infer that $\overline{v} \geq v_j \geq a > \underline{v}$, and update all bid functions accordingly.*

2. *If at some time t, the standing price becomes $S_t = S_0 \geq 0$, but $S_0 \leq \underline{v}$, then infer that $\overline{v} \geq v_j \geq a > \underline{v}$ for some arbitrary given $a > \underline{v}$, for all v_j that have submitted a bid, and update all bid functions accordingly.*

Proof. Suppose all other opponents bid according to the proposed last minute bidding strategy. Then, since nobody else bids, there is no information to update, and hence, playing according to the proposed iteratively defined strategy of the sequential sealed-bid second price auction based on your ex ante information is your best response at the last minute $t = T$ for each stage, because if one deviates, that would trigger other bidders (bidder j) to infer that this bidder (bidder i) has valuation $v_i \in [a, \overline{v}]$, and by affiliation, would raise their last minute bid to

$$\begin{aligned} &E\Big[\beta_{m+1}(Y_{m+1}; z_1, \cdots, z_{m-1}, v)\Big| Y_1 = z_1, \cdots, Y_{m-1} = z_{m-1}, Y_m = v, V_j = v_j, V_i > a\Big] \\ &\geq E\Big[\beta_{m+1}(Y_{m+1}; z_1, \cdots, z_{m-1}, v)\Big| Y_1 = z_1, \cdots, Y_{m-1} = z_{m-1}, Y_m = v, V_j = v_j, V_i \geq \underline{v}\Big]. \end{aligned}$$

□

3 The Experiment

3.1 Theoretical Predictions

In the experiment, we set up sequences of three auctions with both the sealed-bid second price format and the eBay format, and have groups of five subjects with unit demand to bid in each and every auction under the two formats. Under the sealed-bid second price format, there are three stages of bidding and bidders submit a sealed bid in each stage. At the end of each stage, the two highest bids of that stage are revealed, the highest bidder wins the item, and pays the second highest bid. Such procedure is repeated until all three items are sold.

On the other hand, under the eBay auction format, which resembles the rules of the internet auction house, in each stage of bidding, bidders have a fixed time interval (sixty seconds) to submit their "maximum bids." There are no limits on how many bids one can submit, but any bid has to surpass the current "standing bid," which is the second highest maximum bid up to now.[31] The bidder with the highest maximum bid at the end of the bidding stage wins the item and pays the final standing bid. Such procedure is repeated until all three items are sold. Note that instead of paying her maximum bid, the winner pays the final standing bid, which is the highest opponent (maximum) bid. Hence, it is very close to the second price format, except we allow for multiple stages of bid submission, instead of one sealed bid.

Under both formats, subjects are assigned private valuations independently drawn from uniform [0, 100] and rounded to the nearest integer. We record the bids and estimate the bid functions for each auction under both formats. We also record the different timing and amount people bid in the early vs. last auctions for the eBay format. The timing of bids and estimate bidding strategy would lead us to accept or reject the following theoretical predictions:

First of all, as shown in Wang (2006), when there are more than one item sold, we can sustain a last minute bidding equilibrium under the eBay auction format which resembles the symmetric equilibrium in sequential sealed-bid second price auctions. In particular, during the early auctions, no bid is submitted until the last minute. In other words, we have

Hypothesis 1 (Timing of Bids)**.** *Under the eBay format, bidders attempt to bid very late (at the*

[31] When there are less than two bids submitted, the standing bid equals to the reserve price, set at \$0.01 in the experiment.

last minute) in early auctions, but not necessarily in the last auction.[32]

Moreover, given the increasing equilibrium bid functions shown in the previous section, the expected prices of all three auctions are the same, and equal to the highest loser's valuation in the last auction, which is $E[v(4)] = 33.33$, assuming bidders' valuations drawn independently from uniform $[0, 100]$. Note that these bid functions are increasing, and hence, the highest valuation bidder wins the first item, the second highest valuation bidder wins the second item, and the third highest valuation bidder wins the third item. In other words, both auction formats are efficient since higher valuation bidders win the item. Moreover, the expected revenue of both formats are the same as in a one-shot multi-unit Vickrey auction where bidders each submit a sealed bid and the three high bidders each win one item paying the fourth highest bid. This is because in the multi-unit Vickrey auction, each winner pays $E[v(4)]$, equal to the expected winning price of the last auctions (under both formats), and hence, equal to that of early auctions. Thus, we have

Hypothesis 2 (Efficiency and Ascending/Constant Prices)**.** *Both auction formats are efficient since the three highest bidders win the item. Moreover, the winning price is (stochastically) constant across the three auctions if valuations are independent (ascends if affiliated). In particular, with private values independently drawn from* $[0, 100]$*, the expected prices should all equal to* $E[v(4)] = 33.33$*, but the variance of the winning price increases.*[33] *Therefore, expected seller revenue is the same across stages of auctions and total revenue equals to that of a Vickrey auction selling all three items at once.*

Hypothesis 3 (Equilibrium Bidding Strategy)**.** *Under both formats, with independent private val-*

[32]Note that with private values and a minimum increment as small as the minimum difference between valuations, bidders are indifferent between bidding early and late in the last auctions. Hence, it is as optimal to bid early as to bid late.

[33]As shown in Table 2, the average equilibrium winning price (standard deviations in parentheses) using the valuations actually drawn for the experiments are $32.81(8.61)$, $33.20(10.60)$, and $32.76(16.48)$ for the eBay format, and $33.34(9.02)$, $32.88(10.94)$, and $33.59(16.96)$ for the sealed-bid second price format.

ues, the bidding functions for the k-th auction in a sequence of K auctions and N bidders is

$$\beta_k(v) = \frac{N-K}{N-k} \cdot v$$

In the experiment we conduct, $K = 3$ and $N = 5$, so the bid functions are $\beta_1(v) = \frac{v}{2}$, $\beta_2(v) = \frac{2v}{3}$, and $\beta_3(v) = v$.

Note that all of the theoretical predictions stated above do not distinguish between the two auctions formats, which is an implicit hypothesis we test as well.

3.2 Experimental Procedure

The experimental sessions were conducted in the Social Science Experimental Laboratory (SSEL) at California Institute of Technology employing Caltech students, who are highly sophisticated and well-trained in mathematical reasoning. The experiment was programmed and conducted with the software z-Tree (Fischbacher 1999).[34] Figure 1 and 2 are screenshots of the eBay auction treatment, while Figure 3 is that of the sealed-bid second price auction treatment.

Two sessions of each format was conducted from August 2005 to November 2005. Due to a lack of show-ups, the first pair of sessions was conducted with 20 and 25 subjects (4 and 5 groups respectively) and the second pair was conducted with 15 subjects (3 groups).[35] Sessions lasted 1.5-2 hours, and subjects were paid \$2 to \$46 plus a five dollar show up fee.[36]

In each session, we had two practice rounds and fifteen real rounds. In each round, the computer randomly picks five subjects to form a group in which three identical items are sold se-

[34] Although z-Tree is mainly used for "ready-made" experiments, it does come with a "later...do" command for clock auctions. Hence, it was possibly to replicated the eBay auction format using this command.

[35] With pairs of identical number of subject, we use the same valuation draws for both formats to control for potential history effects.

[36] Earning \$2 plus the show-up fee might seem very low, but it is possible if subjects bid above their valuations and/or win more than one item in the same round. In fact, this particular subject incurred a loss of approximated \$13 in the second round due to overbidding, and hence, winning two items in one round, but eventually grew out of debt at the end. Moreover, no other subject earned below \$9.

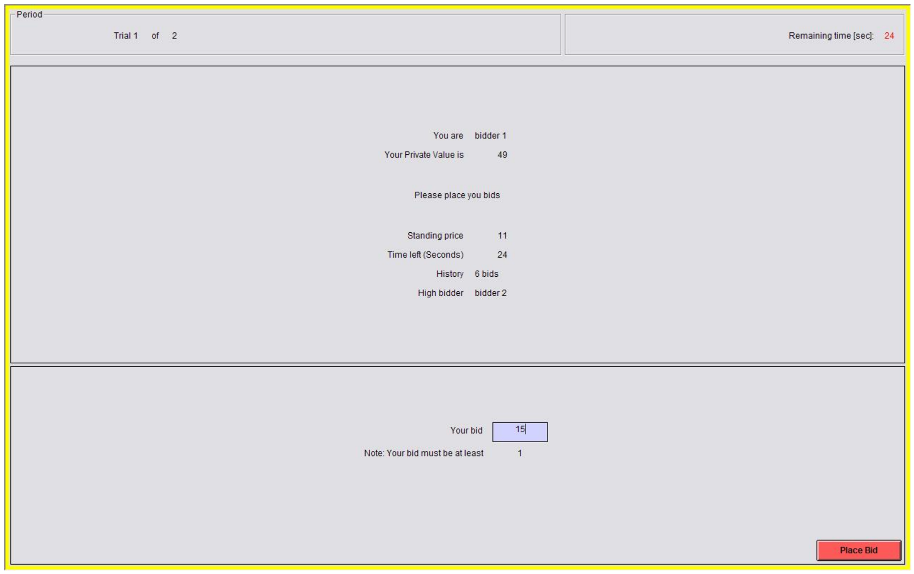

Figure 1: A screenshot of the z-Tree interface when bidding in the eBay auction treatment.

quentially using the same auction format. Buyers' values are independently drawn from uniform $[0, 100]$ and rounded to the nearest integer. To avoid initial learning effects, we drop the first 5 rounds and consider only the last 10 rounds of the 15 rounds.

3.3 Experimental Results

We summarize our experimental findings with their correspondence to the predictions stated as Hypothesis 1 through 3 above.

To begin with, regarding the timing of bids in the eBay format, we have

Result 1 ((Timing of Bids))**.** *Under the eBay format, many bidders attempt to bid very late (at the last minute) in all three auctions. Moreover, other bidders attempt to bid very early.*

In fact, according to Figure 4, during the last 10 rounds, approximately 17% of the bids in the first auction, 28% of the bids in the second auction, and 29% of the bids in the third auction are submitted in the last two seconds. Since bidders are not restricted to one bid, and later bids must be higher than earlier ones (which potentially are less "serious"), this indicates a substantial portion

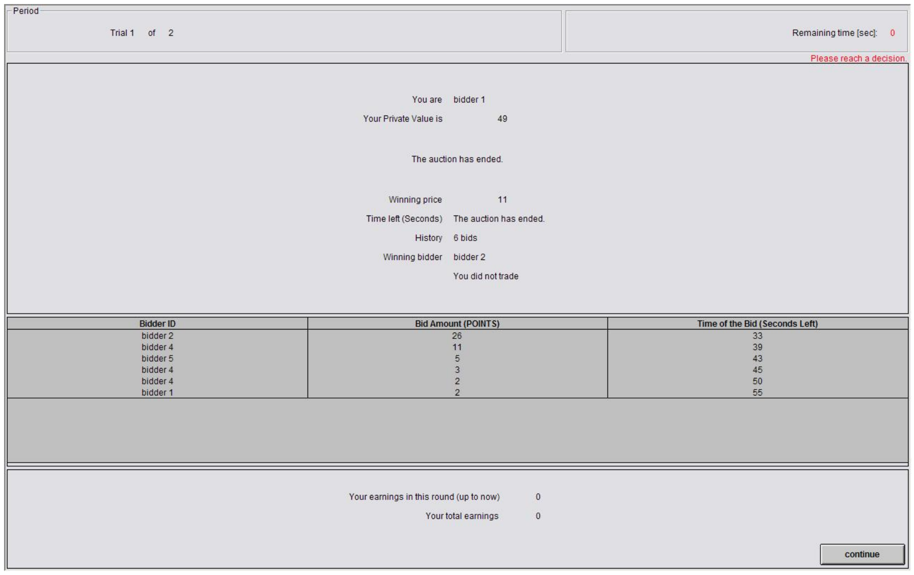

Figure 2: A screenshot of the z-Tree interface when viewing the results in the eBay auction treatment.

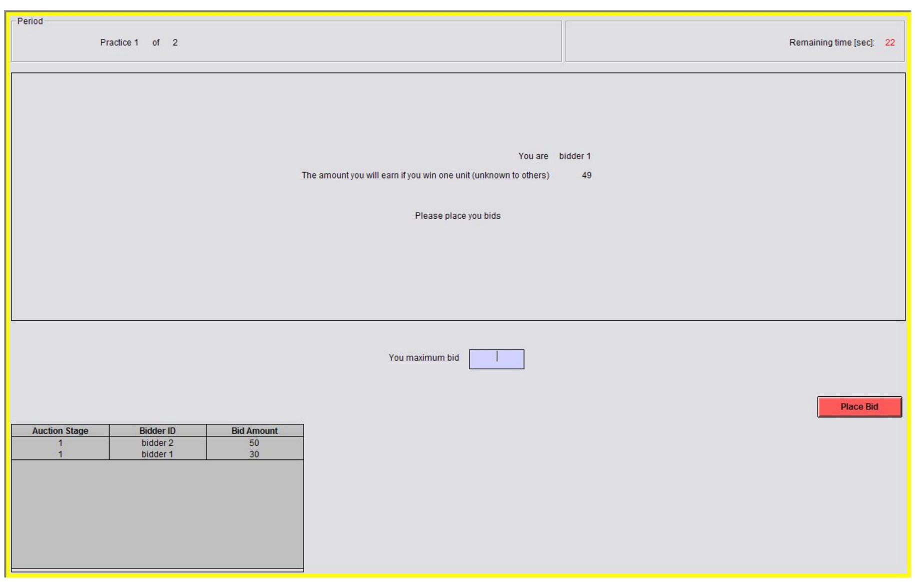

Figure 3: A screenshot of the z-Tree interface when bidding in the sealed-bid second price auction treatment.

of the bids come in very late, confirming what Roth and Ockenfels (2002) report about field data they collected.

Note that since two seconds is barely enough for a subject to read the screen, type in a bid and hit the "Submit" button, this is not just "late bidding," but really "last minute bidding." In fact, as shown in Figure 5, when fitting a polynomial function on the timing of bids, the slopes are all increasing during the last third of the bidding time. Interesting enough, this last minute bidding tendency is also evident (and perhaps the strongest) in the last auction, though theory predicts one would be indifferent between bidding early and late.[37] This might be related to heuristics subjects have learned from past experience such as actually bidding on eBay or participating in other auction experiments.[38]

On the other hand, we also see some intensive bidding behavior at the initial stage of bidding (from 58 seconds to 48 seconds left) where approximately 20% of the bids come in. This is related to the "squatting" behavior in Ely and Hossain (2006). In fact, Figure 5 also shows this tendency since the second order derivatives are negative in the initial stage of bidding.

Secondly, regarding efficiency and the price sequence, we have

Result 2 (Efficiency and Declining Prices)**.** *Both formats have efficiency levels above* 90%*. Moreover, under both formats, the average prices of the first and second auction are significantly higher than* 33.33 *predicted by theory, while that of the last auction is close to* 33.33*. Hence, revenue for the first and second auction are higher than the last auction.*

Under a private setting, efficiency is achieved if the highest bidder(s) win the item(s). In particular, in the current setting, efficiency is achieved if all bidders above the fourth-highest valuation

[37] Note that last minute bidding in the last auction is not "against" theory since in the experiment, the minimum increment for bidding (one cent) is the same as the minimum increment in valuations.

[38] As shown below, almost all bidders bid their valuation in the last auction, contrary to the overbidding literature showing how subject consistently overbid in second price auctions. This suggests that Caltech students are extremely well-trained (in both mathematical reasoning and past experimental experience) to avoid losses by not overbidding. However, individual bidding behaviors below also suggest at least some of the subjects are so well-trained that they bid their valuation even in the early rounds, indicating heuristics instead of actual strategic thinking.

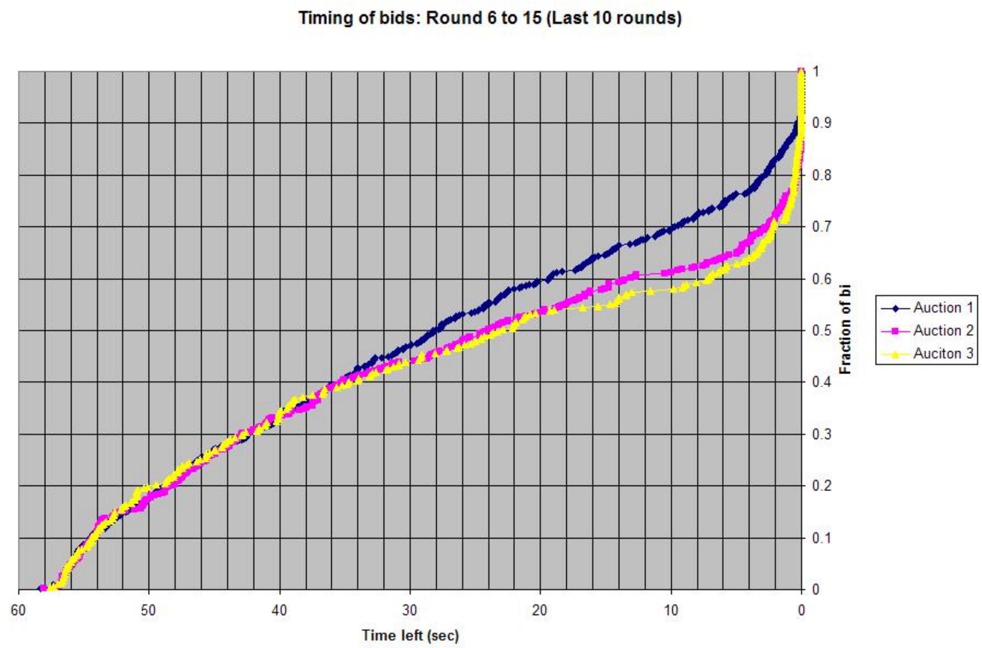

Figure 4: Timing of Bids.

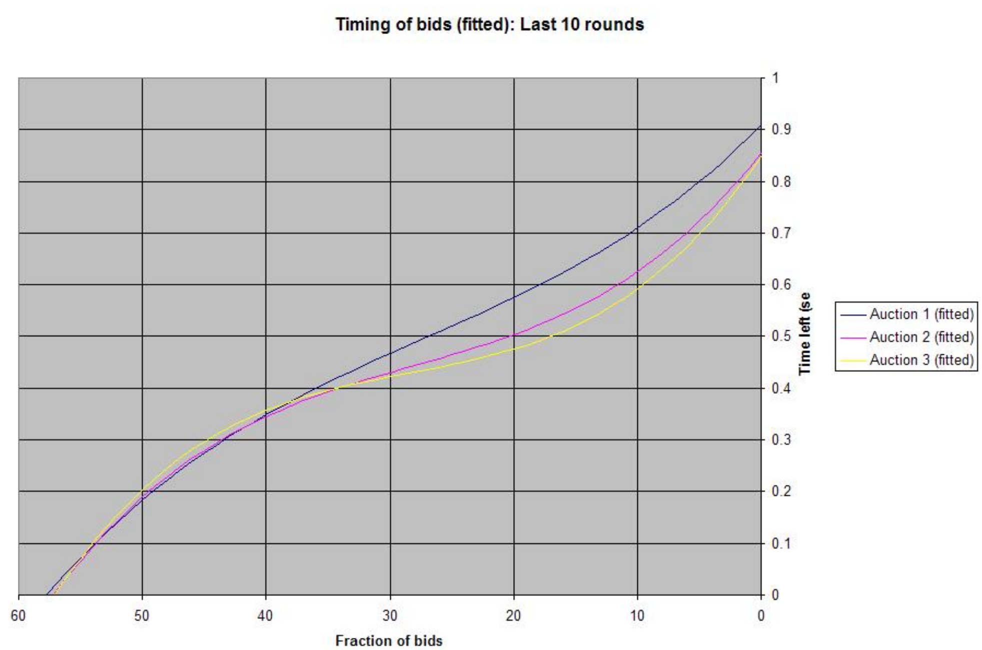

Figure 5: Fitted Timing of Bids.

win an item. We denote these bidders (who have the first, second, or third-highest valuations) as "designated winners." In the experiments, 91.43% of designated winners actually won an item under the eBay auctions, while 94.00% of the designated winners won under the sealed-bid second price auctions.

Regarding the price sequence, as shown in Figure 6, for the eBay auctions, the average winning prices are $36.53(10.25)$ for auction 1, $37.69(14.65)$ for auction 2, and $32.13(19.17)$ for auction 3, in which the first two are significantly different from 33.33 predicted by theory.[39] Moreover, the paired t-tests show that the prices of auction 1 and 2, 1 and 3 are both not significantly different from each other, while that of 2 and 3 are significant at the 0.12% level.[40]

Under the sealed-bid second price format, we have similar results as well. The average winning prices are $38.96(11.97)$ for auction 1, $36.63(12.86)$ for auction 2, and $32.99(18.07)$ for auction 3, in which the first two are significantly different from 33.33 predicted by theory.[41] Moreover, the paired t-tests show that the prices of auction 1 and 2 are not significantly different from each other, while both are significantly larger than that of auction 3 at the 0.3% and 1.6% level.[42]

Comparing the two auction formats, the winning prices of the two formats are also not significantly different from each other. In particular, the t-statistics for the two-sample t-test of the two formats are 1.372 for auction 1, -0.486 for auction 2, and 0.289 for auction 3.

Though these results are not exactly consistent with what Hypothesis 2 predicts, they are in line

[39]The T-statistics are 2.607 (p-value $= 0.011$) for auction 1, 2.485 (p-value $= 0.015$) for auction 2, and -0.526 (p-value $= 0.601$) for auction 3. However, the significance of the first two auctions go away if we only consider last 5 rounds ($t = 1.865$ and 1.881).

[40]T-statistics are 1.941 for auction 1 vs. auction 3, and -0.661 for auction 1 vs. auction 2, both not significant at the 5% level (two-sided). The prices of auction 2 and auction 3 are significantly different from each other ($t = 3.391$, $p = 0.001$) for the paired two-sided test. However, such significance is eliminated if we only consider the last 6 rounds ($t = 1.929$, $p = 0.0607$).

[41]The T-statistics are 4.207 (p-value $= 0.000$) for auction 1, 2.290 (p-value $= 0.025$) for auction 2, and -0.171 (p-value $= 0.865$) for auction 3. However, though the significance of the first auction remains if we consider only the last 5 rounds ($t = 3.510$), the significance of the second auction goes away ($t = 1.377$).

[42]For auction 1 vs. auction 2, T-statistics is 1.797 ($p = 0.076$). For auction 1 and 2 vs. auction 3, the T-statistics are 3.074 ($p = 0.003$) and 2.473 ($p = 0.016$), respectively. However, though the significant difference between the first and last auction is not eliminated even considering only the last 2 or 3 rounds, the significant difference between the last two auctions is eliminated if we only consider the last 8 rounds ($t = 1.570$, $p = 0.121$).

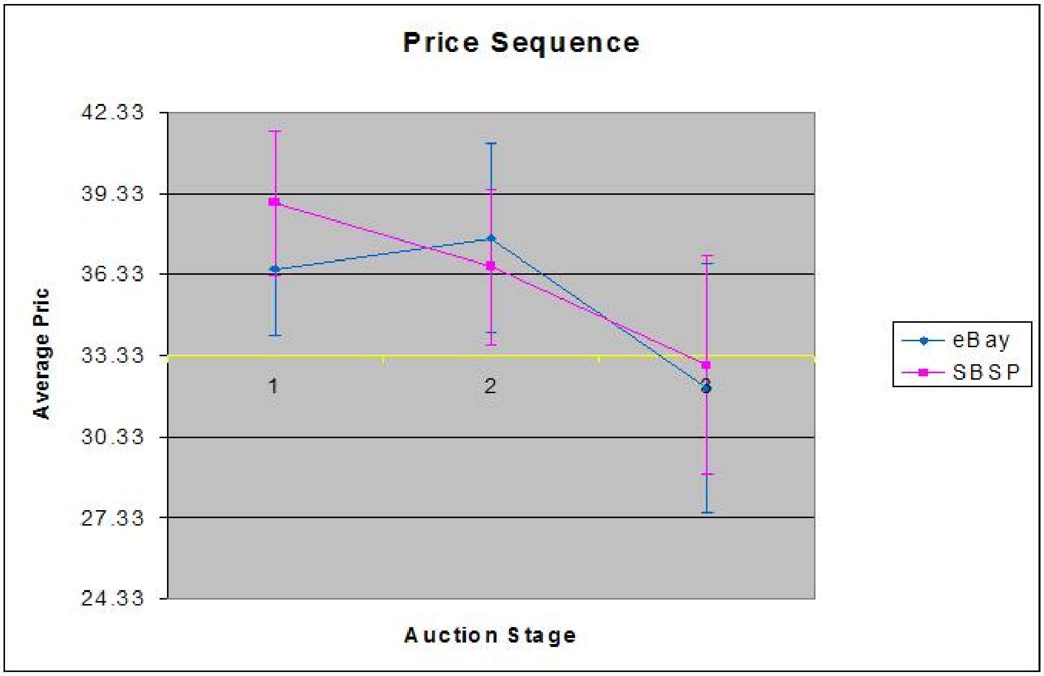

Figure 6: The Winning Price Sequence.

with the empirical evidence such as the "afternoon effect" in wine auctions reported by Ashenfelter (1989), as well as the "declining price anomaly" discussed theoretically by McAfee and Vincent (1993). Nevertheless, while McAfee and Vincent (1993) attribute declining prices to bidder risk aversion, individual behavior results below show that heterogeneity in bidders' level of (bounded) rationality (steps of thinking) might have played an even more important role in creating this anomaly.

One might wonder if learning can wash away this decline in prices. Therefore, we calculated the average winning price for both auction formats under blocks of 5 rounds, and compare them with the predicted price of 33.33. As shown in Table 1, the average prices of auction 1 are still significantly higher than 33.33 (except for the last block of the eBay format), while that of auction 2 are all higher (but mostly not significantly higher) than 33.33.

However, none of the blocks have average prices that are significantly different from each other using a two sample t-test. Even if we consider the difference between early and late auctions under the two formats for each block, only the eBay auction format (in the block of the middle 5 rounds) has a significant difference between auction 2 and 3 ($t = -2.598$, $p = 0.011$) and auction 1 and 3 ($t = -2.668$, $p = 0.010$) since the average price of auction 3 (for that block) is below 30. All other pairs are not significantly different from each other.[43] Therefore, based on the data we have, there is little evidence that learning can "solve" the declining prices puzzle.[44]

On the other hand, when comparing the two formats, none of the blocks were significantly different from each other (at the 5% level). This indicates that the equivalence between the two formats is robust to learning.

Table 1: Average Price Change across Rounds

Auction Number	Auction 1		Auction 2		Auction 3	
and Format	eBay	SBSP	eBay	SBSP	eBay	SBSP
First 5 rounds	40.91*	44.38***	36.63	37.98	32.83	35.83
	(18.10)	(16.29)	(28.24)	(14.95)	(20.65)	(22.00)
Middle 5 rounds	37.29*	38.15*	37.31*	37.58	29.54	34.30
	(8.95)	(12.41)	(9.92)	(14.74)	(14.65)	(18.95)
Last 5 rounds	35.77	39.78**	38.06	35.68	34.71	31.68
	(11.49)	(11.61)	(18.36)	(10.76)	(22.74)	(17.30)

Notes: *, **, *** indicate 5%, 1% and 0.1% significantly away from theoretical mean of 33.33.

Thirdly, we estimate empirical bid functions based on actual bids and their corresponding valuations and compare them with the equilibrium bid functions listed in Hypothesis 3. Using bids from all subjects, we have

Result 3 (Equilibrium Bidding Strategy)**.** *Under both formats, the fitted linear bid functions for each auction are close to what theory predicts. However, individual bids suggest concentrations on either always bidding one's valuation or bidding zero is the first and second auction.*

In particular, under the eBay format, if we fit a linear bid function (without a constant) by

[43] Comparing blocks of (first and last) 7 rounds also yield similar results.

[44] This suggests that we might need to either run more experiments to make sure this non-significance result in due to a smaller sample size ($35 - 40$), or wait and see if longer experiments show a clearer convergence pattern.

regressing bidders' last bids on their valuations, the estimated bid functions are

$$\beta_1(v) = 0.47v, \beta_2(v) = 0.63v, \beta_3(v) = 0.92v.$$

On the other hand, under the sealed-bid second price format, if we fit a linear bid function (without a constant) by regressing bidders' sealed bids on their valuations, we obtain

$$\beta_1(v) = 0.54v, \beta_2(v) = 0.65v, \beta_3(v) = 0.97v.$$

Both are consistent with what equilibrium theory predicts, which are (for $N = 5$ and $K = 3$)

$$\beta_1(v) = \frac{v}{2}, \beta_2(v) = \frac{2v}{3}, \beta_3(v) = v.$$

However, individual behavior suggests a different story.

In particular, under the eBay format, we consider the relation between the "last submitted bids" and their corresponding valuation.[45] In the third auction, which is identical to a standard second price auction, bidders bid their valuation almost all of the time, contrary to the "overbidding" phenomenon reported in the literature. Figure 7, which plots each bidder's last bid over their valuations clearly shows observations populating the 45 degree line.[46] This is due to the highly sophisticated subject pool (Caltech students), as opposed to the average subject pools used in other studies. who sometimes even attempt to collude.[47] In fact, the fitted bid linear bid function (no constant) is $b = 0.92v$, close to, but lower than what theory predicts.

Furthermore, in the second auction, bidders shade their bids and do not always bid their val-

[45]Here, we assign winners of previous auctions to have zero valuations in later auctions.

[46]There are some zero bids, possibly due to the standing price rising too fast for lower valuation bidders to participate.

[47]There were 35 bids in which subjects bid more than 10 points below their valuation, in which Subject #6 of Session 1 contributed 6 times, and Subject #18 of Session 1 contributed 5 times. Hence, the attempt of collusion was not successful, though it did indeed lower the fitted bid function and the winning price of the third auction.

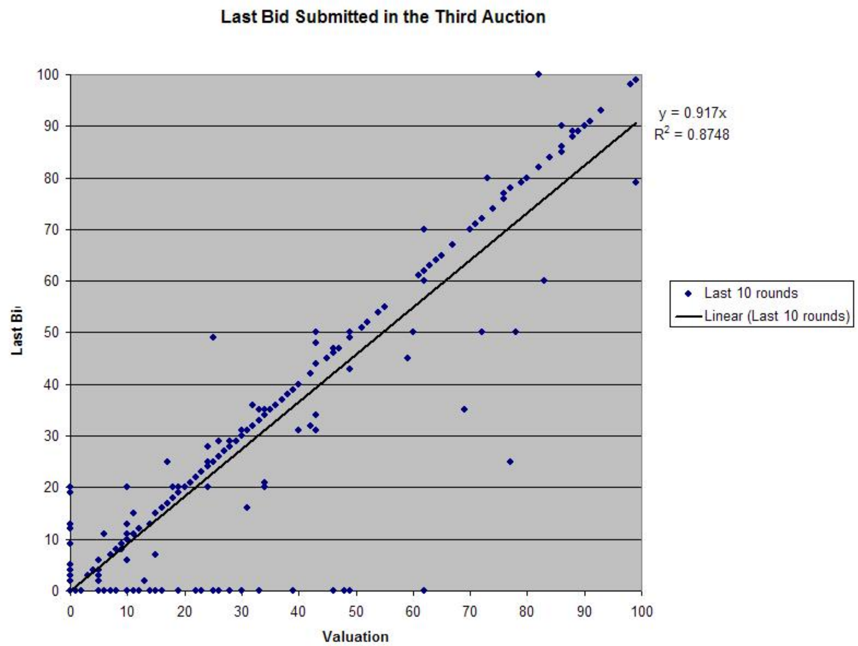

Figure 7: Bidding Strategy of the Third Auction (under the eBay format).

uation. In fact, the fitted linear bid function is $b = 0.63v$, also close to what theory predicts.[48] However, as shown in Figure 8, there are many bids that fall on the 45 degree line and the horizontal axis, meaning that many bidders still bid their valuation (45 degree line) or bid zero/don't bid (horizontal axis), possibly knowing they are sure to win, or have little hope of winning the item.

Finally, in the first auction, the fitted linear bid function is $b = 0.47v$, consistent with what theory predicts. However, the actual bids shown in Figure 9 are either above $b = 0.47v$ or literally zero. This suggests that half of the bidders are "overbidding" in the first auction (and hence, causing a higher winning price), while the other half are avoiding the first auction entirely.[49]

We obtain similar results under the sealed-bid second price format as well. In the third auction, identical to a standard second price auction, bidders bid their valuation almost all of the time, as

[48] Here we excluded one outlier in which Subject #2 bid a proxy bid of $10,000$ in round 6.

[49] Since bidding is voluntary, subjects might have chosen not to bid in the first auction, especially when the have a low valuation. However, there are still many "non-bidding" for valuations higher than 60 in the data, so this may not be the case here.

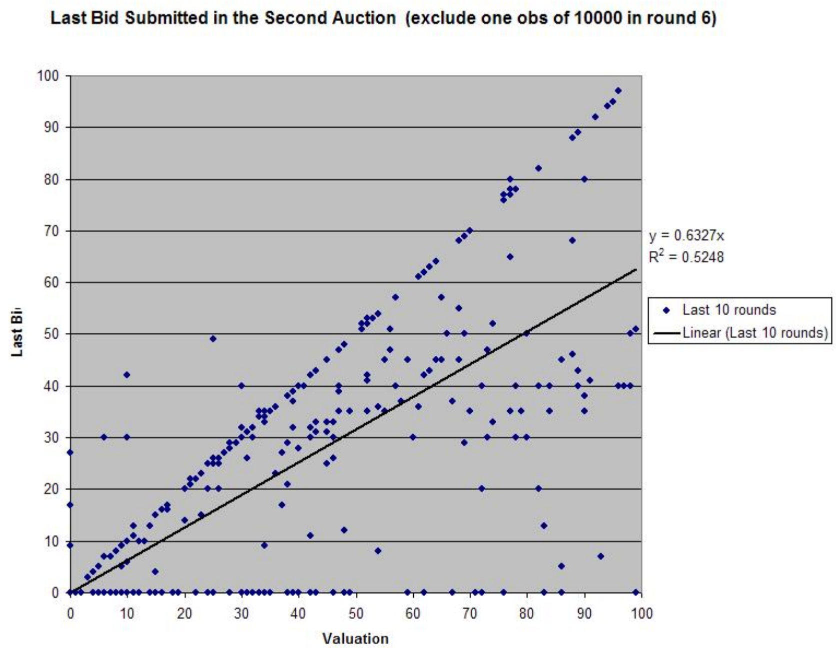

Figure 8: Bidding Strategy of the Second Auction (under the eBay format).

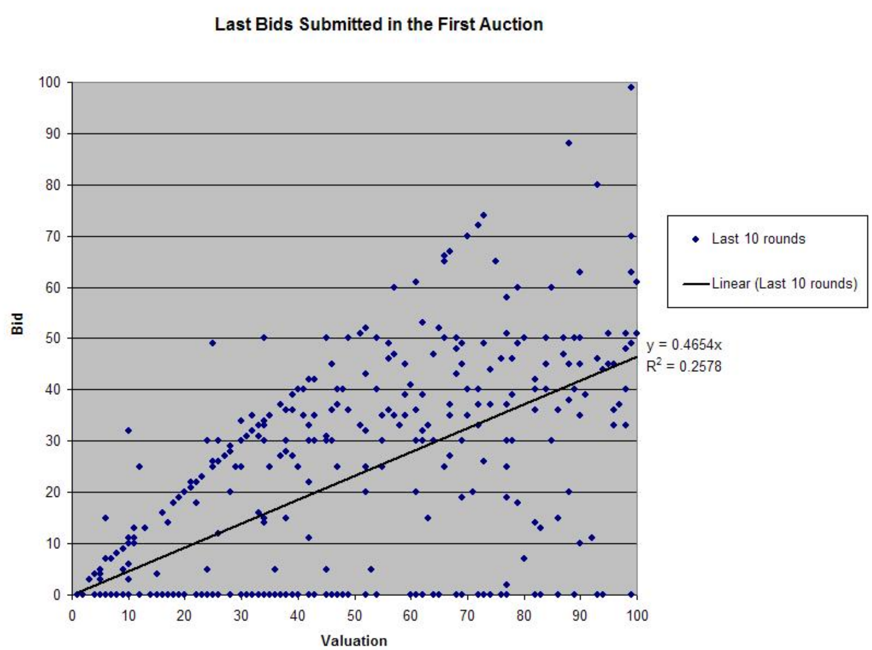

Figure 9: Bidding Strategy of the First Auction (under the eBay format).

shown in Figure 10, resulting in a fitted bid linear bid function (no constant) of $b = 0.97v$, which is almost literally what theory predicts. Although this is much closer to the theoretical prediction than that of the eBay format, we offer a word of caution since we used the "last submitted bid" in the eBay format, which might not be registered if the standing price is above one's bid.[50]

Moreover, in the second auction, bidders again shade their bids and have a fitted linear bid function of $b = 0.65v$. However, as in the eBay format, in Figure 11, we see there are many bids that fall on the 45 degree line and the horizontal axis, meaning that many bidders still bid their valuation (45 degree line) or bid zero (horizontal axis), possibly knowing they are sure to win, or have little hope of winning the item.

Last but not the least, in the first auction, the fitted linear bid function is $b = 0.54v$, still consistent with what theory predicts. Compared to the eBay format, the "all or none" is not as evident here, though there is still evidence that many bidders still bid their valuation in the very first round.

In fact, as shown in Figure 13 if we break down the bidding strategies into histograms, based on subjects' bid/valuation ratio, we find peaks at 0 and 1 for all three auctions, as well as scattered observations close to the equilibrium strategies. The peaks are more evident under the eBay format than that of the sealed-bid second price format, maybe because lower valuation bidders may not have the chance to bid when standing bids are ascending quite fast.[51]

4 A Cognitive Hierarchy Model for eBay Auctions

Based on the experimental data in the previous section, we now construct a steps-of-thinking (cognitive hierarchy) model to explain the discrepancy between theory and experimental evidence.

To begin with, the fully rational Equilibrium (Eq) type bidders are those who bid according to

[50] To do: Try fitting the bid function with only bids submitted by "high valuation bidders" by excluding bidders with valuation below, say, 30.

[51] One way to test this is to see whether the bidders who bid zero have lower valuations than others. TBA.

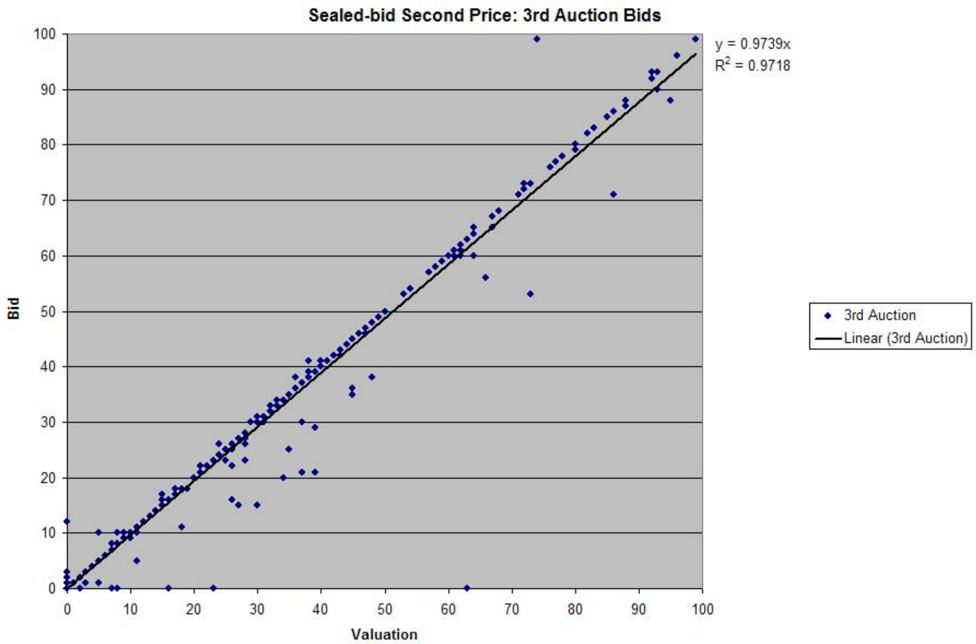

Figure 10: Bidding Strategy of the Third Auction (under the Sealed-bid Second Price format).

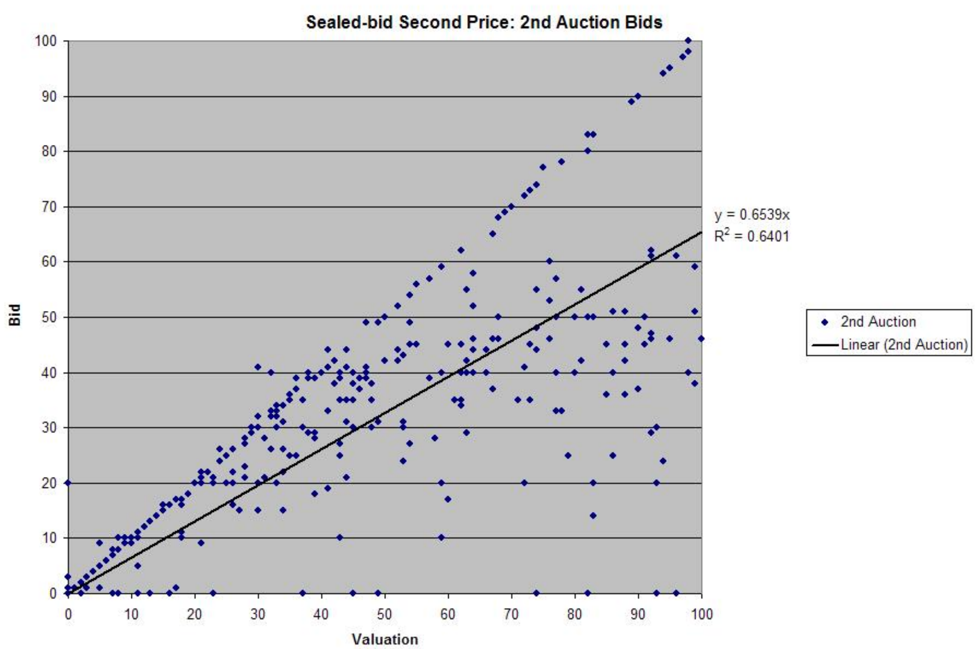

Figure 11: Bidding Strategy of the Second Auction (under the Sealed-bid Second Price format).

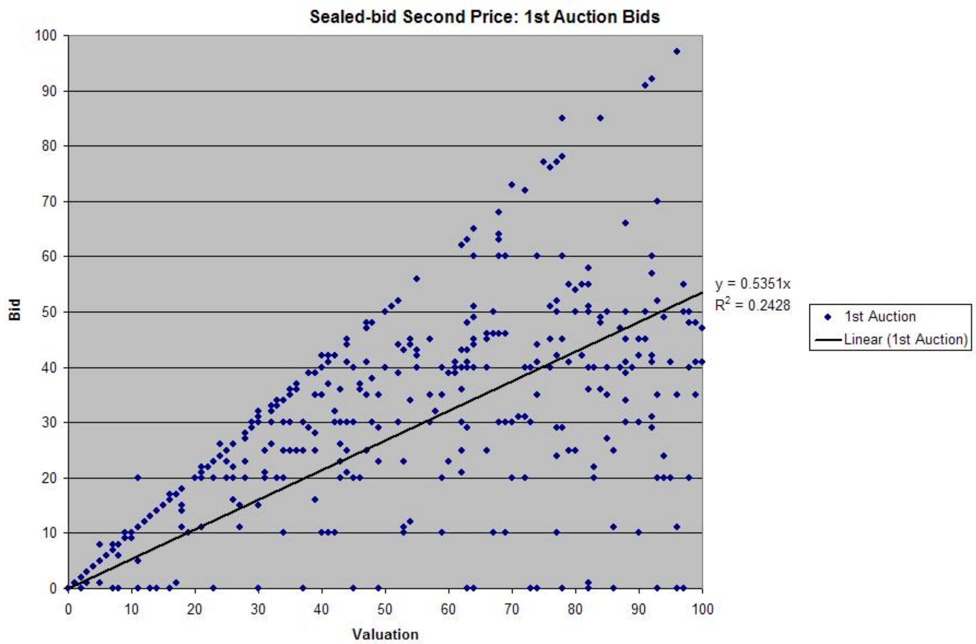

Figure 12: Bidding Strategy of the First Auction (under the Sealed-bid Second Price format).

the bid functions based on standard equilibrium theory, which are (from above sections),

$$\beta_1(v) = \frac{v}{2}, \beta_2(v) = \frac{2v}{3}, \text{ and } \beta_3(v) = v.$$

4.1 The Cognitive Hierarchy Model

Now we consider bidders with bounded rationality in the form of limited steps-of-thinking.

First of all, we coin the anchoring L0 type bidders to bid their valuation v_i in all auctions. This may due to limits in cognitive thinking, or simply acting as if each auction were the last one (for them) due to unavailability or absence in the future. Hence, for L0 type bidders, they have bid functions

$$\beta_1(v) = v, \beta_2(v) = v, \text{ and } \beta_3(v) = v.$$

Secondly, we have the L1 type bidders who are perfectly rational, but perceive that all oppo-

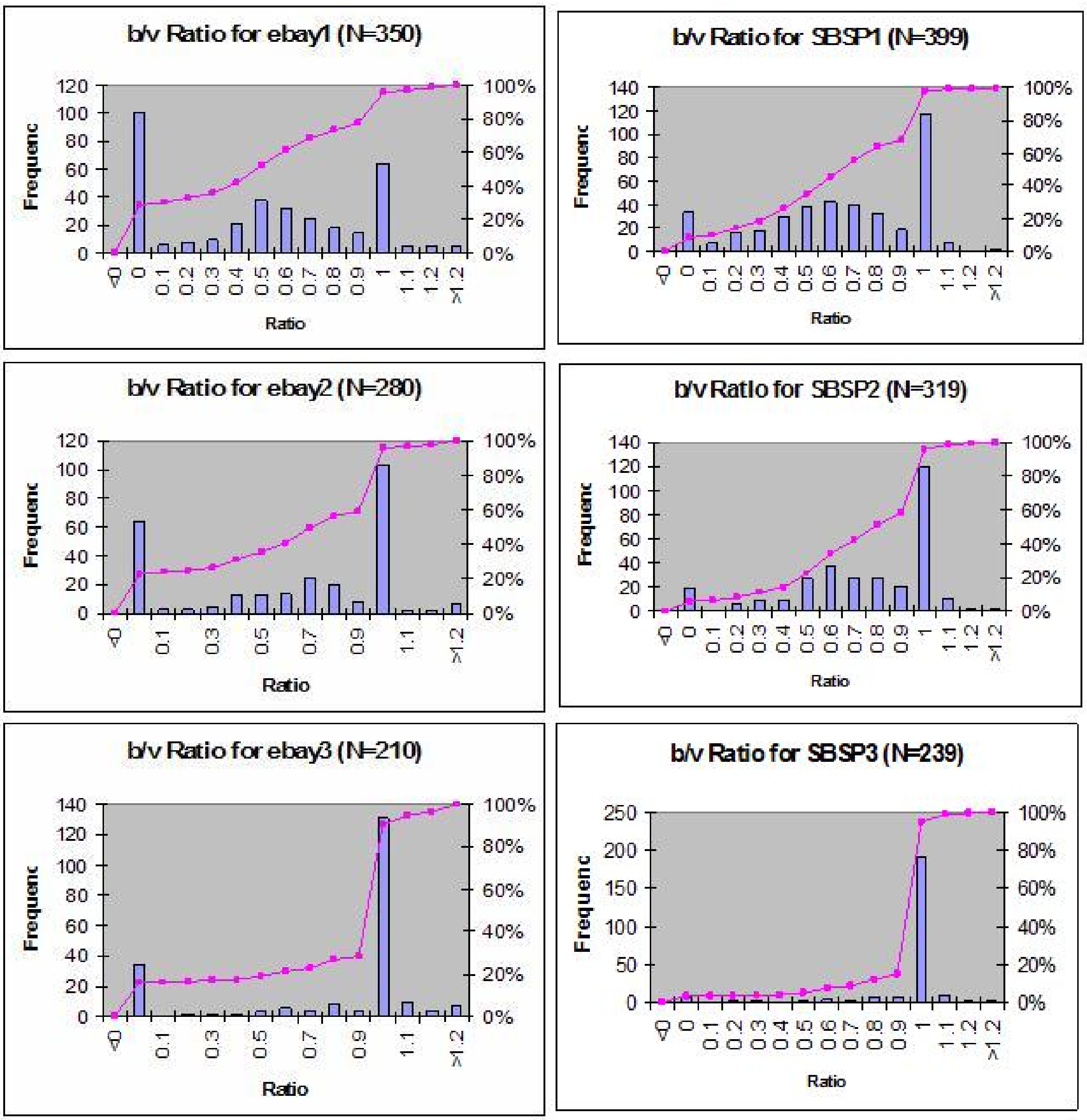

Figure 13: Histograms of Bid/Value Ratio of eBay and Sealed-bid Second Price Auctions.

nents are L0 types. The L1 type bidders would "dodge" L0 type bidders in the first two auctions, knowing that opponents that have the highest and second-highest valuations would win them, and he would face the third-highest valuation opponent in the last auction. This is because it is a dominated strategy to bid in the first two auctions (and pay $b_1 = y_{(1)}$ or $b_2 = y_{(2)}$), knowing you would face opponent bid $b_3 = y_{(3)}$ in the last auction. Hence, the bid functions for L1 type bidders are

$$\beta_1(v) = 0, \beta_2(v) = 0, \text{ and } \beta_3(v) = v.$$

Thirdly, we need to define the L2 type bidders who perceive opponents as L0 and/or L1 type bidders. Though there are several ways to model this, for simplicity we here assume L2 type bidders perceive that there are n_0 L0 type bidders and n_1 L1 type bidders, in which $n_i = 1, 2,$ or 3, and $n_0 + n_1 = 4$ (the total number of opponents in our experiment). Based on these beliefs, we solve for the bid functions for L2 type bidders and obtain the following theorem:

Theorem 5. *In the case of three auctions (M=3) and five bidders (N=5) who have independent private values of uniform* $[0, 1]$*, we have the following types of bidders:*

1. *L0 type bidders: Assume every current auction is the last one (for themselves), and hence, bid according to the bid functions*

$$\beta_1^0(v) = v, \beta_2^0(v) = v, \textit{ and } \beta_3^0(v) = v.$$

2. *L1 type bidders: Assume all opponents are L0 type bidders, and hence, bid according to the bid functions*

$$\beta_1^1(v) = 0, \beta_2^1(v) = 0, \textit{ and } \beta_3^1(v) = v.$$

3. *L2 type bidders: Assume there are* n_0 *L0 type bidders and* n_1 *L1 type bidders, and hence,*

bid according to the bid functions

$$\beta_1^2(v) = 0, \beta_2^2(v) = v - \frac{v^3}{3}, \textit{ and } \beta_3^2(v) = v.$$

if the type distribution belief is $(n_0, n_1) = (2, 2)$*, and*

$$\beta_1^2(v) = 0, \beta_2^2(v) = \sqrt{3(v+1)(3-v)} - 3, \textit{ and } \beta_3^2(v) = v.$$

if the type distribution belief is $(n_0, n_1) = (3, 1)$.[52]

Proof. See Appendix. □

Potentially, we could define even higher level bidders who have even more sophisticated types of thinking. However, since most experimental studies indicate actual experimental subjects rarely exhibit cognitive levels beyond L2, we refrain from the trouble of solving for the bid function of L3 or higher, even if it were feasible.

4.2 Predicting Experimental Data

Based on the cognitive hierarchy model, we predict to see various types of bidders with different steps-of-thinking, or cognitive levels, acting according to each of their bid functions. We now use the experimental data and evaluate how well it predicts subject behavior. Here, we assume perfect classification and use the closest cognitive hierarchy type's bid (over all possible types) as the prediction.

To measure fitness, we use two different measures. One is the square of the sample correlation between the predicted bids (for each model) and the actual bids,or $r^2 = [corr(y_i - (\hat{y}_i)]^2$ where $\hat{y}_i$ are the predicted bids, and y_i are the actual bids. This is equivalent to the unadjusted R^2 under

[52] For $(n_0, n_1) = (3, 1)$, we have the uninteresting case where no opponent would bid in auction 2. In this case, the L2 type bidder can bid anything and win the second auction at the reservation price ($= 0$).

ordinary least square regressions that include a constant. The other is the root mean standard error (RMSE), or $\sqrt{SS/(n-p)}$ where SS is the sum of squares, $n-p$ is the degree of freedom.[53]

In the third auction, all bidder types bid their valuation, which is exactly what Figure 7 and Figure 10 show. In fact, for the eBay auction, the r^2 is 0.87, almost identical to 0.87 with the linear model (fitted linear bid function above). Root mean square errors are also close (9.89 for the behavioral model and 9.45 for the linear model). Similarly, for the sealed-bid second price auction, the r^2 is 0.97, also nearly identical to 0.97 for the linear model, and the root mean square errors are close to each other (4.70 for the behavioral model and 4.60 for the linear model).

In the second auction, the bid function of L0 and L1 correspond to the 45 degree line and the horizontal axis, which is already evident from Figure 8 and Figure 11. Furthermore, the Equilibrium type bidders have a bid function close to that of the fitted linear regression line ($b = 0.67v$ instead of $b = 0.63v$ or $b = 0.65v$), which are also in the two figures. However, the bid function of L2 type bidders are not clearly illustrated in previous graphs. Therefore, Figure 14 and Figure 15 plot the L2 bid functions (for both $(n_0, n_1) = (3, 1)$ and $(2, 2)$) along with the experimental data. Here, in both figures, we see that the L2 bid function with the belief of more L0 type opponents ($(n_0, n_1) = (3, 1)$) fit the data better than both the other L2 bid function with less L0 type opponents and that of the Equilibrium type bid function. Hence, we assume $(n_0, n_1) = (3, 1)$ is L2's belief in the cognitive hierarchy model.

Under this belief, among the 280 bids with valuation > 0 in the eBay auction, we classify bids to their closest types and obtain $(L0, L1, L2, Eq) = (121, 74, 53, 32)$. Using this classification to predict actual bids, the r^2 significantly improves to 0.96, as opposed to 0.53 for the linear model, while root mean square errors improve from 16.84 to 5.14. Similarly, for the sealed-bid second price auction, among the 319 bids with valuation > 0, the numbers of bids close to each type's

[53] Note that since n is very large (350 or 400), using n instead of $n-p$ makes little difference. Here, we use the same degree of freedom (assume $p = 2$) for both models, even though it is not clear how to define the number of parameters in the behavioral model.

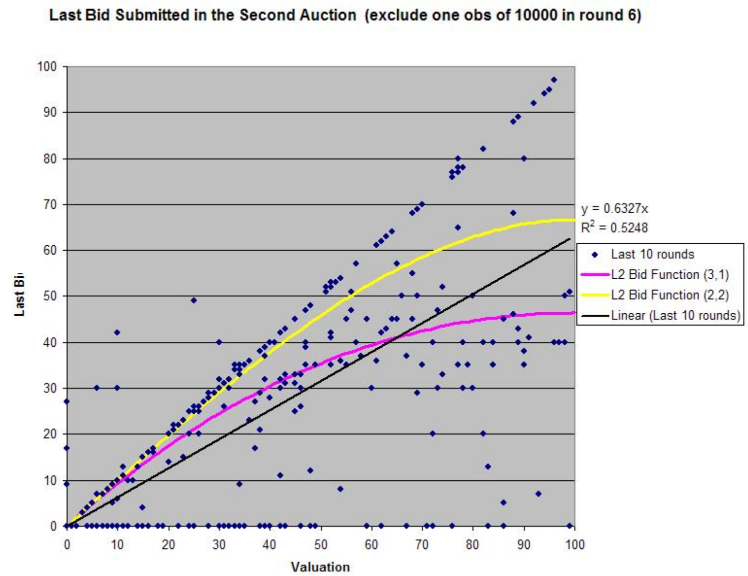

Figure 14: Bidding Strategy of the Second Auction (under the eBay format), with Cognitive Hierarchy prediction plotted.

bid are $(L0, L1, L2, Eq) = (148, 30, 87, 54)$.[54] Moreover, r^2 becomes 0.96, also a significant improvement from the 0.65 for the linear model, and the root mean square error improves from 13.66 to 4.50. Here, we do offer a word of caution that this is the best case scenario for the cognitive hierarchy model, allowing no mistake in type classification at the individual bid level.

Now consider the first auction. The L0 bid functions correspond to the 45 degree line, while the L1, L2 bid functions both correspond to the horizontal axis. The Equilibrium bid function is pretty much identical to the fitted linear regression line ($b = 0.50v$ instead of $b = 0.47v$ or $b = 0.53v$), which seems a closer fit than in the second auction. For the eBay auction, among the 350 bids with valuation > 0, the numbers of bids close to each type's bid are $(L0, L1$ or $L2, Eq) = (111, 115, 124)$. Using this classification, the r^2 drastically improves to 0.91 as opposed to 0.27 for the linear model, while the root mean square error improves from 18.37 to 6.57. On the other hand, for the sealed-bid second price auction, among the 399 bids with valuation > 0, the numbers

[54]Here, we have more people bidding their valuation (L0 types), possibly because this is more like a second price auction (except that there are multiple rounds).

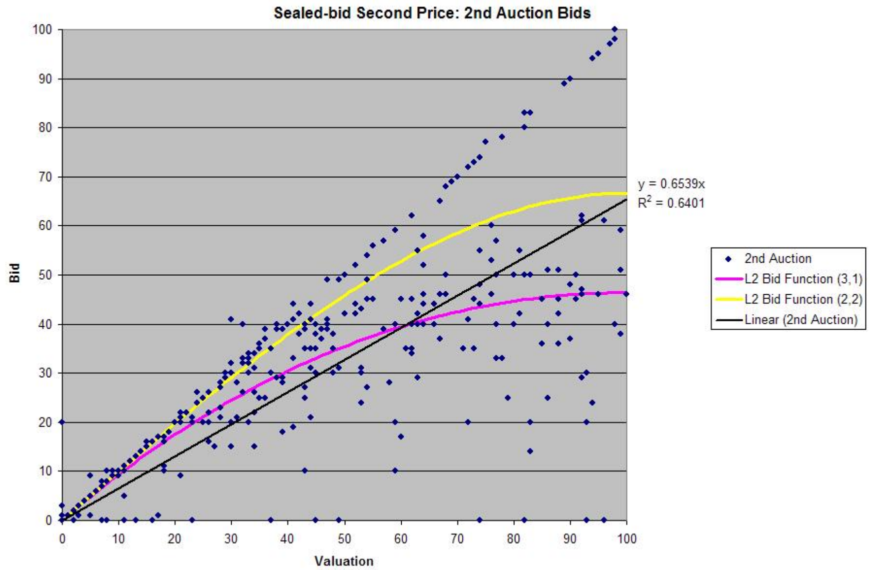

Figure 15: Bidding Strategy of the Second Auction (under the Sealed-bid Second Price format), with Cognitive Hierarchy prediction plotted.

of bids close to each type's bid are $(L0, L1$ or $L2, Eq) = (181, 56, 163)$. Hence, the r^2 is 0.87, also a significant improvement from the 0.31 for the linear model, while root mean square error improves from 16.64 to 7.26. Again, we offer a word of caution that this is the best case scenario for the cognitive hierarchy model.

Finally, we try to simulate empirical price paths and bid functions assuming that there are two L0, one L1, one L2, and one Equilibrium type.[55] We use actual valuations used in the experiment, and randomly assign two bidders to act as L0, one as L1, one as L2, and the last one as in Equilibrium. We then "play out" the strategies and calculate the winning prices.

As shown in Figure 16 and 17, as well as Table 2, the cognitive hierarchy simulation captures the higher prices in the first auction, but still fails to capture the higher prices in the second auction in the eBay Auction. In particular, for the eBay auction, average prices shift from 40.29

[55] Such proportion is based on the fact that in both the first and second auction of the eBay auction, the number of bids close to L0 is approximately the number of bids close to L1 and L2 combined (111 vs. 115 in auction 1, and 121 vs. 127 in auction 2).

to 32.77, and then to 34.89, in which only that of the second auction is significantly different from experimental data. For the sealed-bid second price auction, the average price sequence is $(40.64, 34.63, 34.54)$, none of them significantly different from experimental data. This is an improvement from equilibrium theory which simulates average price sequences of $(32.81, 33.20, 32.89)$ for the eBay auction and $(33.34, 32.88, 33.59)$ for the sealed-bid second price auction, in which prices for the first and second auction under both formats are significantly lower than that of experimental data. Moreover, even though the cognitive hierarchy model does not fully capture the higher prices in the second auction, such fluctuations in prices is actually in line with the conflicting empirical evidence on ascending and declining prices which furnished the controversy on the declining price anomaly.

In addition to prices, we also consider the efficiency level of the cognitive hierarchy model. Simulated 100 times, we obtain an efficiency level of $92.55\%(1.47\%)$ (standard deviation in parenthesis) for the eBay auction and $91.87\%(1.40\%)$ for the sealed-bid second price auction, both in line with the actual efficiency levels in the experimental data (91.43% for the eBay auction and 94.00% for the sealed-bid second price auction).[56]

Table 2: The Price Sequence: Average Prices for Each Auction

Auction Number	Auction 1		Auction 2		Auction 3	
and Format	eBay	SBSP	eBay	SBSP	eBay	SBSP
CH Simulation	40.29	40.64	32.77*	34.63	34.89	34.54
	(17.60)	(18.91)	(11.36)	(11.73)	(18.12)	(17.82)
Equilibrium Theory	32.81*	33.34***	33.20*	32.88*	32.89	33.59
	(8.61)	(9.02)	(10.60)	(10.94)	(16.38)	(16.96)
Experimental Data	36.53	38.96	37.69	36.62	32.13	32.99
	(10.25)	(11.97)	(14.65)	(12.86)	(19.17)	(18.07)

Note: *,**, and *** in the first two rows (predictions) indicate 5%, 1% and 0.1% significantly away from the third rows (actual data) using an unpaired two-side t-test. (We do not use a paired test since simulated types can be drastically difference from actual types.)

[56]However, the average simulated efficiency level is significantly away from the actual efficiency level.

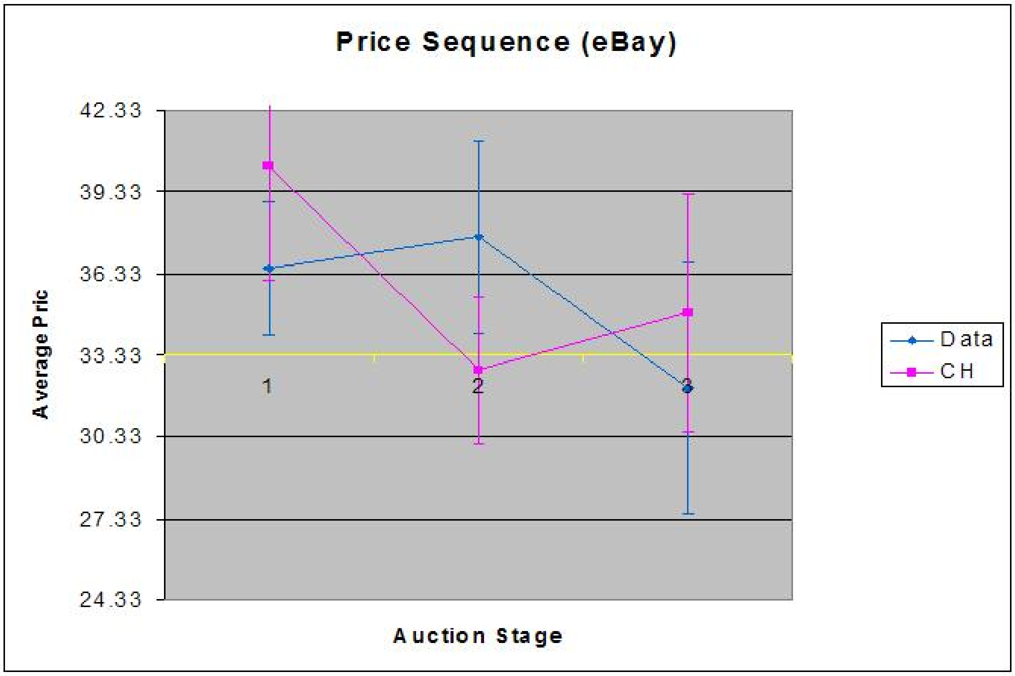

Figure 16: The Winning Price Sequence Predicted by the Cognitive Hierarchy Model (eBay Auction).

5 Conclusion

The similarity between the eBay auction rules and the sealed-bid second price auction has been widely perceived. In this paper, we formally investigate this similarity in a sequential auction setting. Specifically, we consider a sequential (sealed-bid) second price auction in which the highest and second-highest bids are announced after each stage with unit demand bidders of affiliated private values. We find an efficient, symmetric sequential (perfect Bayesian) equilibrium in monotonic strategies where in each stage, bidders bid the expected winning price of the next stage, conditional on being the tied winner in this stage. Interestingly, equilibrium bids do not depend on the previous second-highest bid, which contains information about one's current opponents. Nonetheless, the outcome is efficient, and the expected price sequence ascends, contrary to what many empirical evidence show. Moreover, such an equilibrium also survives in an eBay auction setting, in which bidders are allowed to submit multiple proxy (second-price) bids in a given fixed

Figure 17: The Winning Price Sequence Predicted by the Cognitive Hierarchy Model (Sealed-bid Second Price Auction).

time period, when all bidders bid only at the last minute, which is usually referred to as sniping.

We then conduct controlled laboratory experiments with the special case of three items sold one after the other to bidders with independent private values, using either a sealed-bid second price auction or an eBay auction. We show the timing of bidding replicates the “last minute bidding(sniping)” phenomenon reported by Roth and Ockenfels (2002), as well as the early “squatting” behavior coined by Ely and Hossain (2006). Moreover, both auction formats achieve more than 90% efficiency and have estimated bid functions close to the theoretical predictions, showing little difference between the two.

However, the average prices are higher for the first two items sold and decline to theoretical prediction in the last auction. Hence, we have declining prices, but higher revenue (than theory predicts). Investigating such discrepancy leads us to find a significant portion of subjects either bid their valuations or simply bid zero in the first two auctions, even though all bidders almost always bid their valuations in the last auction. Hence, besides the fully rational Equilibrium type bidders (Eq) who follow the standard theory prediction we initially derived, we consider heterogeneous bidders who have different levels of reasoning, and construct a steps-of-thinking (cognitive hierarchy) model to explain the experimental results. In particular, based on the anchoring of L0 type bidders who always bid their valuation, we define L1 type bidders who best respond to L0 types and "dodge" the first two auction. We then solve for the L2 type bidders who hold a belief of the proportion of L0 and L1 type opponents. The combination of these types of bidders explain the data better than that of equilibrium theory and shed light on the "declining price anomaly" documented by Ashenfelter (1989) and previously attributed to (extreme) risk aversion by McAfee and Vincent (1993).

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Appendix: Proofs

Proof of Lemma 3. The first order condition requires, given $x_3^* = v$, at $x_2 = v$,

$$\frac{\partial u}{\partial x_2} = \frac{\partial W_2}{\partial x_2}(v, x_1, x_2) + \frac{\partial W_3}{\partial x_2}(v, x_1, x_2, x_3^*) = 0$$

Since

$$W_2 = \int_{x_1}^{\overline{v}}\int_{\underline{v}}^{\min\{x_2,y_1\}} \Big[v - \beta_2(y_2; y_1, x_1)\Big] f_{12}(y_1, y_2|v)dy_2dy_1$$

$$W_3 = \int_{x_1}^{\overline{v}}\int_{x_2}^{\max\{x_2,y_1\}}\int_{\underline{v}}^{\min\{x_3,y_2\}} \big(v - \beta_3(y_3; y_1, y_2, x_2)\big) f_{123}(y_1, y_2, y_3|v)dy_3dy_2dy_1$$

we have

$$
\begin{aligned}
\frac{\partial W_2}{\partial x_2} &= \int_{\max\{x_1,x_2\}}^{\overline{v}} \Big[v - \beta_2(x_2; y_1, x_1)\Big] \cdot f_{12}(y_1, x_2|v) dy_1 \\
\frac{\partial W_3}{\partial x_2} &= -\int_{\max\{x_1,x_2\}}^{\overline{v}} \int_{\underline{v}}^{\min\{x_3,x_2\}} \big[v - \beta_3(y_3; y_1, x_2, x_2)\big] f_{123}(y_1, x_2, y_3|v) dy_3 dy_1 \\
&\quad - \int_{x_1}^{\overline{v}} \int_{x_2}^{\max\{x_2,y_1\}} \int_{\underline{v}}^{\min\{x_3,y_2\}} \frac{\partial \beta_3}{\partial z_3}(y_3; y_1, y_2, x_2) f_{123}(y_1, y_2, y_3|v) dy_3 dy_2 dy_1
\end{aligned}
$$

Note that $\frac{\partial \beta_3}{\partial z_3} = 0$ since $\beta_3(v; z_1, z_2, z_3) = v$. Hence, when $x_3^* = v$, the first order condition requires, at $x_2 = v$,

$$
\begin{aligned}
\frac{\partial u}{\partial x_2} &= \int_{\max\{x_1,x_2\}}^{\overline{v}} \Bigg\{ \Big[v - \beta_2(x_2; y_1, x_1)\Big] f_{12}(y_1, x_2|v) \\
&\qquad - \int_{\underline{v}}^{\min\{v,x_2\}} \big[v - \beta_3(y_3; y_1, x_2, x_2)\big] f_{123}(y_1, x_2, y_3|v) dy_3 \Bigg\} dy_1 \\
&= \int_{\max\{x_1,v\}}^{\overline{v}} \Bigg\{ \Big[v - \beta_2(v; y_1, x_1)\Big] f_{12}(y_1, v|v) \\
&\qquad - \int_{\underline{v}}^{v} \big[v - \beta_3(y_3; y_1, v, v)\big] f_{123}(y_1, v, y_3|v) dy_3 \Bigg\} dy_1 \\
&= 0
\end{aligned}
$$

and is satisfied if

$$
\begin{aligned}
\beta_2(v; z_1) &= \frac{1}{f_{12}(z_1, v|v)} \int_{\underline{v}}^{v} \underbrace{\beta_3(y_3; z_1, v, v)}_{y_3} f_{123}(z_1, v, y_3|v) dy_3 \\
&= E[\beta_3(Y_3; z_1, v, v)|Y_1 = z_1, Y_2 = v, V_i = v] \\
&= E[\beta_3(V_{(4)}; z_1, v, v)|V_{(1)} = z_1, V_{(2)} = v, V_{(3)} = v]
\end{aligned}
$$

To show that $x_2^* = v$ is indeed optimal for these β_2, we need to show that $\frac{\partial u}{\partial x_2} \geq 0$ for all $x_2 < v$, and $\frac{\partial u}{\partial x_2} \leq 0$ for all $x_2 > v$.

For $x_2 < v$,

$$\begin{aligned}\frac{\partial u}{\partial x_2} = \int_{\max\{x_1,x_2\}}^{\bar{v}} \Bigg\{ & v \cdot \left[f_{12}(y_1, x_2|v) - \int_{\underline{v}}^{x_2} f_{123}(y_1, x_2, y_3|v) dy_3 \right] \\ & - \beta_2(x_2; y_1, x_1) f_{12}(y_1, x_2|v) + \int_{\underline{v}}^{x_2} \beta_3(y_3; y_1, x_2, x_2) f_{123}(y_1, x_2, y_3|v) dy_3 \Bigg\} dy_1 \\ & \geq 0\end{aligned}$$

since by affiliation,[57] for $x_2 < v$,

$$\begin{aligned}\beta_2(x_2; y_1) &= E[\beta_3(Y_3; y_1, x_2, x_2)|Y_1 = y_1, Y_2 = x_2, V_i = x_2] \\ &\leq E[\beta_3(Y_3; y_1, x_2, x_2)|Y_1 = y_1, Y_2 = x_2, V_i = v] \\ &= \frac{1}{f_{12}(y_1, x_2|v)} \int_{\underline{v}}^{x_2} y_3 f_{123}(y_1, x_2, y_3|v) dy_3\end{aligned}$$

[57] Milgrom and Weber (1982) provide several important properties of affiliation. The one used here and through out is

Theorem 5. *Let $Z_1, \cdot, Z_k$ be affiliated and let H be any nondecreasing function. Then the function h defined by*

$$h(a_1, b_1; \cdots; a_k, b_k) = E\Big[H(Z_1, \cdots, Z_k)\Big| a_1 \leq Z_1 \leq b_1, \cdots, a_k \leq Z_k \leq b_k\Big]$$

is nondecreasing in all of its arguments.

For $x_2 > v$, we have $v \leq y_3 = \beta_3(y_3; y_1, x_2, x_2)$ for $y_3 \in [v, x_2]$. Hence,

$$
\begin{aligned}
\frac{\partial u}{\partial x_2} = &\int_{\max\{x_1,x_2\}}^{\overline{v}} \Bigg\{ v \cdot \left[f_{12}(y_1, x_2|v) - \int_{\underline{v}}^{v} f_{123}(y_1, x_2, y_3|v) dy_3 \right] \\
&\quad - \beta_2(x_2; y_1, x_1) f_{12}(y_1, x_2|v) + \int_{\underline{v}}^{v} \beta_3(y_3; y_1, x_2, x_2) f_{123}(y_1, x_2, y_3|v) dy_3 \Bigg\} dy_1 \\
= &\int_{\max\{x_1,x_2\}}^{\overline{v}} \Bigg\{ \int_{v}^{x_2} v f_{123}(y_1, x_2, y_3|v) dy_3 - \beta_2(x_2; y_1, x_1) f_{12}(y_1, x_2|v) \\
&\quad + \int_{\underline{v}}^{v} \beta_3(y_3; y_1, x_2, x_2) f_{123}(y_1, x_2, y_3|v) dy_3 \Bigg\} dy_1 \\
\leq &\int_{\max\{x_1,x_2\}}^{\overline{v}} \Bigg\{ - \beta_2(x_2; y_1, x_1) f_{12}(y_1, x_2|v) \\
&\quad + \left(\int_{v}^{x_2} + \int_{\underline{v}}^{v} \right) \beta_3(y_3; y_1, x_2, x_2) f_{123}(y_1, x_2, y_3|v) dy_3 \Bigg\} dy_1 \\
\leq &\, 0
\end{aligned}
$$

since by affiliation, for $x_2 > v$,

$$
\begin{aligned}
\beta_2(x_2; y_1, x_1) &= E[\beta_3(Y_3; y_1, x_2, x_2)|Y_1 = y_1, Y_2 = x_2, V_i = x_2] \\
&\geq E[\beta_3(Y_3; y_1, x_2, x_2)|Y_1 = y_1, Y_2 = x_2, V_i = v] \\
&= \frac{1}{f_{12}(y_1, x_2|v)} \int_{\underline{v}}^{x_2} \beta_3(y_3; y_1, x_2, x_2) f_{123}(y_1, x_2, y_3|v) dy_3
\end{aligned}
$$

Thus, we have shown that

$$
\begin{aligned}
\frac{\partial u}{\partial x_2} &\geq 0 \text{ if } x_2 < v \\
\frac{\partial u}{\partial x_2} &\leq 0 \text{ if } x_2 > v
\end{aligned}
$$

and $x_2^* = v$ is indeed a global maximum. In other words, it is optimal to choose $x_2 = v$ knowing one would choose $x_3^* = v$ in stage 3. This completes the proof of the positive part of the lemma.

To show that in general there is no $\beta_2(v)$ that is independent of both z_1, and z_2, recall the first order condition requires, if $\beta_2(v; z_1, z_2) = \beta_2(v)$, at $x_2 = v$,

$$\begin{aligned}
\frac{\partial u}{\partial x_2} &= \int_{\max\{x_1,v\}}^{\overline{v}} \bigg\{ \Big[v - \beta_2(v; y_1, x_1)\Big] f_{12}(y_1, v|v) \\
&\qquad\qquad - \int_{\underline{v}}^{v} \big[v - \beta_3(y_3; y_1, v, v)\big] f_{123}(y_1, v, y_3|v) dy_3 \bigg\} dy_1 \\
&= \int_{\max\{x_1,v\}}^{\overline{v}} \bigg\{ - \beta_2(v) f_{12}(y_1, v|v) + \int_{\underline{v}}^{v} \beta_3(y_3; y_1, v, v) f_{123}(y_1, v, y_3|v) dy_3 \bigg\} dy_1 \\
&= 0
\end{aligned}$$

Hence, we need

$$\begin{aligned}
\beta_2(v) &= \frac{1}{\int_{\max\{x_1,v\}}^{\overline{v}} f_{12}(y_1, v|v) dy_1} \int_{\max\{x_1,v\}}^{\overline{v}} \int_{\underline{v}}^{v} \beta_3(y_3; y_1, v, v) f_{123}(y_1, v, y_3|v) dy_3 dy_1 \\
&= E\Big[\beta_3(Y_3; Y_1, v, v) \Big| Y_1 > \max\{x_1, v\}, Y_2 = v, V_i = v\Big] \\
&= E\Big[Y_3 \Big| Y_1 > \max\{x_1, v\}, Y_2 = v, V_i = v\Big] \\
&= E[V_{(4)} | V_{(1)} > \max\{x_1, v\}, V_{(2)} = v, V_{(3)} = v]
\end{aligned}$$

However, this $\beta_2(v)$ is well-defined only if

$$E[V_{(4)} | V_{(1)} > \max\{x_1, v\}, V_{(2)} = v, V_{(3)} = v]$$

is the same for all $x_1 > v$. A typical example is when bidder valuations are independent private values(IPV), in which case $\beta_2(v) = E[V_{(4)} | V_{(2)} = v, V_{(3)} = v]$. □

Proof of Lemma 4. Since

$$\begin{aligned}
W_1 &= \int_{\underline{v}}^{x_1} \Big[v - \beta_1(y_1)\Big] f_1(y_1|v) dy_1 \\
W_2 &= \int_{x_1}^{\overline{v}} \int_{\underline{v}}^{\min\{x_2,y_1\}} \Big[v - \beta_2(y_2; y_1, x_1)\Big] f_{12}(y_1, y_2|v) dy_2 dy_1 \\
W_3 &= \int_{x_1}^{\overline{v}} \int_{x_2}^{\max\{x_2,y_1\}} \int_{\underline{v}}^{\min\{x_3,y_2\}} \big[v - \beta_3(y_3; y_1, y_2, x_2)\big] f_{123}(y_1, y_2, y_3|v) dy_3 dy_2 dy_1
\end{aligned}$$

we have

$$
\begin{aligned}
\frac{\partial W_1}{\partial x_1} &= \Big[v - \beta_1(x_1)\Big] f_1(x_1|v) \\
\frac{\partial W_2}{\partial x_1} &= -\int_{\underline{v}}^{\min\{x_2,x_1\}} \Big[v - \beta_2(y_2; x_1, x_1)\Big] f_{12}(x_1, y_2|v) dy_2 \\
&\quad - \int_{x_1}^{\overline{v}} \int_{\underline{v}}^{\min\{x_2,y_1\}} \frac{\partial \beta_2}{\partial z_2}(y_2; y_1, x_1) f_{12}(y_1, y_2|v) dy_2 dy_1 \\
\frac{\partial W_3}{\partial x_1} &= -\int_{x_2}^{\max\{x_2,x_1\}} \int_{\underline{v}}^{\min\{x_3,y_2\}} \big[v - \beta_3(y_3; x_1, y_2, x_2)\big] f_{123}(x_1, y_2, y_3|v) dy_3 dy_2
\end{aligned}
$$

and hence, given $x_2^* = x_3^* = v$, the first order condition of x_1 is

$$
\begin{aligned}
\frac{\partial u}{\partial x_1} = v \cdot \Bigg[& \underbrace{\int_{\underline{v}}^{x_1} f_{12}(x_1, y_2|v) dy_2}_{f_1(x_1|v)} - \int_{\underline{v}}^{\min\{v,x_1\}} f_{12}(x_1, y_2|v) dy_2 \\
& - \int_{v}^{\max\{v,x_1\}} \int_{\underline{v}}^{\min\{v,y_2\}} f_{123}(x_1, y_2, y_3|v) dy_3 dy_2 \Bigg] \\
& - \beta_1(x_1) f_1(x_1|v) + \int_{\underline{v}}^{\min\{v,x_1\}} \beta_2(y_2; x_1, x_1) f_{12}(x_1, y_2|v) dy_2 \\
& + \int_{v}^{\max\{v,x_1\}} \int_{\underline{v}}^{\min\{v,y_2\}} \beta_3(y_3; x_1, y_2, x_2) f_{123}(x_1, y_2, y_3|v) dy_3 dy_2 \\
& - \int_{x_1}^{\overline{v}} \int_{\underline{v}}^{\min\{v,y_1\}} \frac{\partial \beta_2}{\partial z_2}(y_2; y_1, x_1) f_{12}(y_1, y_2|v) dy_2 dy_1 \\
= & -\beta_1(v) f_1(v|v) + \int_{\underline{v}}^{v} \beta_2(y_2; v, v) f_{12}(v, y_2|v) dy_2 \\
& - \int_{v}^{\overline{v}} \int_{\underline{v}}^{v} \frac{\partial \beta_2}{\partial z_2}(y_2; y_1, v) f_{12}(y_1, y_2|v) dy_2 dy_1 \\
= & \ 0
\end{aligned}
$$

at $x_1^* = v$, if

$$
\begin{aligned}
\beta_1(v) = \frac{1}{f_1(v|v)} \Bigg[& \int_{\underline{v}}^{v} \beta_2(y_2; v, v) f_{12}(v, y_2|v) dy_2 \\
& - \int_{v}^{\overline{v}} \int_{\underline{v}}^{v} \frac{\partial \beta_2}{\partial z_2}(y_2; y_1, v) f_{12}(y_1, y_2|v) dy_2 dy_1 \Bigg]
\end{aligned}
$$

is a (strictly) increasing function.[58]

In particular, consider

$$\beta_2(v; z_1, z_2) = \beta_2(v; z_1) = E\big[Y_3\big|Y_1 = z_1, Y_2 = v, V_i = v\big]$$

where $\frac{\partial \beta_2}{\partial z_2} = 0$, then

$$\beta_1(v) = \frac{1}{f_1(v|v)}\left[\int_{\underline{v}}^{v} \beta_2(y_2; v, v) f_{12}(v, y_2|v) dy_2\right] = E\Big[\beta_2(Y_2; v, v)\Big|Y_1 = v, V_i = v\Big]$$

and is indeed increasing by affiliation.

Moreover, if $x_1 < v$, we have

$$\frac{\partial u}{\partial x_1} = -\beta_1(x_1) f_1(x_1|v) + \int_{\underline{v}}^{x_1} \beta_2(y_2; x_1, x_1) f_{12}(x_1, y_2|v) dy_2 \geq 0$$

since by affiliation, for $x_1 < v$,

$$\begin{aligned}
\beta_1(x_1) &= E\Big[\beta_2(Y_2; x_1, x_1)\Big|Y_1 = x_1, V_i = x_1\Big] \\
&\leq E\Big[\beta_2(Y_2; x_1, x_1)\Big|Y_1 = x_1, V_i = v\Big] \\
&= \frac{1}{f_1(x_1|v)}\left[\int_{\underline{v}}^{x_1} \beta_2(y_2; x_1, x_1) f_{12}(x_1, y_2|v) dy_2\right]
\end{aligned}$$

[58]Note that if $\frac{\partial \beta_2}{\partial z_2} \neq 0$, then β_1 may not be increasing, and it is not clear whether $\frac{\partial u}{\partial x_1} \geq (\leq) 0$ for all $x_1 < (>) v$. This is why other possible $\beta_2(v; z_2)$ in lemma 3 might be difficult to generalize.

If $x_1 > v$,

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= v \cdot \Bigg[\underbrace{f_1(x_1|v) - \int_{\underline{v}}^{v} f_{12}(x_1, y_2|v)dy_2 - \int_{v}^{x_1}\int_{\underline{v}}^{v} f_{123}(x_1, y_2, y_3|v)dy_3dy_2}_{\int_v^{x_1}\int_v^{y_2} f_{123}(x_1,y_2,y_3|v)dy_3dy_2}\Bigg] \\ &\quad - \beta_1(x_1)f_1(x_1|v) + \int_{\underline{v}}^{v} \beta_2(y_2; x_1, x_1)f_{12}(x_1, y_2|v)dy_2 \\ &\quad + \int_{v}^{x_1}\int_{\underline{v}}^{v} \beta_3(y_3; x_1, y_2, x_2)f_{123}(x_1, y_2, y_3|v)dy_3dy_2 \\ &\le -\beta_1(x_1)f_1(x_1|v) + \int_{\underline{v}}^{v} \beta_2(y_2; x_1, x_1)f_{12}(x_1, y_2|v)dy_2 \\ &\quad + \int_{v}^{x_1}\left(\int_{\underline{v}}^{v} + \int_{v}^{y_2}\right) \underbrace{\beta_3(y_3; x_1, y_2, x_2)}_{y_3} f_{123}(x_1, y_2, y_3|v)dy_3dy_2 \\ &\quad \text{(since } v \le y_3 = \beta_3(y_3; x_1, y_2, x_2) \text{ for } y_3 \in [v, y_2]\text{)}\end{aligned}$$

Moreover, since by affiliation, for $y_2 \ge v$,

$$\begin{aligned}\beta_2(y_2; x_1, x_1) &= E\Big[\beta_3(Y_3; x_1, y_2, y_2)\Big| Y_1 = x_1, Y_2 = y_2, V_i = y_2\Big] \\ &\ge E\Big[\beta_3(Y_3; x_1, y_2, y_2)\Big| Y_1 = x_1, Y_2 = y_2, V_i = v\Big] \\ &= \frac{1}{f_{12}(x_1, y_2|v)} \int_{\underline{v}}^{y_2} \beta_3(y_3; x_1, y_2, x_2)f_{123}(x_1, y_2, y_3|v)dy_3\end{aligned}$$

We have, for all $x_1 > v$,

$$\begin{aligned}\frac{\partial u}{\partial x_1} &\le -\beta_1(x_1)f_1(x_1|v) + \int_{\underline{v}}^{v} \beta_2(y_2; x_1, x_1)f_{12}(x_1, y_2|v)dy_2 \\ &\quad + \int_{v}^{x_1}\left(\int_{\underline{v}}^{y_2} \beta_3(y_3; x_1, y_2, x_2)f_{123}(x_1, y_2, y_3|v)dy_3\right) dy_2 \\ &\le -\beta_1(x_1)f_1(x_1|v) + \left(\int_{\underline{v}}^{v} + \int_{v}^{x_1}\right) \beta_2(y_2; x_1, x_1)f_{12}(x_1, y_2|v)dy_2 \\ &\le 0\end{aligned}$$

since by affiliation, for $x_1 > v$,

$$\begin{aligned}\beta_1(x_1) &= E\Big[\beta_2(Y_2; x_1, x_1)\Big| Y_1 = x_1, V_i = x_1\Big] \\ &\geq E\Big[\beta_2(Y_2; x_1, x_1)\Big| Y_1 = x_1, V_i = v\Big] \\ &= \frac{1}{f_1(x_1|v)}\left[\int_{\underline{v}}^{x_1} \beta_2(y_2; x_1, x_1) f_{12}(x_1, y_2|v) dy_2\right]\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}\frac{\partial u}{\partial x_1} &\geq 0 \text{ if } x_1 < v \\ \frac{\partial u}{\partial x_1} &\leq 0 \text{ if } x_1 > v,\end{aligned}$$

and hence, $x_1^* = v$ is indeed a global maximum. In other words, given this β_1, it is optimal to choose $x_1 = v$ knowing one would choose $x_2^* = x_3^* = v$ in future stages. □

Proof of Lemma 5. Let W_i denote the expected utility of winning stage i, then the bidder's utility is

$$\begin{aligned}&u(x_1, \cdots, x_k, x_{k+1}^*, x_{k+2}^*, \cdots, x_M^*|v) = \\ &\qquad W_1(x_1|v) + W_2(x_1, x_2|v) + \cdots + W_k(x_1, \cdots, x_k|v) \\ &\qquad\quad + W_{k+1}(x_1, \cdots, x_k, x_{k+1}^*|v) + W_{k+2}(x_1, \cdots, x_k, x_{k+1}^*, x_{k+2}^*) \\ &\qquad\qquad + \cdots + W_M(x_1, \cdots, x_k, x_{k+1}^*, x_{k+2}^*, \cdots, x_M^*|v)\end{aligned}$$

where

$$\begin{aligned}W_1(x_1|v) &= \int_{\underline{v}}^{x_1} \Big[v - \beta_1(y_1)\Big] f_1(y_1|v) dy_1 \\ W_2(x_1, x_2|v) &= \int_{x_1}^{\overline{v}}\int_{\underline{v}}^{\min\{x_2, y_1\}} \Big[v - \beta_2(y_2; y_1, x_1)\Big] f_{12}(y_1, y_2|v) dy_2 dy_1\end{aligned}$$

and

$$\begin{aligned}W_m(x_1, \cdots, x_m|v) = &\int_{x_1}^{\overline{v}}\int_{x_2}^{\max\{x_2, y_1\}} \cdots \int_{x_{m-1}}^{\max\{x_{m-1}, y_{m-2}\}} \int_{\underline{v}}^{\min\{x_m, y_{m-1}\}} \\ &\Big[v - \beta_m(y_m; y_1, \cdots, y_{m-1}, x_{m-1})\Big] f_{1\cdots m}(y_1, \cdots, y_m|v) dy_m \cdots dy_1\end{aligned}$$

for $m = 3, 4, \cdots, M$.[59] Note that W_1 through W_{k-1} all have derivative $\frac{\partial W_i}{\partial x_k} = 0$.

Hence, for the choice of x_k, the first order condition requires, at $x_k = v$,

$$\frac{\partial u}{\partial x_k} = \frac{\partial W_k}{\partial x_k} + \frac{\partial W_{k+1}}{\partial x_k} + \cdots + \frac{\partial W_M}{\partial x_k}$$

where[60]

$$\begin{aligned}
\frac{\partial W_k}{\partial x_k} = & \int_{x_1}^{\overline{v}} \int_{x_2}^{\max\{x_2, y_1\}} \cdots \int_{\max\{x_{k-1}, x_k\}}^{\max\{x_{k-1}, y_{k-2}\}} \Big\{ \Big[v - \beta_k(x_k; y_1, \cdots, y_{k-1}, x_{k-1}) \Big] \\
& \qquad \cdot f_{1\cdots k}(y_1, \cdots, y_{k-1}, x_k | v) \Big\} dy_{k-1} \cdots dy_1 \\
\frac{\partial W_{k+1}}{\partial x_k} = & - \int_{x_1}^{\overline{v}} \int_{x_2}^{\max\{x_2, y_1\}} \cdots \int_{\max\{x_{k-1}, x_k\}}^{\max\{x_{k-1}, y_{k-2}\}} \int_{\underline{v}}^{\min\{x_{k+1}, x_k\}} \\
& \Big[v - \beta_{k+1}(y_{k+1}; y_1, \cdots, y_{k-1}, x_k, x_k) \Big] \\
& \qquad \cdot f_{1\cdots(k+1)}(y_1, \cdots, y_{k-1}, x_k, y_{k+1} | v) dy_{k+1} dy_{k-1} \cdots dy_1 \\
& - \int_{x_1}^{\overline{v}} \cdots \int_{\underline{v}}^{\min\{x_{k+1}, x_k\}} \frac{\partial \beta_{k+1}}{\partial z_{k+1}} (y_{k+1}; y_1, \cdots, y_k, x_k) \\
& \qquad \cdot f_{1\cdots(k+1)}(y_1, \cdots, y_{k+1} | v) dy_{k+1} \cdots dy_1
\end{aligned}$$

and, for $m = (k+2), (k+3), \cdots, M$,

$$\begin{aligned}
\frac{\partial W_m}{\partial x_k} = & - \int_{x_1}^{\overline{v}} \cdots \int_{\max\{x_{k-1}, x_k\}}^{\max\{x_{k-1}, y_{k-2}\}} \Bigg(\int_{x_{k+1}}^{\max\{x_{k+1}, x_k\}} \int_{x_{k+2}}^{\max\{x_{k+2}, y_{k+1}\}} \cdots \int_{\underline{v}}^{\min\{x_m, y_{m-1}\}} \\
& \Big\{ \Big[v - \beta_m(y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, x_{m-1}) \Big] \\
& \quad \cdot f_{1\cdots m}(y_1, \cdots, x_k, \cdots, y_m | v) \Big\} dy_m \cdots dy_{k+1} \Bigg) dy_{k-1} \cdots dy_1
\end{aligned}$$

[59] This is because the integration area for W_m is $y_i \geq y_{i+1}$ and $y_i \geq x_i$ for $i = 1, \cdots, (m-1)$, and $y_m \leq x_m$.

[60] Since

$$\int_{x_k}^{\max\{x_k, y_{k-1}\}} \cdots dy_k = \begin{cases} \int_{x_k}^{x_k} \cdots dy_k & \text{if } x_k \geq y_{k-1} \\ \int_{x_k}^{y_{k-1}} \cdots dy_k & \text{if } x_k \leq y_{k-1}, \end{cases}$$

for $y_{k-1} < x_k$, the integration is zero for that y_{k-1}. Hence, in $\frac{\partial W_m}{\partial x_k}$, we integrate y_{k-1} from $\max\{x_{k-1}, x_k\}$ to $\max\{x_{k-1}, y_{k-2}\}$.

Hence, if $\frac{\partial \beta_{k+1}}{\partial z_{k+1}} = 0$,[61] at $x^*_{k+1} = x^*_{k+2} = \cdots = x^*_M = v$,

$$\begin{aligned}
\frac{\partial u}{\partial x_k} = \int_{x_1}^{\overline{v}} \cdots \int_{\max\{x_{k-1},x_k\}}^{\max\{x_{k-1},y_{k-2}\}} \Bigg\{ & -\beta_k(x_k; y_1, \cdots, y_{k-1}, x_{k-1}|v) f_{1\cdots k}(y_1, \cdots, y_{k-1}, x_k|v) \\
+ \int_{\underline{v}}^{\min\{v,x_k\}} & \beta_{k+1}(y_{k+1}; y_1, \cdots, y_{k-1}, x_k, x_k) f_{1\cdots(k+1)}(y_1, \cdots, x_k, y_{k+1}|v) dy_{k+1} \\
& + v\Bigg[f_{1\cdots k}(y_1, \cdots, y_{k-1}, x_k|v) - \int_{\underline{v}}^{\min\{v,x_k\}} f_{1\cdots(k+1)}(y_1, \cdots, x_k, y_{k+1}|v) dy_{k+1} \Bigg] \\
& - \Bigg[\sum_{m=k+2}^{M} \int_{v}^{\max\{v,x_k\}} \cdots \int_{\underline{v}}^{\min\{x_m,y_{m-1}\}} \Big[v - \beta_m(y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, x_{m-1}) \Big] \\
& \qquad \cdot f_{1\cdots m}(y_1, \cdots, x_k, \cdots, y_m|v) \Big) dy_m \cdots dy_{k+1} \Bigg] \Bigg\} dy_{k-1} \cdots dy_1
\end{aligned}$$

$= 0$ at $x_k = v$ if

$$\begin{aligned}
& \beta_k(v; z_1, \cdots, z_k) \\
& = E\Big[\beta_{k+1}(Y_{k+1}; z_1, \cdots, z_{k-1}, v, v) \Big| Y_1 = z_1, \cdots, Y_{k-1} = z_{k-1}, Y_k = v, V_i = v \Big] \\
& = \int_{\underline{v}}^{v} \beta_{k+1}(y_{k+1}; z_1, \cdots, z_{k-1}, v, v) \frac{f_{1\cdots(k+1)}(z_1, \cdots, z_{k-1}, v, y_{k+1}|v)}{f_{1\cdots k}(z_1, \cdots, z_{k-1}, v|v)} dy_{k+1}
\end{aligned}$$

Note that β_k is indeed independent of z_k.

To show that $x_k = v$ is indeed optimal, we need to show that

$$\begin{aligned}
\frac{\partial u}{\partial x_k} &\geq 0 && \text{if } x_k < v, \\
\frac{\partial u}{\partial x_k} &\leq 0 && \text{if } x_k > v.
\end{aligned}$$

[61] This is satisfied for the proposed bid functions.

If $x_k < v$, then $\max\{v, x_k\} = v$, $\min\{v, x_k\} = x_k$, and hence,

$$\frac{\partial u}{\partial x_k} = \int_{x_1}^{\overline{v}} \cdots \int_{\max\{x_{k-1},x_k\}}^{\max\{x_{k-1},y_{k-2}\}} \Big\{ \int_{\underline{v}}^{x_k} \beta_{k+1}(y_{k+1}; y_1, \cdots, y_{k-1}, x_k, x_k) f_{1\cdots(k+1)}(y_1, \cdots, x_k, y_{k+1}|v) dy_{k+1} - \beta_k(x_k; y_1, \cdots, y_{k-1}, x_{k-1}|v) f_{1\cdots k}(y_1, \cdots, y_{k-1}, x_k|v) \Big\} dy_{k-1} \cdots dy_1 \geq 0$$

since by affiliation, for $x_k < v$,

$$\begin{aligned}
&\beta_k(x_k; y_1, \cdots, y_{k-1}, x_{k-1}) \\
&= E\Big[\beta_{k+1}(Y_{k+1}; y_1, \cdots, y_{k-1}, x_k, x_k)\Big| Y_1 = y_1, \cdots, Y_{k-1} = y_{k-1}, Y_k = x_k, V_i = x_k\Big] \\
&\leq E\Big[\beta_{k+1}(Y_{k+1}; y_1, \cdots, y_{k-1}, x_k, x_k)\Big| Y_1 = y_1, \cdots, Y_{k-1} = y_{k-1}, Y_k = x_k, V_i = v\Big] \\
&= \int_{\underline{v}}^{x_k} \beta_{k+1}(y_{k+1}; y_1, \cdots, y_{k-1}, x_k, x_k) \frac{f_{1\cdots(k+1)}(y_1, \cdots, y_{k-1}, x_k, y_{k+1}|v)}{f_{1\cdots k}(y_1, \cdots, y_{k-1}, x_k|v)} dy_{k+1}
\end{aligned}$$

If $x_k > v$, we have

$$\frac{\partial u}{\partial x_k} = \int_{x_1}^{\overline{v}} \cdots \int_{\max\{x_{k-1},x_k\}}^{\max\{x_{k-1},y_{k-2}\}} \Big\{ A_{k+1} + A_{k+2} + \cdots + A_M + v \cdot B - \beta_k(x_k; y_1, \cdots, y_{k-1}, x_{k-1}|v) f_{1\cdots k}(y_1, \cdots, y_{k-1}, x_k|v) \Big\} dy_{k-1} \cdots dy_1$$

where

$$\begin{aligned}
A_{k+1} &= \int_{\underline{v}}^{v} \beta_{k+1}(y_{k+1}; y_1, \cdots, y_{k-1}, x_k, x_k) f_{1\cdots(k+1)}(y_1, \cdots, x_k, y_{k+1}|v) dy_{k+1} \\
A_{k+2} &= \int_{v}^{x_k} \int_{\underline{v}}^{v} \beta_{k+2}(y_{k+2}; y_1, \cdots, x_k, y_{k+1}, v) \\
&\qquad \cdot f_{1\cdots(k+2)}(y_1, \cdots, x_k, \cdots, y_{k+2}|v) dy_{k+2} dy_{k+1} \\
A_m &= \int_{v}^{x_k} \int_{v}^{y_{k+1}} \cdots \int_{v}^{y_{m-2}} \int_{\underline{v}}^{v} \beta_m(y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, v) \\
&\qquad \cdot f_{1\cdots m}(y_1, \cdots, x_k, \cdots, y_m|v) dy_m \cdots dy_{k+1}
\end{aligned}$$

for $m = 3, 4, \cdots, M$, and

$$
\begin{aligned}
B &= f_{1\cdots k}(y_1, \cdots, y_{k-1}, x_k|v) - \int_{\underline{v}}^{v} f_{1\cdots(k+1)}(y_1, \cdots, x_k, y_{k+1}|v)dy_{k+1} \\
&\quad - \sum_{m=k+2}^{M} \int_{v}^{x_k}\int_{v}^{y_{k+1}}\cdots\int_{v}^{y_{m-2}}\int_{\underline{v}}^{v} f_{1\cdots m}(y_1, \cdots, x_k, \cdots, y_m|v)dy_m \cdots dy_{k+1} \\
&= \int_{v}^{x_k} f_{1\cdots(k+1)}(y_1, \cdots, x_k, y_{k+1}|v)dy_{k+1} \\
&\quad - \int_{v}^{x_k}\int_{\underline{v}}^{v} f_{1\cdots(k+2)}(y_1, \cdots, x_k, \cdots, y_{k+2}|v)dy_{k+2}dy_{k+1} \\
&\quad - \sum_{m=k+3}^{M} \int_{v}^{x_k}\int_{v}^{y_{k+1}}\cdots\int_{v}^{y_{m-2}}\int_{\underline{v}}^{v} f_{1\cdots m}(y_1, \cdots, x_k, \cdots, y_m|v)dy_m \cdots dy_{k+1} \\
&= \cdots \\
&= \int_{v}^{x_k}\int_{v}^{y_{k+1}}\cdots\int_{v}^{y_{M-2}}\int_{v}^{y_{M-1}} f_{1\cdots M}(y_1, \cdots, x_k, \cdots, y_M|v)dy_M \cdots dy_{k+1}
\end{aligned}
$$

Let

$$
\begin{aligned}
C_m = \int_{v}^{x_k}\int_{v}^{y_{k+1}}\cdots\int_{v}^{y_{m-2}}\int_{\underline{v}}^{y_{m-1}} &\beta_m(y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, v) \\
&\cdot f_{1\cdots m}(y_1, \cdots, x_k, \cdots, y_m|v)dy_m \cdots dy_{k+1}
\end{aligned}
$$

Then, since $v < y_M = \beta_M(y_M; y_1, \cdots, x_k, \cdots, y_{M-1}, v)$ for $y_M \in [v, x_k]$,

$$
\begin{aligned}
A_M + v \cdot B = \int_{v}^{x_k}\int_{v}^{y_{k+1}}\cdots\int_{v}^{y_{M-2}} \Big[&\int_{v}^{y_{M-1}} v f_{1\cdots M}(y_1, \cdots, x_k, \cdots, y_M|v)dy_M \\
&+ \int_{\underline{v}}^{v} \beta_M(y_M; y_1, \cdots, x_k, \cdots, y_{M-1}, v) \\
&\qquad \cdot f_{1\cdots M}(y_1, \cdots, x_k, \cdots, y_M|v)dy_M \Big] dy_{M-1} \cdots dy_{k+1} \\
\le C_M
\end{aligned}
$$

and iteratively, for all $m = M-1, M-2, \cdots, k+2$,[62]

$$
\begin{aligned}
A_{m-1} + C_m = \int_v^{x_k}\int_v^{y_{k+1}}\cdots\int_v^{y_{m-3}} \Bigg[\int_v^{y_{m-2}} \bigg(\int_v^{y_{m-1}} &\beta_m(y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, v) \\
&\cdot f_{1\cdots m}(y_1, \cdots, x_k, \cdots, y_m|v) dy_m \bigg) dy_{m-1} \\
+ \int_{\underline{v}}^{v} &\beta_{m-1}(y_{m-1}; y_1, \cdots, x_k, \cdots, y_{m-2}, v) \\
&\cdot f_{1\cdots(m-1)}(y_1, \cdots, x_k, \cdots, y_{m-1}|v) dy_{m-1} \Bigg] dy_{m-2} \cdots dy_{k+1} \\
\le \int_v^{x_k}\int_v^{y_{k+1}}\cdots\int_v^{y_{m-3}} \bigg(\int_v^{y_{m-2}} + \int_{\underline{v}}^{v} \bigg) &\beta_{m-1}(y_{m-1}; y_1, \cdots, x_k, \cdots, y_{m-2}, v) \\
&\cdot f_{1\cdots m}(y_1, \cdots, x_k, \cdots, y_m|v) dy_m \cdots dy_{k+1} \\
= C_{m-1}&
\end{aligned}
$$

since by affiliation for $y_{m-1} \ge v$,[63]

$$
\begin{aligned}
&\beta_{m-1}(y_{m-1}; y_1, \cdots, x_k, \cdots, y_{m-2}, v) \\
&= E\Big[\beta_m(Y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, y_{m-1}) \Big| Y_1 = y_1, \cdots, Y_{m-1} = y_{m-1}, V_i = y_{m-1}\Big] \\
&= E\Big[\beta_m(Y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, v) \Big| Y_1 = y_1, \cdots, Y_{m-1} = y_{m-1}, V_i = y_{m-1}\Big] \\
&\ge E\Big[\beta_m(Y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, v) \Big| Y_1 = y_1, \cdots, Y_{m-1} = y_{m-1}, V_i = v\Big] \\
&= \int_{\underline{v}}^{x_k} \beta_m(y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, v) \frac{f_{1\cdots(k+1)}(y_1, \cdots, y_{k-1}, x_k, y_{k+1}|v)}{f_{1\cdots m}(y_1, \cdots, y_{k-1}, x_k|v)} dy_{k+1}
\end{aligned}
$$

[62]Note that for any z_m,

$$\beta_m(y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, z_m) = \beta_m(y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, v)$$

[63]Since $\frac{\partial \beta_m}{\partial z_m} = 0$, $\beta_m(Y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, y_{m-1}) = \beta_m(Y_m; y_1, \cdots, x_k, \cdots, y_{m-1}, v)$.

Hence,

$$
\begin{aligned}
& A_{k+1} + \cdots + (A_M + v \cdot B) \\
\leq\ & A_{k+1} + \cdots + A_{M-1} + C_M \\
\leq\ & A_{k+1} + \cdots + A_{M-2} + C_{M-1} \leq \cdots \\
\leq\ & A_{k+1} + C_{k+2} = C_{k+1} \\
=\ & \int_{\underline{v}}^{x_k} \beta_{k+1}(y_{k+1}; y_1, \cdots, x_k, v) f_{1\cdots(k+1)}(y_1, \cdots, x_k, y_{k+1}|v) dy_{k+1}
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\partial u}{\partial x_k} \\
\leq\ & \int_{x_1}^{\overline{v}} \cdots \int_{\max\{x_{k-1}, x_k\}}^{\max\{x_{k-1}, y_{k-2}\}} \bigg\{ - \beta_k(x_k; y_1, \cdots, y_{k-1}, x_{k-1}|v) f_{1\cdots k}(y_1, \cdots, y_{k-1}, x_k|v) \\
& + \int_{\underline{v}}^{x_k} \beta_{k+1}(y_{k+1}; y_1, \cdots, x_k, v) f_{1\cdots(k+1)}(y_1, \cdots, x_k, y_{k+1}|v) dy_{k+1} \bigg\} dy_{k-1} \cdots dy_1 \\
\leq\ & 0
\end{aligned}
$$

since again by affiliation, for $x_k > v$,

$$
\begin{aligned}
& \beta_k(x_k; y_1, \cdots, y_{k-1}, x_{k-1}) \\
=\ & E\Big[\beta_{k+1}(Y_{k+1}; y_1, \cdots, y_{k-1}, x_k, x_k) \Big| Y_1 = y_1, \cdots, Y_{k-1} = y_{k-1}, Y_k = x_k, V_i = x_k\Big] \\
\geq\ & E\Big[\beta_{k+1}(Y_{k+1}; y_1, \cdots, y_{k-1}, x_k, x_k) \Big| Y_1 = y_1, \cdots, Y_{k-1} = y_{k-1}, Y_k = x_k, V_i = v\Big] \\
=\ & \int_{\underline{v}}^{x_k} \beta_{k+1}(y_{k+1}; y_1, \cdots, y_{k-1}, x_k, x_k) \frac{f_{1\cdots(k+1)}(y_1, \cdots, y_{k-1}, x_k, y_{k+1}|v)}{f_{1\cdots k}(y_1, \cdots, y_{k-1}, x_k|v)} dy_{k+1}
\end{aligned}
$$

Therefore, we have shown that

$$
\begin{aligned}
\frac{\partial u}{\partial x_k} &\geq 0 && \text{if } x_k < v, \\
\frac{\partial u}{\partial x_k} &\leq 0 && \text{if } x_k > v.
\end{aligned}
$$

and hence, $x_k = v$ is indeed a global maximum. □

Proof of Theorem 5. It is suffice to prove the L2 type bid function are as stated above.

First note that since L1 type bidders will not bid in the first two auctions, only L0 type bidders will bid in the first two auctions. Hence, it is a dominated strategy for the L2 type bidders to win the first auction and pay the valuation of the highest L0 type bidder, since in every case he could bid (the same amount) and win the second auction and pay the valuation of the second-highest L0 type bidder. Therefore, L2 type bidders would bid $b_1 = 0$ in the first auction.

We now solve for the bid function b_2 and b_3 in the second and third auction, in which bidder's utility (of winning) is

$$\max_{b_2,b_3} u(v, b_2, b_3) = \underbrace{W_2(v, b_2)}_{\text{win 2nd}} + \underbrace{W_3(v, b_2, b_3)}_{\text{win 3rd}}$$

where W_2 and W_3 are the expected utility of winning the second and third auction, respectively.

For the case of $(n_0, n_1) = (2, 2)$, we denote the valuation of the two L0 type bidders as y_1, y_2 where $y_1 \geq y_2$, and the valuation of the two L1 bidders as z_1, z_2 where $z_1 \geq z_2$. We assume v, y_i and z_i are independent across different groups of types, and have full support $[\underline{v}, \overline{v}]$.

Given the following conditional pdf

$$Z_1\Big|_V \sim f_1(z_1)\text{with cdf } F_1, \qquad Y_2\Big|_V \sim f_2(y_2)\text{with cdf } F_2$$

the expected utility of bidder v bidding b_2, b_3, winning the second and third auction are:

$$\begin{aligned}
W_2 &= \int_{\underline{v}}^{b_2} (v - y_2) f_2(y_2) dy_2 \\
W_3 &= \int_{b_2}^{\overline{v}}\int_{\underline{v}}^{b_3} (v - z_1) f_2(y_2) f_1(z_1) dy_2 dz_1 \\
&= \Big[1 - F_2(b_2)\Big] \int_{\underline{v}}^{b_3} (v - z_1) f_1(z_1) dz_1
\end{aligned}$$

To show that $b_3^* = v$, we observe that

$$\begin{aligned}
\frac{\partial u}{\partial b_3} &= \frac{\partial W_3}{\partial b_3} \\
&= \Big[1 - F_2(b_2)\Big](v - b_3) f_1(b_3) \\
&\begin{cases} \geq 0 & b_3 \leq v \\ = 0 \quad \text{if} & b_3 = v \\ \leq 0 & b_3 \geq v \end{cases}
\end{aligned}$$

Hence, $b_3^* = v$ is indeed a global maximum.

For the choice of b_2^*, assuming $b_3^* = v$, we derive:

$$\frac{\partial W_2}{\partial b_2} = (v - b_2) f_2(b_2)$$
$$\frac{\partial W_3}{\partial b_2} = -f_2(b_2) \int_{\underline{v}}^{v} (v - z_1) f_1(z_1) dz_1$$

Hence,

$$\frac{\partial u}{\partial b_2} = f_2(b_2) \left\{ v\Big[1 - F_1(v)\Big] - b_2 + \int_{\underline{v}}^{v} z_1 f_1(z_1) dz_1 \right\}$$

Assuming that v, y_i and z_i are independent even within each group of types, and they all come from the same distribution $F(\cdot)$ with pdf $f(\cdot)$,[64] we obtain

$$F_1(z_1) = F(z_1)^2,$$
$$f_1(z_1) = 2F(z_1) f(z_1).$$

Therefore,

$$\begin{aligned}
\frac{\partial u}{\partial b_2} &= f_2(b_2) \left\{ \Big[1 - F(v)^2\Big] v - b_2 + \int_{\underline{v}}^{v} z_1 \cdot 2F(z_1) f(z_1) dz_1 \right\} \\
&= f_2(b_2) \left\{ v - b_2 - \int_{\underline{v}}^{v} F(z_1)^2 dz_1 \right\} \\
&\begin{cases} \geq 0 & b_2 \leq v - \int_{\underline{v}}^{v} F(z_1)^2 dz_1 \\ = 0 \quad \text{if} & b_2 = v - \int_{\underline{v}}^{v} F(z_1)^2 dz_1 \\ \leq 0 & b_2 \geq v - \int_{\underline{v}}^{v} F(z_1)^2 dz_1 \end{cases}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
b_2^* &= v - \int_{\underline{v}}^{v} F(z_1)^2 dz_1 \\
&= v - \frac{v^3}{3}
\end{aligned}$$

[64] This is where we need the independent private value assumption.

for the distribution of uniform over $[0, 1]$ where $F(v) = v$.[65]

For the case of $(n_0, n_1) = (3, 1)$, we denote the valuation of the three L0 type bidders as y_1, y_2, y_3 where $y_1 \geq y_2 \geq y_3$, and the valuation of the L1 bidders as z_1. Again, we assume v, y_i and z_i are independent across different groups of types, and have full support $[\underline{v}, \overline{v}]$.

Given the following conditional pdf

$$Y_2\Big|_V \sim f_2(y_2) \text{with cdf } F_2, \qquad Z_1\Big|_V \sim f_1(z_1) \text{with cdf } F_1, \qquad Y_2, Y_3\Big|_V \sim f_{23}(y_2, y_3) \text{with cdf } F_{23}$$

the expected utility of bidder v bidding b_2, b_3, winning the second and third auction are:

$$\begin{aligned}
W_2 &= \int_{\underline{v}}^{b_2} (v - y_2) f_2(y_2) dy_2 \\
W_3 &= \int_{b_2}^{\overline{v}} \int_{\underline{v}}^{\min\{y_2, b_3\}} \int_{\underline{v}}^{y_3} (v - y_3) f_{23}(y_2, y_3) f_1(z_1) dz_1 dy_3 dy_2 \qquad (y_3 > z_1) \\
&\quad + \int_{b_2}^{\overline{v}} \int_{\underline{v}}^{b_3} \int_{\underline{v}}^{\min\{y_2, z_1\}} (v - z_1) f_{23}(y_2, y_3) f_1(z_1) dy_3 dz_1 dy_2 \qquad (y_3 < z_1)
\end{aligned}$$

To show that $b_3^* = v$, we observe that

$$\begin{aligned}
\frac{\partial u}{\partial b_3} &= \int_{\max\{b_2, b_3\}}^{\overline{v}} \int_{\underline{v}}^{b_3} (v - b_3) f_{23}(y_2, b_3) f_1(z_1) dz_1 dy_2 \\
&\quad + \int_{b_2}^{\overline{v}} \int_{\underline{v}}^{\min\{y_2, b_3\}} (v - b_3) f_{23}(y_2, y_3) f_1(b_3) dy_3 dy_2 \\
&= (v - b_3) \cdot K \\
&\begin{cases} \geq 0 & b_3 \leq v \\ = 0 \quad \text{if} & b_3 = v \\ \leq 0 & b_3 \geq v \end{cases}
\end{aligned}$$

Hence, $b_3^* = v$ is indeed a global maximum.

[65] This is where we need the assumption of uniform distribution.

For the choice of b_2^*, assuming $b_3^* = v$, we derive:

$$\begin{aligned}
\frac{\partial W_2}{\partial b_2} &= (v - b_2) f_2(b_2) \\
\frac{\partial W_3}{\partial b_2} &= -\int_{\underline{v}}^{\min\{b_2, v\}} \int_{\underline{v}}^{y_3} (v - y_3) f_{23}(b_2, y_3) f_1(z_1) dz_1 dy_3 \\
&\quad - \int_{\underline{v}}^{v} \int_{\underline{v}}^{\min\{b_2, z_1\}} (v - z_1) f_{23}(b_2, y_3) f_1(z_1) dy_3 dz_1
\end{aligned}$$

Assuming that v, y_i and z_i are independent even within each group of types, and they all come from the same distribution $F(\cdot)$ with pdf $f(\cdot)$,[66] we obtain

$$\begin{aligned}
f_1(z_1) &= f(z_1), \\
f_{23}(y_2, y_3) &= 3!\Big[1 - F(y_2)\Big] f(y_2) f(y_3) \quad = \quad K(y_2) f(y_3), \\
f_2(y_2) &= 3!\Big[1 - F(y_2)\Big] f(y_2) F(y_2) \quad = \quad K(y_2) F(y_2).
\end{aligned}$$

Therefore, for $K(b_2) = 3!\Big[1 - F(b_2)\Big] f(b_2)$, since $b_2 \leq v$,[67], we have

$$\begin{aligned}
\frac{\partial u}{\partial b_2} &= -b_2 F(b_2) K(b_2) + \int_{\underline{v}}^{b_2} y_3 K(b_2) f(y_3) F(y_3) dy_3 + \int_{\underline{v}}^{v} z_1 K(b_2) f(z_1) F(\min\{b_2, z_1\}) dz_1 \\
&\quad + v \left[F(b_2) K(b_2) - \int_{\underline{v}}^{b_2} K(b_2) f(y_3) F(y_3) dy_3 - \int_{\underline{v}}^{v} K(b_2) f(z_1) F(\min\{b_2, z_1\}) dz_1 \right] \\
&= K(b_2) \left\{ -b_2 F(b_2) + \left[\frac{b_2}{2} F(b_2)^2 - \int_{\underline{v}}^{b_2} \frac{1}{2} F(y_3)^2 dy_3 \right] + \left[\underbrace{\int_{b_2}^{v} z_1 f(z_1) F(b_2) dz_1}_{z_1 > b_2} + \underbrace{\int_{\underline{v}}^{b_2} z_1 f(z_1) F(z_1) dz_1}_{z_1 < b_2} \right] \right\} \\
&\quad + v K(b_2) \left[F(b_2) - \frac{F(b_2)^2}{2} - \underbrace{\int_{b_2}^{v} f(z_1) F(b_2) dz_1}_{z_1 > b_2} - \underbrace{\int_{\underline{v}}^{b_2} f(z_1) F(z_1) dz_1}_{z_1 < b_2} \right] \\
&= K(b_2) \left\{ -b_2 F(b_2) + b_2 F(b_2)^2 - \int_{\underline{v}}^{b_2} F(y_3)^2 dy_3 + F(b_2) \left[v F(v) - b_2 F(b_2) - \int_{b_2}^{v} F(z_1) dz_1 \right] \right\} \\
&\quad + v K(b_2) F(v) \left[1 - F(v) \right]
\end{aligned}$$

[66] This is where we need the independent private value assumption.

[67] Note that since all opponents are "passive" and would not chance their strategy according to one's behavior, bidding above one's valuation is dominated, since the bidder would incur a loss for all additional chances of winning.

For the distribution of uniform $[0,1]$ where $F(v) = v$ and $f(v) = 1$, we can further simplify the above first order condition:[68]

$$\begin{aligned}
\frac{\partial u}{\partial b_2} &= K(b_2)\left\{-b_2^2 + b_2^3 - \frac{b_2^3}{3} + b_2\left[v^2 - b_2^2 - \frac{v^2}{2} + \frac{b_2^2}{2}\right]\right\} + vK(b_2)b_2(1-v) \\
&= K(b_2)b_2\left\{v(1-v) - b_2 - \frac{b_2^2}{6} + \frac{v^2}{2}\right\} \\
&= K(b_2)b_2\left\{v - \frac{v^2}{2} - b_2 - \frac{b_2^2}{6}\right\} \\
&\begin{cases} \geq 0 & \quad b_2 \leq \sqrt{3(v+1)(3-v)} - 3 \\ = 0 & \text{if} \quad b_2 = \sqrt{3(v+1)(3-v)} - 3 \\ \leq 0 & \quad b_2 \geq \sqrt{3(v+1)(3-v)} - 3 \end{cases}
\end{aligned}$$

Thus, we have

$$b_2^* = \sqrt{3(v+1)(3-v)} - 3$$

which concludes the proof. □

[68] This is where we need the assumption of uniform distribution.