Game Theory: Lecture 1

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1 Introduction: foundational issues in game theory

1.1 History of game theory

0. Initiator: John von Neumann (1903-1957). He has three major publications on game theory and related subject:

0.1. von Neumann (1928): Minimax Theorem for a two-person zero-sum game

0.2. von Neumann (1937): Balanced Economic Growth and Minimax Theorem

He reproved the minimax theorem using Brouwer's fixed point theorem, and then applied the minimax theorem to the balanced economic model to obtain the existence of a balanced growth path.

0.3. von Neumann and Oskar Morgenstern (1944) published the book entitled *Theory of Games and Economic Behavior*, Princeton University Press. This book is the real beginning of the present theory of games:

a. Expected utility theory

b. Description of a game in extensive form; normalization by the concept of a strategy

c. Normal form game — we will start with this form of a game

d. For a normalized form game with 2 players and the sum of payoffs is always zero, when mixed strategies are allowed to play, the Minimax Theorem holds

e. Cooperative game theory: the three quarters of the book by von Neumann-Morgenstern were devoted to cooperative game theory. They investigated the mathematical concept of a stable set, interpreting it as a socially accepted stable standard of behavior. Nevertheless, this part is a failure of their great trials.

After von Neumann-Morgenstern (1944):

1. 1st period to 1964: mathematical studies by mathematicians including John F. Nash—Princeton's red books

2. 2nd period from 1965 to 1975: cooperative game theory and its applications to market economies. The Debreu-Scarf (1963) limit theorem is an important contribution in this period.

3. 3rd period from 1975 to 1990: recognition of the importance of noncooperative game theory, especially, extensive games. The tendency in this period is characterized by a study of "rational behavior".

4. 4th period from 1990 to present: proliferation of applications of game theory to many fields such as in economics (auction, voting, etc.), biology, etc.

1.2 Game theory in crisis

Influence of game theory in economics:

1. emphasis of strategic considerations: rational expectations, strategic voting, etc.

2. powerful solution concept: Nash equilibrium and incentives

As a mathematical tool, game theory is rather successful

How about as a theory of rational behavior?

1.3 Theory of rational behavior

1. Formulation of physical situation

2. Theory of preferences

3. Theory of probability

4. Theory of information/knowledge

4.1. information process

4.2. logical ability and inference

4.3. belief formation and revision

2 Expected Utility Theory

2.1 Standard Theory

$X = \{x_1, ..., x_n\}$ is a finite set of outcomes. The vN-M expected utility theory has two components: the set of lotteries over X, and a preference relation $\succ$ over those lotteries.

The set of lotteries is $\Delta(X) = \{p \in ([0,1])^{|X|} : \sum_{x \in X} p_x = 1\}$.

vN-M axioms for expected utility

EU1 (*Ordering*)$\succ$ is a preference relation.

EU2 (*Independence*) For all $p,\ q,\ r \in \Delta(X)$ and $\alpha \in (0,1)$,

$$\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r \text{ if and only if } p \succ q.$$

EU3 (The *Continuity* axiom) For all $p,\ q,\ r \in \Delta(X)$, if $p \succ q \succ r$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)r \succ q \succ \beta p + (1-\beta)r$.

Theorem 2.1. *A preference relation $\succ$ satisfies EU1-EU3 if and only if there exists a function $h : X \to \mathbb{R}$ such that for all $p, q \in \Delta(X)$,*

$$p \succ q \ \Leftrightarrow \ \sum_{x \in X} p_x h(x) > \sum_{x \in X} q_x h(x).$$

Moreover, if h' satisfies the above condition, then there exist $a > 0$ and b such that for all $x \in X$, $h'(x) = ah(x) + b$.

Remarks.

- The interpretation of $h(x)$ is different from the utility function obtained in the ordinal utility representation. Strictly speaking, $h(x)$ is *not* the utility of the outcome x, but of the lottery that assigns probability 1 to x (what is the difference?).

- Uniqueness (up to linear transformations) only applies to representations with the form of expected utilities. Why is uniqueness important?

The theorem is proved with the following three lemmas:

Lemma 2.1. *If $p \succ q$, then $\alpha > \beta \Leftrightarrow \alpha p + (1-\alpha)q \succ \beta p + (1-\beta)q$ for all $\alpha, \beta \in [0,1]$.*

Proof. $\beta < \alpha \leq 1$. Hence, by (EU2),

$$p = (1-\beta)p + \beta p \succ (1-\beta)q + \beta p = \beta p + (1-\beta)q.$$

Therefore, by (EU2) again,

$$\alpha p + (1-\alpha)q = \frac{\alpha - \beta}{1-\beta} p + \frac{1-\alpha}{1-\beta}(\beta p + (1-\beta)q) \succ \beta p + (1-\beta)q.$$

On the other hand, if $\alpha = \beta$, then $\alpha p + (1-\alpha)q \sim \beta p + (1-\beta)q$. $\square$

Lemma 2.2. *Let $x_1 \precsim x_2 \precsim .... \precsim x_n$ and $x_1 \prec x_n$. For any $p \in \Delta(X)$, there exists a unique $\alpha(p) \in [0,1]$ such that*

$$p \sim \alpha(p)x_n + (1-\alpha(p))x_1.$$

Proof. First note that $x_n \succsim p \succsim x_1$. This can be proved by induction on $|\operatorname{supp}(p)|$. It is obvious when $|\operatorname{supp}(p)| = 1$. Assume that this holds for all p with $|\operatorname{supp}(p)| \leq m$. Let $\operatorname{supp}(q) = \{y_1, ..., y_{m+1}\}$ and let $A = \{y_1, ..., y_m\}$. Then $q = (1-p[y_{m+1}])q^A + p[y_{m+1}]y_{m+1}$, where $q^A[y_i] = \frac{q[y_i]}{1-q[y_{m+1}]}$ for $i = 1, ..., m$. By the induction hypothesis, $x_n \succsim q^A \succsim x_1$ and $x_n \succsim y_{m+1} \succsim x_1$. By (EU2) we have $x_n \succsim q \succsim x_1$.

If $p \sim x_n$, then $\alpha(p) = 1$; if $p \sim x_1$, then $\alpha(p) = 0$. Suppose that $x_n \succ p \succ x_1$. Let

$$C = \{\alpha \in [0,1] : \alpha x_n + (1-\alpha)x_1 \succ p\}.$$

Then $\alpha(p) = \inf C$.

To see this, suppose that $\alpha(p)x_n + (1-\alpha(p))x_1 \prec p \prec x_n$. By (EU3) there exists some $\beta > \alpha(p)$ such that $\beta x_n + (1-\beta)x_1 \prec p$. By Lemma 2.1, $\gamma < \beta$ implies that $\gamma \notin C$ and hence $\beta \leq \alpha(p)$, a contradiction.

On the other hand, suppose that $\alpha(p)x_n + (1-\alpha(p))x_1 \succ p \succ x_1$. By (EU3) there exists some $\beta < \alpha(p)$ such that $\beta x_n + (1-\beta)x_1 \succ p$ and so $\beta \in C$. Hence $\beta \geq \alpha(p)$, a contradiction. $\square$

Lemma 2.3. *Let $u(x_i) = \alpha(x_i)$. Then for any $p \in \Delta(X)$, $\sum_{i=1}^n p[x_i]u(x_i) = \alpha(p)$.*

Proof. We prove by induction on $|\operatorname{supp}(p)|$. It is obvious when $|\operatorname{supp}(p)| = 1$. Assume that this holds for all p with $|\operatorname{supp}(p)| \leq m$. Let $\operatorname{supp}(q) = \{y_1, ..., y_{m+1}\}$ and let $A = \{y_1, ..., y_m\}$. Then $q = (1-p[y_{m+1}])q^A + p[y_{m+1}]y_{m+1}$. By the induction hypothesis, $\sum_{i=1}^{m} q^A[y_i]u(y_i) = \alpha(q^A)$. Moreover, $q^A \sim \alpha(q^A)x_n + (1-\alpha(q^A))x_1$ and $y_{m+1} \sim \alpha(y_{m+1})x_n + (1-\alpha(y_{m+1}))x_1$. By appropriate applications of (EU2),

$$q = (1 - q[y_{m+1}])q^A + q[y_{m+1}]y_{m+1} \sim$$

$$((1-q[y_{m+1}])\alpha(q^A)+q[y_{m+1}]u(y_{m+1}))x_n+(1-((1-q[y_{m+1}])\alpha(q^A)+q[y_{m+1}]u(y_{m+1})))x_1,$$

and hence

$$\alpha(q) = (1 - q[y_{m+1}])\alpha(q^A) + q[y_{m+1}]u(y_{m+1}) = \sum_{i=1}^{m+1} q[y_i]u(y_i).$$

□

2.2 The Frequentist perspective

What is probability in the vN-M expected utility theory?

- vague answer available
  - use probability values as primitives
    (in contrast with Savage's subjective probability)
  - von Neumann-Morgenstern emphasized frequentist theory
    (without direct formulation, forgotten by most people)

- ignored by most literature
  - one suggested answer: probability is belief
    - no external counterpart of probability values
  - why worried about a mere interpretation?
    - different interpretations suggest different decision criteria

The frequentist theory of probability (von Mises, 1939; Kolmogorov, 1965)

Probability values are properties of infinite sequences

- Lottery $\alpha x + (1-\alpha)y$:
  - x and y—*pure outcomes*; α—a number in $[0,1]$

von Mises: consider infinite sequence $(x,\ y,\ x,\ x,\ y,\ \dots)$

- **R1 frequency requirement**: the relative frequency of x is α
- **R2 randomness requirement**: no detectable pattern in this sequence

The decision maker's choice problem becomes

$$\left[\begin{array}{l} \xi = (x, y, x, x, y,\ \dots) \\ \zeta = (x, x, y, x, x,\ \dots) \end{array}\right. \quad \text{instead of} \quad \left[\begin{array}{l} \alpha x + (1-\alpha)y \\ \beta x + (1-\beta)y \end{array}\right.$$

where in ξ the frequency of x is α and in ζ the frequency of x is β

Frequentist compound lotteries

Compound lottery: $(p, q, \alpha) \mapsto \alpha p + (1-\alpha)q$

$(p, q \in \Delta(X);\ \alpha \in [0,1])$

Shuffle operator: $(\xi, \zeta, \nu) \mapsto \xi \oslash_\nu \zeta$

- $\xi, \zeta \in X^{\mathbb{N}}$; $\nu \in \{0,1\}^{\mathbb{N}}$
- math: $(\xi \oslash_\nu \zeta)_t = (1-\nu_t)\xi_{t-f^\nu(t)} + \nu_t \zeta_{f^\nu(t)}$
  - $f^\nu(0) = 0$ and $f^\nu(t) = \sum_{s=0}^{t-1} \nu_s$ for $t > 0$
- $\xi \oslash_\nu \zeta$ contains all information of the original sequences
- visualize the idea of compound lotteries
- alternatives: deleting the outcome not chosen

Domain of preference

A *collective* is an infinite sequence that satisfies R1.

- $\xi \in X^{\mathbb{N}}$ is a p-sequence if

$$\lim_{T\to\infty} \frac{|\{t : 0 \le t \le T-1,\ \xi_t = x\}|}{T} = p_x \text{ for each } x \in X.$$

- Set of collectives:

$$\Omega_X = \{\xi \in X^{\mathbb{N}} : \xi \text{ is a } p\text{-sequence for some } p \in \Delta(X)\}.$$

Lemma 2.4. *If ξ is a p-sequence, ζ a q-sequence, and ν an $(\alpha, 1-\alpha)$-sequence, then $\xi \oslash_\nu \zeta$ is an $(\alpha p + (1-\alpha)q)$-sequence.*

Axioms

A1 $\precsim$ is a complete and transitive binary relation

A2 For all ξ, ζ, η in Ω_X, if $\xi \prec \zeta \prec \eta$, then there is a number α and an $(\alpha, 1-\alpha)$-sequence ν in $\{0,1\}^{\mathbb{N}}$ such that $\zeta \sim \xi \oslash_\nu \eta$

A3 For all ξ, ζ, η in Ω_X and all $(\alpha, 1-\alpha)$-sequences ν^1, ν^2 in $\{0,1\}^{\mathbb{N}}$ with $\alpha > 0$, $\xi \oslash_{\nu^1} \eta \precsim \zeta \oslash_{\nu^2} \eta$ if and only if $\xi \precsim \zeta$

Frequentist transaltion

To translate axioms from $(\Omega_X, \precsim)$ to $(\Delta(X), \precsim^P)$, define $\Gamma : \Omega_X \to \Delta(X)$ as $\Gamma(\xi) = p$ if ξ is a p-sequence. Then $\precsim$ over Ω_X is translated into $\precsim^P$ over $\Delta(X)$ if for all $\xi,\ \zeta \in \Omega_X$,

$$\xi \precsim \zeta \text{ if and only if } \psi(\xi) \precsim^P \psi(\zeta). \tag{1}$$

Theorem 2.2 (Frequentist translation)**.** *(a) Suppose that $\precsim$ over Ω_X satisfies A1 and A3. There is a $\precsim^P$ over $\Delta(X)$ that satisfies (1).*

(b) Suppose that $\precsim$ over Ω_X and $\precsim^P$ over $\Delta(X)$ satisfy (1). For $i = 1, 2, 3$,

$\precsim$ satisfies Ai if and only if $\precsim^P$ satisfies EUi.

Representation theorem

Theorem 2.3 (Frequentist axiomatization of expected utility)**.** *$\precsim$ satisfies A1-A3 if and only if there exists a function h such that for all $\xi,\ \zeta \in \Omega_X$,*

$$\xi \precsim \zeta \ \Leftrightarrow \lim_{T\to\infty} \sum_{t=0}^{T-1} \frac{h(\xi_t)}{T} \le \lim_{T\to\infty} \sum_{t=0}^{T-1} \frac{h(\zeta_t)}{T}.$$

3 Two-Person Games and Various Concepts

3.1 2-Person Normal Form Games

A 2-person *normal form game* is given as a triple:

$$G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N}),$$

where

(1): $N = \{1, 2\}-$ the set of players;

(2): $S_i = \{\mathbf{s}_{i1}, ..., \mathbf{s}_{i\ell_i}\}$ $-$ the set of pure strategies for player $i = 1, 2$;

(3): $h_i : S_1 \times S_2 \to \mathbb{R}$ $-$ the payoff function of player $i = 1, 2$.

The set $S_1 \times S_2$ is the set of ordered pairs of elements in $S_1 \times S_2$, and is called the Cartesian product of S_1 and S_2.

A 2-person normal form game $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ is often described by a matrix form:

Prisoner's Dilemma

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(5, 5)$	$(1, 6)$
$\mathbf{s}_{12}$	$(6, 1)$	$(3, 3)$

Matching Pennies

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(1, -1)$	$(-1, 1)$
$\mathbf{s}_{12}$	$(-1, 1)$	$(1, -1)$

We say that a 2-person game is *zero-sum* iff

$$h_1(s_1, s_2) + h_2(s_1, s_2) = 0 \text{ for all } (s_1, s_2) \in S_1 \times S_2. \tag{2}$$

3.2 Maximin Decision Criterion

We consider the individual decision making criterion for player i called the *maximin decision criterion*. It has two steps:

(1): Player i evaluates each of his strategies by its worst possible payoff;

(2): Player i maximizes the evaluation by controlling his strategies.

These two steps are formulated mathematically as follows. Let $i = 1$.

(1*): for each $s_1 \in S_1$, the evaluation of s_1 is defined by $\min_{s_2} h_1(s_1, s_2)$;

(2*): Player 1 maximizes $\min_{s_2} h_1(s_1, s_2)$ by controlling s_1.

These two steps are expressed by

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) = \max_{s_1 \in S_1} (\min_{s_2 \in S_2} h_1(s_1, s_2)). \tag{3}$$

We say that s_1^* is a *maximin strategy* iff it is a solution of (3).

Question 4: Can we regard $\min_{s_2 \in S_2} h_1(s_1, s_2)$ as a function of s_1? In what sense?

Question 5: Formulate the maximin decision criterion for player 2.

Example ??.1: Consider the following zero-sum game:

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\min_{s_2} h_1(s_1, s_2)$
$\mathbf{s}_{11}$	$(5, -5)$	$(3, -3)$	3
$\mathbf{s}_{12}$	$(2, -2)$	$(6, -6)$	2
$\min_{s_1} h_2(s_1, s_2)$	?	?	

When the zero-sum condition (2) holds, the maximization of h_1 is equivalent to the minimization of h_2, i.e.,

$$h_1(s_1, s_2) \to \max_{s_1} \quad \Longleftrightarrow \quad h_2(s_1, s_2) \to \min_{s_1} \tag{4}$$

and the minimization of h_1 is equivalent to the maximization of h_2, i.e.,

$$h_1(s_1, s_2) \to \min_{s_2} \quad \Longleftrightarrow \quad h_2(s_1, s_2) \to \max_{s_2}. \tag{5}$$

By (4) and (5), the maximin decision criterion for player 2 will be formulated as follows:

(1*-2): for each $s_2 \in S_2$, the evaluation of s_2 is defined by $\max_{s_1} h_1(s_1, s_2)$;

(2*-2): Player 2 minimizes $\max_{s_1} h_1(s_1, s_2)$ by controlling s_2.

These two steps are expressed by

$$\min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2) = \min_{s_2 \in S_2} (\max_{s_1 \in S_1} h_1(s_1, s_2)). \tag{6}$$

Merit: Since (3) and (6) are described using the same payoff function h_1, we can compare these two. In general, we have the following inequality.

Lemma ??.1. $\max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) \leq \min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2)$.

Proof. Let $t_1 \in S_1$ and $t_2 \in S_2$ be arbitrary strategies for players 1 and 2. First, we have

$$h_1(t_1, t_2) \leq \max_{s_1 \in S_1} h_1(s_1, t_2)$$

Looking at the latter inequality, we have

$$\min_{s_2 \in S_2} h_1(t_1, s_2) \leq \min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2).$$

Since the right-hand side is constant, we have

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) \leq \min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2).$$

□

In the following example, the assertion of Lemma **??**.1 holds in inequality.

Example ??.2: Consider the zero-sum game of Example **??**.1:

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\min_{s_2} h_1(s_1, s_2)$
$\mathbf{s}_{11}$	$5\ (-5)$	$3\ (-3)$	3
$\mathbf{s}_{12}$	$2\ (-2)$	$6\ (-6)$	2
			$\max_{s_1} \min_{s_2} h_1(s_1, s_2) = 3$
$\max_{s_1} h_1(s_1, s_2)$	5	6	$\min_{s_2} \max_{s_1} h_1(s_1, s_2) = 5$

In the following example, the assertion of Lemma **??**.1 holds in equality.

Example ??.3: Consider the zero-sum game:

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\min_{s_2} h_1(s_1, s_2)$
$\mathbf{s}_{11}$	5	3	3
$\mathbf{s}_{12}$	6	4	4
			$\max_{s_1} \min_{s_2} h_1(s_1, s_2) = 4$
$\max_{s_1} h_1(s_1, s_2)$	6	4	$\min_{s_2} \max_{s_1} h_1(s_1, s_2) = 4$

Example ??.4: The Scissors-Rok-Paper is formulated as follows:

	Sc	**Ro**	**Pa**
Sc	0	-1	1
Ro	1	0	-1
Pa	-1	1	0

Calculate the maximin value and minimax value.

3.3 Strictly Determined Games

We say that a 2-person zero-sum game $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ is *strictly determined* iff

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) = \min_{s_2 \in S_2} \max_{s_1 \in S_1} h_1(s_1, s_2). \tag{7}$$

Then the game of Example **??**.3 is strictly determined, while the game of Example **??**.2 is not.

Question 6: Explain why a game is said to be strictly determined. In other words, what is the intended meaning of "strictly determined".

In fact, the equation (7) is related to some concept of an equilibrium. In a 2-person zero-sum game, it is the saddle point, but in general, it is the Nash equilibrium.

Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a 2-person normal form game. We say that (s_1^*, s_2^*) is a *saddle point* with respect to h_1 iff for all $s_1 \in S_1$ and $s_2 \in S_2$

$$h_1(s_1, s_2^*) \leq h_1(s_1^*, s_2^*) \leq h_1(s_1^*, s_2). \tag{8}$$

Example ??.5: In the game of Table ??.4, the pair $(\mathbf{s}_{12}, \mathbf{s}_{22})$ is a unique saddle point. On the other hand, the game of Table ??.3 has no saddle point.

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	5	3
$\mathbf{s}_{12}$	2	6

Table ??.3

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	5	3
$\mathbf{s}_{12}$	6	4

Table ??.4

In fact, this coincidence of having a saddle point and (7) is not mere accidental for the above example. We have the following theorem.

Theorem ??.2.(1): A 2-person game G has a saddle point if and only if (7) holds.

(2): Suppose that G has a saddle point. Then a pair (s_1^*, s_2^*) of strategies is a saddle point if and only if s_1^* and s_2^* are maximin strategies for players 1 and 2, respectively.

Proof. We prove the Only-If part and If part of (1). These are also proofs of the Only-If part and If part of (2).

(1)(*Only-If*): Let (s_1^*, s_2^*) be a saddle point of game G. Then, $h_1(s_1^*, s_2^*) = \max_{s_1} h_1(s_1, s_2^*) = \min_{s_2} h_1(s_1^*, s_2)$ by (8). Consider $\min_{s_2} \max_{s_1} h_1(s_1, s_2^*)$. Then, $\min_{s_2} \max_{s_1} h_1(s_1, s_2^*) \leq h_1(s_1^*, s_2^*)$. Similarly, we have $\max_{s_1} \min_{s_2} h_1(s_1, s_2^*) \geq h_1(s_1^*, s_2^*)$. By Lemma ??.1, we have (7).

(*If*): Suppose that (7) holds. Let s_1^* and s_2^* be maximin strategies for players 1 and 2. By (7), we have

$$\max_{s_1} h_1(s_1, s_2^*) = \min_{s_2} \max_{s_1} h_1(s_1, s_2) = \max_{s_1} \min_{s_2} h_1(s_1, s_2) = \min_{s_2} h_1(s_1^*, s_2)$$

If $h_1(s_1^*, s_2^*) \leq \max_{s_1} h_1(s_1, s_2^*)$ and $\min_{s_2} h_1(s_1^*, s_2) \leq h_1(s_1^*, s_2^*)$, we have $\max_{s_1} h_1(s_1, s_2^*) \leq h_1(s_1^*, s_2^*) \leq \min_{s_2} h_1(s_1^*, s_2)$. This implies (8).□

The concept of a saddle point is, in fact, equivalent to the concept of a Nash equilibrium. We say that a strategy pair (s_1^*, s_2^*) is a *Nash equilibrium* iff

$$h_1(s_1, s_2^*) \leq h_1(s_1^*, s_2^*) \text{ for all } s_1 \in S_1, \tag{9}$$

$$h_2(s_1^*, s_2) \leq h_2(s_1^*, s_2^*) \text{ for all } s_2 \in S_{2.}$$

We have the following theorem on the relationship between the saddle point and Nash equilibrium.

Theorem ??.3. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a zero-sum 2-person game. Then a pair (s_1^*, s_2^*) of strategies is a saddle point for h_1 if and only if (s_1^*, s_2^*) is a Nash equilibrium in G.

Proof. The first half of (9) is the same as the first half of (8). Hence, it suffices to compare the latter half of (9) with that of (8). By the zero-sum condition (2), the latter half of (9) is rewritten as

$$-h_1(s_1^*, s_2) \leq -h_1(s_1^*, s_2^*) \text{ for all } s_2 \in S_{2.}$$

This is equivalent to

$$h_1(s_1^*, s_2^*) \leq h_1(s_1^*, s_2) \text{ for all } s_2 \in S_{2.}$$

This is the latter half of (8). □

3.4 Nash Equilibrium and Pareto-Optimality in Two-person Games

Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a two-person normal form game. Then, we say that a strategy pair (s_1, s_2) *Pareto-dominates* another strategy pair (t_1, t_2) iff

$$h_1(s_1, s_2) > h_1(t_1, t_2) \tag{10}$$

$$h_2(s_1, s_2) > h_2(t_1, t_2).$$

We say that (s_1, s_2) is a *Pareto optimal* iff no strategy pair Pareto-dominates (s_1, s_2). Note that Pareto optimality is an attribute of a pair of strategies.

Question 7. Find one example of a Pareto dominance in Table 1 and Table 2.

Table **??**.1: Prisoner's Dilemma

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(5,5)$	$(1,6)$
$\mathbf{s}_{12}$	$(6,1)$	$(3,3)$

Table **??**.2: Matching Pennies

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(1,-1)$	$(-1,1)$
$\mathbf{s}_{12}$	$(-1,1)$	$(1,-1)$

Also, obtain Pareto optimal strategy pairs in each of Tables **??**.1 and **??**.2.

Question 8. What are the Nash equilibria in Table **??**.1 and Table **??**.2?

According to the above calculations, some people may conjecture that a Nash equilibrium is not Pareto optimal and vice versa. This conjecture is incorrect. First, we have the game called the *Battle of the Sexes:*

Table **??**.5: the Battle of the Sexes

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(2,1)$	$(0,0)$
$\mathbf{s}_{12}$	$(0,0)$	$(1,2)$

Also, we have the following proposition.

Proposition ??.4. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a zero-sum 2-person game. Then, any strategy pair (s_1, s_2) is Pareto-optimal in G.

Proof. Let (s_1, s_2) and (t_1, t_2) be any strategy pairs in game G. By (2), $h_1(s_1, s_2) > h_1(t_1, t_2)$ if and only if $h_2(s_1, s_2) < h_2(t_1, t_2)$. Thus, there is no Pareto-dominance relation between (s_1, s_2) and (t_1, t_2). Since (s_1, s_2) and (t_1, t_2) are arbitrary, there is no Pareto-dominance relation in game G. Thus, (s_1, s_2) is Pareto optimal. $\square$

4 Extensive Games

4.1 Extensive Games

An extensive game Γ has the following constituents:

(1): the set of players $N = \{1, ..., n\}$;

(2): a *finite tree* T consisting of the sets of *nodes* and *branches*;

(3): the set of nodes is divided into the set of non-terminal nodes X and

the set of terminal nodes Z;

each node in X is called a *decision node*; and each in Z is called an *endnode*;

(4): a player assignment π which assigns one player in N to each node $x \in X$;

(5): for $i \in N$, the set $\{x \in X : \pi(x) = i\}$ is partitioned into the set of

information sets $\{I_{i1}, ..., I_{im_i}\}$;

(6): player i has the set of *available actions* at each I_i, which is denoted by $A(I_i)$;

(7): payoff assignments to the players attached to the endnodes.

Let us look at those structure by Prisoner's Dilemma.

Example 4.1.(Prisoner's Dilemma): Figure 4.1 is the representation of Prisoner's Dilemma in extensive form. Here,

the player set $N = \{1, 2\}$:

the decision nodes $X = \{x_o, x_1, x_2\}$ and the endnodes $Z = \{x_3, x_4, x_5, x_6\}$;

two information sets: $I_1 = \{x_0\}$ and $I_2 = \{x_1, x_2\}$;

the available actions $A(I_1) = \{L, R\}$ and $A(I_2) = \{L, R\}$;

the payoff assignments are attached at each endnode.

In Figure 4.1, player 2 moves after player 1. But this is an artificial order. From the viewpoint of decision making, this order does not matter, since player 2 would not receive any information about what player 1 chooses.

In this example, the concept of a strategy is not affected by an information structure, since no information is available in this game. Player 1 has two strategies L and R, and player 2 has also L and R. A strategy, say L, for player 2 can be described in Figure 4.1 by putting the left arrows from x_1 to x_3 and from x_2 to $x_{5.}$

Question 1. In Figure 4.1, the arrows from x_1 to x_3 and from x_2 to x_6 do not describe a strategy for player 2. What is wrong with this description?

The next example has a different information structure.

Example 4.2.(Prisoner's Dilemma with Perfect Information): Change Figure 4.1 to Figure 4.2 so that the information set $\{x_1, x_2\}$ is divided into $I_2 = \{x_1\}$ and $I_3 = \{x_2\}$. In this game, player 2 would observe what player 1 does. The two separated information sets $I_2 = \{x_1\}$ and $I_3 = \{x_2\}$ mean that his received information at x_1 is distinguished from that at x_2. Of course, player 1 makes a choice at x_0 without any information about 2's choice. In this game, player 2 can make a contingent plan such as

$$\sigma_2(I) = \begin{cases} R & \text{if } I = I_2 = \{x_1\} \\ \\ L & \text{if } I = I_3 = \{x_2\}. \end{cases}$$

In words, player 2 would choose R if he receives information I_2, and he would choose L if he receives I_3.

Let us write down all the strategies for player 2 :

$$\sigma_2^1(I) = \begin{cases} L & \text{if } I = I_2 \\ \\ L & \text{if } I = I_3 \end{cases} \qquad \sigma_2^2(I) = \begin{cases} L & \text{if } I = I_2 \\ \\ R & \text{if } I = I_3 \end{cases}$$

$$\sigma_2^3(I) = \begin{cases} R & \text{if } I = I_2 \\ \\ L & \text{if } I = I_3 \end{cases} \qquad \sigma_2^4(I) = \begin{cases} R & \text{if } I = I_2 \\ \\ R & \text{if } I = I_3. \end{cases}$$

Question 2. How do you describe the strategy σ_2^2 graphically in Figure 4.2? Also, describe σ_2^2 in words.

4.2 The Derived Normal Game and Nash Equilibrium

Let us continue the game of Example 4.2. In this game, players 1 and 2 have the following strategy sets:

$$S_1 = \{L, R\} \text{ and } S_2 = \{\sigma_2^1, \sigma_2^2, \sigma_2^3, \sigma_2^4\}. \tag{11}$$

Then, we have the normal form game $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$, where $N = \{1, 2\}, S_1$ and S_2 are given by (11), the payoff functions h_1 and h_2 are given by the following matrix:

Table 4.1

	σ_2^1	σ_2^2	σ_2^3	σ_2^4
L	$(5, 5)$	$(5, 5)$	$(1, 6)$	$(1, 6)$
R	$(6, 1)$	$(3, 3)$	$(6, 1)$	$(3, 3)$

Indeed, suppose that players 1 and 2 choose L and σ_2^2, respectively. This situation is described by the arrows in Figure 4.3. Following the arrows indicating the strategies, we have one path from the root to a leaf is determined, which is called the *realization path* by (L, σ_2^2). In this case, the realization path is the sequence $\{x_0, x_1, x_2\}$. At the leaf, x_2, determined, we have payoff vector $(5, 5)$.

Once a normal form game $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ is obtained, the concept of a Nash equilibrium is defined in the way before, (9) of page 7.

Question 3. What are Nash equilibria in the game of Table 4.1?

Question 4. Describe the Nash equilibria graphically in the tree description of the game.

We consider the Battle of the Sexes:

Table **??**.5

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(2, 1)$	$(0, 0)$
$\mathbf{s}_{12}$	$(0, 0)$	$(1, 2)$

This game is expressed as the extensive game in Figure 4.4.

Question 5. Describe the situation of the Battle of the Sexes with the information structure that player 2 would observe what player 1 chooses.

Question 6. Is the strategy pair indicated by the arrow in Figure 4.5 a Nash equilibrium?

Question 7. What are Nash equilibria in the modified Battle of the Sexes of Question 5?

4.3 Subgame Perfect Equilibrium

In fact, the strategy pair indicated by the arrows in Figure 4.5 is also a Nash equilibrium.

However, we can regard the subtree indicated by the smaller box including x_2 as a one-person problem, where player 2 is only the player. The indicated strategy is not a Nash equilibrium in this one-person game.

Actually, the game of Figure 4.5 has three subgames, one already indicated, the second one started from x_1, and the last one is the entire game. We say that a strategy pair is a *subgame perfect equilibrium* iff a strategy pair indicates a Nash equilibrium to every subgame.

Question 8. Calculate the subgame perfect equilibrium in the game of Figure 4.5.

Question 9. The game of Figure 4.5 has a unique subgame perfect equilibrium, and two non-subgame perfect equilibria. If you are player 2 (boy), think about which strategy you play?

Question 10. How many subgames does the game of Figure 4.4 have?

Let us go to another famous example.

Example 4.3.(Chian-Store Game): Look at Figure 4.6. In the town in question, Chain store has already a branch, and now (potential) Retailer makes a decision to open a store or not. If Retailer chooses "in", then Chain store can choose either "aggressive" policy or "cooperative" policy. The profits for these players are given at the endnodes of the tree.

Each player has only two strategies: Retailer has "out" and "in", and Chain store has "aggressive" and "cooperative".

Let an extensive game Γ be given. Take one decision node x in Γ, and consider the upper part Γ_x of x including x itself. When all the structures of Γ are closed in Γ_x, this Γ_x is called a *subgame* of Γ. The entire game Γ itself is a subgame but not a proper subgame.

Let $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ be a strategy profile in an extensive game Γ. Let Γ' be a subgame of Γ. Then we denote the restriction of σ^* to Γ' by $\sigma^{*\prime}$. We say that $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ is a *subgame perfect equilibrium* iff for any subgame Γ' of Γ, the restriction of $\sigma^{*\prime}$ to Γ' is a Nash equilibrium in Γ'.

Question 11. How many subgames does the chain-store game have?

Question 12. Calculate the subgame perfect equilibrium in the chain-store game.

Question 13. Obtain all the Nash equilibria in this game.

Example 4.4.(Chian-Store Game with two towns): Now, consider the situation where there are two towns retailers $R1$ and $R2$. The game is played as follows: In town 1, retailer $R1$ and chain-store C play the game of Example 4.3. Then, the result of the first town is informed to town 2. Then retailer $R2$ and chain-store C play the same game. This situation is described as Figure 4.8.

The entire situation is described as an extensive game in Figure 4.8.

Question 14. How many subgames does this game have?

Question 15. Calculate the subgame perfect equilibrium in this game.

Question 16. How many strategies does Chain-store have?

Question 17. Think about the chain-store game with 3 towns.

Figure 3.9 is an extensive game description of the chain-store with two-towns. How many subgames are in this extensive game?

The calculation of the subgame perfect equilibrium is easy. Indicate the subgame perfect equilibrium.

However, the number of strategies for player C is now quite large. How many strategies does C have? Then, how many strategies does each R1 and R2 have?

4.4 Twice-Repeated Prisoner's Dilemma Γ_2

Let consider the situation where the game of "Prisoner's Dilemma" is repeated twice with the same players. We denote this situation by Γ_2. After the first round is finished, both players observe what are played in the first round. In the entire game Γ_2, the payoff for each player is assumed to be the sum of the outcome of each round. The entire situation is described as the extensive game of Fig.3.10.

Question 18. What are the subgames in this twice-repeated prisoner's dilemma?

Question 19. How many strategies does each player have?

Question 20. Calculate the subgame perfect equilibrium in this game.

4.5 The Number of Strategies in the k-times Repeated Prisoner's Dilemma Γ_k

We have adopted the definition of a *strategy* for a player to be a function over the set of his information sets. However, a strategy defined in this manner may contain some redundancy. For example, the twice-repeated Prisoner's Dilemma Γ_2 depicted in the following. Consider the strategy σ_1 defined by

$$\sigma_1(I_{10}) = L, \sigma_1(I_{11}) = L, \sigma_1(I_{12}) = L, \text{ and } \sigma_1(I_{13}) = R, \sigma_1(I_{14}) = R.$$

In this strategy, the specifications $\sigma_1(I_{13}) = R$ and $\sigma_1(I_{14}) = R$ are redundant, since this part is avoided by σ_1 itself and does not happen at all. In this sense, the following strategy σ_1' can be regarded as "equivalent" to σ_1:

$$\sigma_1'(I_{10}) = L, \sigma_1'(I_{11}) = L, \sigma_1'(I_{12}) = L, \text{ and } \sigma_1'(I_{13}) = L, \sigma_1'(I_{14}) = L.$$

One procedure of eliminating this redundancy is to consider the equivalence classes of strategies.

Let σ_1 be a strategy. We say that an information I for player 1 is *compatible with* σ_1 iff there is another strategy σ_2 such that the realization path of (σ_1, σ_2) intersects I. We say that σ_1 is r-equivalent to σ_1' iff $\sigma_1(I) = \sigma_1'(I)$ for any information set I compatible with σ_1, in which case we write $\sigma_1 \triangleq \sigma_1'$.

Question 1. The relation $\triangleq$ is an equivalence relation, i.e., it satisfies (1) Reflexivity: $\sigma_1 \triangleq \sigma_1$ for all σ_1; (2) Symmetry: $\sigma_1 \triangleq \sigma_1'$ implies $\sigma_1' \triangleq \sigma_1$;and (3) Transitivity: $\sigma_1 \triangleq \sigma_1'$ and $\sigma_1' \triangleq \sigma_1''$ imply $\sigma_1 \triangleq \sigma_1''$.

We would like to count r-equivalent strategies only once. This can be done by considering the equivalent classes:

$$[\sigma_1] = \{\sigma_1' : \sigma_1' \triangleq \sigma_1\} \text{ for strategy } \sigma_1. \tag{12}$$

Now, our question is to count the number of the equivalent classes of strategies for player 1.

Question 2. How many equivalence classes of strategies for 1 in Γ_2?

In general, the number of equivalence classes of strategies for 1 in the k-time repeated Prisoner's Dilemma Γ_k is

$$2^{2^{k-1}+2^{k-2}+\ldots+2+1} \ (= 2^{2^k-1}). \tag{13}$$

Question 3. What is this number for $k = 7$?

Question 4. Prove (13) (Hint: use induction on k).

5 Mixed Strategies and Existence of A Nash Equilibrium

5.1 Mixed Extension

Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be an n-person normal form game. Here, $S_i = \{\mathbf{s}_{i1}, ..., \mathbf{s}_{i\ell_i}\}$ is the finite set of pure strategies for $i = 1, ..., n$.

Now, we allow each player i to choose a pure strategy by using a random mechanism. A random mechanism is called a *mixed strategy*. In other words, it is a probability distribution $x_i = (x_{i1}, ..., x_{i\ell_i})$ over S_i. Since x_i is a probability distribution, it holds:

$$\sum_{t=1}^{\ell_i} x_{it} = 1 \text{ and } x_{it} \geq 0 \text{ for } t = 1, ..., \ell_i. \tag{14}$$

We denote the set of all mixed strategies (probability distributions over S_i) by $\Delta(S_i)$.

We take the expected payoff for each player i : When each player i chooses his mixed strategy x_i for $i \in N$, the probability of a pure strategy combination $(\mathbf{s}_{1t_1}, ..., \mathbf{s}_{nt_n})$ to happen is

$$x_{1t_1} \times \cdots \times x_{nt_n}. \tag{15}$$

With this probability, player i gets his payoff $h_i(\mathbf{s}_{1t_1}, ..., \mathbf{s}_{nt_n})$. We take the expected sum of these payoffs:

$$\hat{h}_i(x_1, ..., x_n) = \sum_{t_1=1}^{\ell_1} \cdots \sum_{t_n=1}^{\ell_n} (x_{1t_1} \times \cdots \times x_{nt_n}) \times h_i(\mathbf{s}_{1t_1}, ..., \mathbf{s}_{nt_n}) \tag{16}$$

for each $(x_1, ..., x_n) \in \Delta(S_1) \times \cdots \times \Delta(S_n)$.

Now, we have the triple:

$$\hat{G} = (N, \{\Delta(S_i)\}_{i \in N}, \{\hat{h}_i\}_{i \in N}), \tag{17}$$

which is called the *mixed extension* of the game $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$.

Let us see one example:

Example 5.1. Consider the matching pennies:

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(1,-1)$	$(-1,1)$
$\mathbf{s}_{12}$	$(-1,1)$	$(1,-1)$

The mixed extension of this game is as $N = \{1,2\}$, $\Delta(S_1) = \{(x_{11}, x_{12}) : x_{11} + x_{12} = 1$ and $x_{11}, x_{12} \geq 0\}, \Delta(S_2) = \{(x_{21}, x_{22}) : x_{21} + x_{22} = 1$ and $x_{21}, x_{22} \geq 0\}$, and

$$\hat{h}_1(x_1, x_2) = 1 \times x_{11} \times x_{21} + (-1) \times x_{11} \times x_{22} + (-1) \times x_{12} \times x_{21} + 1 \times x_{12} \times x_{22}.$$

Question 1. Formulate $\hat{h}_2(x_1, x_2)$ by yourself.

Question 2. What is $\Delta(S_1)$ of this example, geometrically?

Example 5.2. Consider the Scissors-Rock-Paper:

	Sc	**Ro**	**Pa**
Sc	0	-1	1
Ro	1	0	-1
Pa	-1	1	0

The mixed extension of this game is given as $N = \{1,2\}$, $\Delta(S_1) = \{(x_{11}, x_{12}, x_{13}) : x_{11} + x_{12} + x_{13} = 1$ and $x_{11}, x_{12}, x_{23} \geq 0\}, \Delta(S_2) = \{(x_{21}, x_{22}, x_{23}) : x_{21} + x_{22} + x_{23} = 1$ and $x_{21}, x_{22}, x_{23} \geq 0\}$, and

$$\begin{aligned}\hat{h}_1(x_1, x_2) &= 0 \times x_{11} \times x_{21} + (-1) \times x_{11} \times x_{22} + 1 \times x_{11} \times x_{23} \\ &\quad +1 \times x_{12} \times x_{21} + 0 \times x_{12} \times x_{22} + (-1) \times x_{12} \times x_{23} \\ &\quad +(-1) \times x_{13} \times x_{21} + 1 \times x_{13} \times x_{22} + 0 \times x_{13} \times x_{23}.\end{aligned}$$

The sets $\Delta(S_1)$ and $\Delta(S_2)$ of mixed strategies are identical to

$$\Delta^2 = \{(a_1, a_2, a_3) : a_1 + a_2 + a_3 = 1 \text{ and } a_1, a_2, a_3 \geq 0\}.$$

It is described by the regular triangle of height 1 with coordinates a_1, a_2, a_3, where each a_i is the height of the perpendicular from the point a to the side i. In this case, we can prove $a_1 + a_2 + a_3 = 1$ and $a_1, a_2, a_3 \geq 0$.

Question 3. Prove $a_1 + a_2 + a_3 = 1$ and $a_1, a_2, a_3 \geq 0$. In what sense, is Δ^2 identical to the regular triangle of height 1? This Δ^2 is called the 2-*dimensional simplex.* Why is this called 2-dimensional rather than 3-dimensional?

Question 4. The 3-dimensional simplex is given as

$$\Delta^3 = \{(a_1, a_2, a_3, a_4) : a_1 + a_2 + a_3 + a_4 = 1 \text{ and } a_1, a_2, a_3, a_4 \geq 0\}.$$

Think about the geometric representation of Δ^3 in the 3-dimensional space.

We regard pure strategy $\mathbf{s}_{ik}$ in S_i as equivalent to the ℓ_i-dimensional unit vector $e^k(\ell_i) = (0, ..., 0, 1, 0, ..., 0)$ of $k-1$ number of 0 and of the k-th component 1 in that in the latter, pure strategy $\mathbf{s}_{ik}$ is chosen with probability 1 and the other pure strategies are not chosen. Rigorously, we define the function $\psi : S_i \to \Delta(S_i)$ $(i \in N)$ by

$$\psi(s_{ik}) = e^k(\ell_i) \text{ for } k = 1, ..., \ell_i.$$

By this function, we can regard S_i as a subset of $\Delta(S_i)$, i.e.,

$$\psi(S_i) \subseteq \Delta(S_i). \tag{18}$$

Also, we have

$$h_i(s_1, ..., s_n) = \hat{h}_i(\psi_1(s_1), ..., \psi_n(s_n)) \text{ for all } (s_1, ..., s_n) \in S_1 \times ... \times S_n.$$

In this respect, the mixed extension $\hat{G}$ is really an extension of the original finite game G.

5.2 Nash Equilibrium in Mixed Strategies

Let $\hat{G} = (N, \{\Delta(S_i)\}_{i \in N}, \{\hat{h}_i\}_{i \in N})$ be the mixed extension of a finite game $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$. A profile of mixed strategies $x^* = (x_1^*, ..., x_n^*)$ $(x_i^* \in \Delta(S_i)$ for $i \in N)$ is called a *Nash equilibrium* iff for all $i \in N$,

$$\hat{h}_i(x_i, x_{-i}^*) \leq \hat{h}_i(x_i^*, x_{-i}^*) \text{ for all } x_i \in \Delta(S_i). \tag{19}$$

Here, we are using the notation: $x^*_{-i} = (x^*_1, ..., x^*_{i-1}, x^*_{i+1}, ..., x^*_n)$ and $(x_i, x^*_{-i}) = (x^*_1, ..., x^*_{i-1}, x_i, x^*_{i+1}, ..., x^*_n)$. Thus, (x^*_i, x^*_{-i}) is x^* itself.

The following is the famous theorem due to John F. Nash.

Theorem 5.1 (Nash (1951)). Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be an n-person finite normal form game. Then, the mixed extension $\hat{G} = (N, \{\Delta(S_i)\}_{i\in N}, \{\hat{h}_i\}_{i\in N})$ has a Nash equilibrium.

Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be a 2-person zero-sum game. Then, the mixed extension $\hat{G}$ also satisfies the zero-sum condition. Since a Nash equilibrium becomes a saddle point, it follows from Theorem 5.1 that the mixed extension $\hat{G}$ has a saddle point.

Corollary 5.2. Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be an 2-person zero-sum game. Then, the mixed extension $\hat{G} = (N, \{\Delta(S_i)\}_{i\in N}, \{\hat{h}_i\}_{i\in N})$ has a saddle point with respect to $\hat{h}_1$.

From this corollary and the mixed strategy analogue of Theorem **??**.2, we have the following theorem.

Theorem 5.3. **(von Neumann (1928))**. Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be an 2-person zero-sum game. In the mixed extension $\hat{G} = (N, \{\Delta(S_i)\}_{i\in N}, \{\hat{h}_i\}_{i\in N})$, we have

$$\max_{x_1\in\Delta(S_1)} \min_{x_2\in\Delta(S_2)} \hat{h}_1(x_1, x_2) = \min_{x_2\in\Delta(S_2)} \max_{x_1\in\Delta(S_1)} \hat{h}_1(x_1, x_2). \tag{20}$$

Theorem 5.1 is proved by applying Brouwer's fixed point theorem (or Kakutani's fixed point theorem). Now, we present Brouwer's fixed point theorem.

Let (R^m, d) be the m-dimensional Euclidean space with the Euclidean metric d, where

$$d(x, y) = \sqrt{\sum_{t=1}^{m}(x_t - y_t)^2} \text{ for } x, y \in R^m.$$

We say that a sequence $\{x^\nu\}$ *converges to* x^0 iff the sequence of real numbers $\{d(x^\nu, x^0)\}$ converges to 0.

Let T be a subset of R^m, i.e., $T \subseteq R^m$. We say that T is *closed* (in the topological sense) iff for any sequence $\{x^\nu\}$ in T, if $\{x^\nu\}$ converges to x^0 (in R^m), then x^0 belongs to

T.

Example 5.1. The interval $[0, 1]$ is closed, but $(0, 1]$ is not closed.

Example 5.2. The m dimensional simplex Δ^m is closed.

Let T be a subset of R^m. We say that T is *bounded* iff there is a number M such that $d(0, x) \leq M$ for all $x \in T$.

Example 5.3. The interval $[0, 1]$ is bounded, but $[0, +\infty)$ is not bounded.

Example 5.4. The m dimensional simplex Δ^m is bounded.

We say that a subset T of R^m is *compact* iff T is closed and bounded. Hence, the interval $[0, 1]$ is compact, and the m-dimensional simplex is compact, too. What are not compact?

Let T be a subset of R^m. We say that T is *convex* iff for any $x, y \in T$ and $\lambda \in [0, 1]$, the convex combination $\lambda x + (1 - \lambda)y$ belongs to T.

Example 5.5. The intervals $[0, 1]$ and $[0, +\infty)$ are convex, but the union $[0, 1) \cup (1, 2]$ is not convex.

Example 5.6. The m -dimensional simplex Δ^m is convex.

Now, let f be a function from T to T. We say that f is continuous iff for any sequence $\{x^\nu\}$ in T, if $\{x^\nu\}$ converges $x^0 \in T$, then $\{f(x^\nu)\}$ converges $f(x^0)$.

Now, we can present Brouwer's fixed point.

Theorem 5.4 (Brouwer (1908?)). Let T be a nonempty compact convex subset of R^m, and let f be a continuous function from T to T. Then f has a fixed point x^0 in T, i.e., $f(x^0) = x^0$.

Example 5.7. The 1-dimensional version of Brouwer's fixed point theorem is depicted in Figure 5.2. What are fixed points in this figure?

Some of you learned the intermediate value theorem in the calculus class:

Intermediate Value Theorem. Let f be a continuous function from the closed interval

$[a, b]$ to R. Let $f(a) > f(b)$ ($f(a) < f(b)$). For any value y_0 with $f(a) > y_0 > f(b)$ ($f(a) < y_0 < f(b)$), there is a $c \in [a, b]$ such that $f(c) = y_0$.

5.3 The Best-Response Diagram

For a 2-person game with two pure strategies for each player, it is possible calculate all the Nash equilibria by the best response diagram. Consider the matching pennies:

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(1, -1)$	$(-1, 1)$
$\mathbf{s}_{12}$	$(-1, 1)$	$(1, -1)$

We denote x_{11} by p and x_{21} by q. Then, a mixed strategy pair (x_1, x_2) is written as

$$((p, 1-p), (q, 1-q)).$$

Then, we abbreviate the payoff functions $\hat{h}_1((p, 1-p), (q, 1-q))$ and $\hat{h}_2((p, 1-p), (q, 1-q))$ as

$$\hat{h}_1(p, q) \quad \text{and} \quad \hat{h}_2(p, q).$$

Then we can calculate

$$\begin{aligned} \hat{h}_1(p, q) &= pq + (-1)p(1-q) + (-1)(1-p)q + (1-p)(1-q) \\ &= 4pq - 2p - 2q + 1 \\ &= p(4q-2) - 2q + 1. \end{aligned} \tag{21}$$

Now, we are ready to obtain the *best response correspondence*.

Recall that player 1 controls p but q is just given for him. To maximize the last term of (21)by controlling p, we have the following condition:

$$\begin{aligned} 4q - 2 &> 0 \Longrightarrow p = 1 \\ 4q - 2 &= 0 \Longrightarrow p \text{ is arbitrary} \\ 4q - 2 &< 0 \Longrightarrow p = 0. \end{aligned}$$

We depict this condition in Figure 5.4.

We repeat the parallel calculation for player 2 : First, we have

$$\begin{aligned}\hat{h}_2(p,q) &= (-1)pq + p(1-q) + (1-p)q + (-1)(1-p)(1-q) \quad (22)\\ &= -4pq + 2p + 2q - 1\\ &= q(-4p+2) + 2p - 1.\end{aligned}$$

Then, we have the condition for the best response:

$$\begin{aligned}-4p+2 &> 0 \Longrightarrow q = 1\\ -4q+2 &= 0 \Longrightarrow q \text{ is arbitrary}\\ -4q+2 &< 0 \Longrightarrow q = 0.\end{aligned}$$

Again, we depict this condition in the Figure 5.4.

Question 5. Obtain the Nash equilibria in the Battle of the Sexes within mixed strategies, using the best-response diagram.

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(2,0)$	$(0,0)$
$\mathbf{s}_{12}$	$(0,0)$	$(1,2)$

Question 6. Obtain the Nash equilibria in the Prisoner's Dilemma within mixed strategies, using the best-response diagram.

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(5,5)$	$(1,6)$
$\mathbf{s}_{12}$	$(6,1)$	$(3,3)$

5.4 A Game with Incomplete Information

Usually, a person does not know other people's preferences, while he himself knows his own preferences. This idea is described by a game with incomplete information, formulated by Harsanyi (1967). Here, we consider only one example.

Let us denote the Prisoner's Dilemma and the Battle of the Sexes by (g_1, g_2) and (h_1, h_2), respectively. The game with incomplete information is formulated as

(1): the chance move chooses one pair from $\{(g_1, g_2), (g_1, h_2), (h_1, g_2), (h_1, h_2)\}$ with the same probability 1/4;

(2): each player $i = 1, 2$ gets information about what his payoff is, i.e., g_i or h_i;

(3): then each player chooses (pure or mixed) strategy;

(4): both players get payoffs.

This game is formulated as an extensive game of Figure 5.5.

Question 7. How many pure strategies of each player does have in this game?

Question 8. Think about the normal form game for the extensive game of Figure 5.5.

The above game is also formulated as a normal form game: $(N, \{S_1, S_2, \{h_1, h_2\})$ given as:

(1): $N = \{1, 2\}$;

(2): $S_1 = S_2 = \{(L, L), (L, R), (R, L), (R, R)\}$ (since each player has two information sets);

(3): $(h_1((L, L), (L, L)), h_2((L, L), (L, L)) = \frac{1}{4}(5, 5) + \frac{1}{4}(5, 1) + \frac{1}{4}(2, 5) + \frac{1}{4}(2, 1) = (\dfrac{14}{4}, \dfrac{12}{4})$

$(h_1((L, R), (L, L)), h_2((L, R), (L, L)) = \frac{1}{4}(5, 5) + \frac{1}{4}(5, 1) + \frac{1}{4}(6, 0) + \frac{1}{4}(0, 0) = (\dfrac{11}{4}, \dfrac{6}{4})$

$(h_1((R, L), (L, L)), h_2((R, L), (L, L)) = \frac{1}{4}(6, 1) + \frac{1}{4}(6, 0) + \frac{1}{4}(2, 5) + \frac{1}{4}(2, 1) = (3.5, 1.5)$

$(h_1((L, R), (L, R)), h_2((L, R), (L, R)) = \frac{1}{4}(5, 5) + \frac{1}{4}(1, 0) + \frac{1}{4}(6, 0) + \frac{1}{4}(1, 2) = (\dfrac{13}{4}, \dfrac{7}{4})$

. . .

$$(h_1((R,R),(R,R)), h_2((R,R),(R,R)) = \tfrac{1}{4}(3,3) + \tfrac{1}{4}(3,2) + \tfrac{1}{4}(1,3) + \tfrac{1}{4}(1,2) = (2,5)$$

	(L,L)	(L,R)	(R,L)	(R,R)
(L,L)	$(3.5,\ 3)$			
(L,R)		$(\frac{13}{4}, \frac{7}{4})$		
(R,L)			$(3.5,\ 1.5)$	
(R,R)				$(2,\ 5)$

5.5 The Region of Attainable Expected Payoffs

In game theoretical decision making, it is a basic assumption that each player makes independent choice. In the case of mixed strategies, each player uses an independent mixed strategy. This independence choice may lead to a mathematical interesting problem of the region of the attainable expected payoffs. In this subsection, we will discuss this fact.

First, consider the Prisoner's Dilemma of Table 1:

Table 1

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(5,5)$	$(1,6)$
$\mathbf{s}_{12}$	$(6,1)$	$(2,2)$

Table 2: Battle of the Sexes

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$
$\mathbf{s}_{11}$	$(2,1)$	$(0,0)$
$\mathbf{s}_{12}$	$(0,0)$	$(1,2)$

When players 1 and 2 use mixed strategies $(p, 1-p)$ and $(q, 1-q)$, the pair of their expected payoffs is given as:

$$\begin{aligned}(\hat{h}_1(p,q), \hat{h}_2(p,q)) &= pq(5,5) + p(1-q)(1,6) + (1-p)q(6,1) + (1-p)(1-q)(2,2) \\ &= p[q(5,5) + (1-q)(1,6)] + (1-p)[q(6,1) + (1-q)(2,2)].\end{aligned}$$

First, look at two expressions: $q(5,5) + (1-q)(1,6)$ and $q(6,1) + (1-q)(2,2)$. The first expression $q(5,5) + (1-q)(1,6)$ is the convex combination of the two points $(5,5)$ and $(1,6)$ with weights q and $1-q$. This is depicted in Figure 5.6 with $q = \frac{1}{3}$. Also, the expected payoff axes are taken as u_1 and u_2.

Question 9. Where is the location of $q(6,1) + (1-q)(2,2)$ in Figure 5.6?

Now, consider the pair of final expected payoffs: $p[q(5,5)+(1-q)(1,6)]+(1-p)[q(6,1)+(1-q)(2,2)]$. It is the convex combination of the two points $A(q) = [q(5,5)+(1-q)(1,6)]$ and $B(q) = [q(6,1) + (1-q)(2,2)]$ with weights p and $1-p$. When $p = \frac{2}{3}$, it is depicted in Figure 5.6.

When p varies from 0 to 1, the point of the pair of expected payoffs moves from $B(q)$ to $A(q)$. When q varies from 0 to 1,the point $pA + (1-p)q$ moves from the base line to the upper line of the parallelogram. Therefore, all points in the region determined by the four points are achievable with some strategies $(p, 1-p)$ and $(q, 1-q)$. One consequence is that the set of achievable points is a convex set.

Question 10. Calculate the Nash equilibrium within mixed strategies in the above Prisoner's Dilemma. Plot the payoff vector given by the NE.

Now, consider the Battle of the Sexes: When players 1 and 2 use mixed strategies $(p, 1-p)$ and $(q, 1-q)$, the pair of their expected payoffs is given as:

$$
\begin{aligned}
& (\hat{h}_1(p,q), \hat{h}_2(p,q)) && (23)\\
= \;& pq(2,1) + p(1-q)(0,0) + (1-p)q(0,0) + (1-p)(1-q)(1,2) \\
= \;& p[q(2,1) + (1-q)(0,0)] + (1-p)[q(0,0) + (1-q)(1,2)].
\end{aligned}
$$

Draw the diagram of the region of attainable payoffs following the above procedure using the formula (23).

Question 11. For the Battle of the Sexes, the region of attainable payoffs is not convex. Think about why.

Question 12. Calculate the Nash equilibrium within mixed strategies in the Battle of the Sexes. Plot the payoff vectors given by those equilibria.

Consider (23). We denote $(\hat{h}_1(p,q), \hat{h}_2(p,q))$ by (u_1, u_2). Then a pair of expected payoffs achievable by a mixed strategy pair is expressed by

$$u_1 = 3pq - p - q + 1 \qquad (1)$$

$$u_2 = 3pq - 2p - 2q + 2 \qquad (2)$$

for some $p \in [0,1]$ and $q \in [0,1]$. Our interest is to consider what region is generated by such a pair (u_1, u_2).

From (1) we have $p = (u_1 + q - 1)/(3q - 1)$ assuming $q \neq 1/3$. Plugging this to (2), we have

$$u_2 = \frac{u_1 + q - 1}{3q - 1}(3q - 2) - 2q + 2,$$

which is changed into

$$0 = -3q^2 + q(3u_1 - 3u_2 + 3) + (-2u_1 + u_2). \tag{24}$$

When the simultaneous equation of (1) and (2) has a solution, equation (24) has also a solution. We regard (24) as an quadratic equation with an unknown q. Then the condition for this quadratic equation to have a solution is:

$$(3u_1 - 3u_2 + 3)^2 - 4 \times (-3) \times (-2u_1 + u_2) \geq 0.$$

Sometimes, (24) has a unique solution. The condition for a unique solution is:

$$\begin{aligned} 9u_1^2 + 9u_2^2 - 18u_1u_2 - 6u_1 - 6u_2 + 9 &= 0, \text{ that is,} \qquad (25) \\ 3u_1^2 + 3u_2^2 - 6u_1u_2 - 2u_1 - 2u_2 + 3 &= 0. \end{aligned}$$

This is the equation describing the *envelop curve* determined by (1) and (2).

Coordination: Now, let us allow the two players to coordinate their probability distributions. That is, they can use a joint strategy. They have now only one strategy set. In the case of Battle of the Sexes, they have

$$\{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1 \text{ and } \alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0\}. \tag{26}$$

Here, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ means a probability distribution so that $(\mathbf{s}_{11}, \mathbf{s}_{21}), (\mathbf{s}_{11}, \mathbf{s}_{22}), (\mathbf{s}_{12}, \mathbf{s}_{21})$ and $(\mathbf{s}_{12}, \mathbf{s}_{22})$ are chosen with probabilities $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, respectively. This set is described as

$$\Delta(S_1 \times S_2). \tag{27}$$

Now, the set $\Delta(S_1 \times S_2)$ differs from $\Delta(S_1) \times \Delta(S_2)$.

Question 13. Depict the sets $\Delta(S_1) \times \Delta(S_2)$ and $\Delta(S_1 \times S_2)$ in geometric ways. Also, $\Delta(S_1) \times \Delta(S_2)$ can be regarded as a (proper) subset of $\Delta(S_1 \times S_2)$. Think about why.

Now, consider the attainable set of payoff vectors: For a 2-person game with 2 pure strategies for each player, the attainable set by joint strategies are given as

$$U(G) = \{\alpha_1 \mathbf{a}^1 + \alpha_2 \mathbf{a}^2 + \alpha_3 \mathbf{a}^3 + \alpha_4 \mathbf{a}^4 : (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Delta(S_1 \times S_2)\}, \tag{28}$$

where $\mathbf{a}^1 = (h_1(\mathbf{s}_{11}, \mathbf{s}_{21}), h_2(\mathbf{s}_{11}, \mathbf{s}_{21}))$, $\mathbf{a}^2 = (h_1(\mathbf{s}_{11}, \mathbf{s}_{22}), h_2(\mathbf{s}_{11}, \mathbf{s}_{22}))$, $\mathbf{a}^3 = (h_1(\mathbf{s}_{12}, \mathbf{s}_{21}),$ $h_2(\mathbf{s}_{12}, \mathbf{s}_{21}))$ and $\mathbf{a}^4 = (h_1((\mathbf{s}_{12}, \mathbf{s}_{22}), h_2(\mathbf{s}_{12}, \mathbf{s}_{22}))$.

Question 14. Prove that $U(G)$ is a convex subset of R^2.

When we allow only independent mixed strategies, the attainable set becomes

$$\begin{aligned} U_I(G) &= \{pq\mathbf{a}^1 + p(1-q)\mathbf{a}^2 + (1-p)q\mathbf{a}^3 + (1-p)(1-q)\mathbf{a}^4 \\ &: (p, 1-p) \in \Delta(S_1) \text{ and } (q, 1-q) \in \Delta(S_2)\}. \end{aligned} \tag{29}$$

Question 15. Give a game where the set U_I of attainable payoffs without coordination is nonconvex.