Derivatives in Economics

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Derivatives appear in economics! The term "**marginal**" are used to denote the rate of change of a quantity with respect to the variable on which it depends. Therefore, the "marginal" of a function means the "derivative" of a function! For example, a manufacturer produces a product and the **cost of product**, $C(x)$, depends on the number of units of produced products, x. The "**marginal cost of production**" is the extra cost of producing one more unit, or

$$\Delta C = C(x+1) - C(x) = \frac{C(x+1) - C(x)}{1}.$$

If the product is divisible, we can let it be as small as we want and the marginal cost becomes

$$\lim_{\Delta x \to 0} \Delta C = \frac{C(x+\Delta x) - C(x)}{\Delta x} = C'(x).$$

In the same way, we define the "**marginal revenue**" as the derivative of the revenue function, and the "**marginal utility**" as the derivative of the utility function, etc.

Example: In a mining operation the cost C (in dollars) of extracting each tonne of ore is

$$C(x) = \frac{2000}{x} + 20x - 400$$

where x is the number of tonnes extracted each day.

1. When does the mining operation have negative marginal cost? In this range of x, the mining operation has "economies of scale."

2. When does the mining operation have positive marginal cost? In this range of x, equipments are overloaded and labour work overtime.

3. If each tonnes of ore can be sold for \$15, how many tonnes of ore should be extracted each day to maximize the daily profit of the mine?

Economists usually use different scales of measurement when counting production quantity (in thousands vs. in millions), or pricing (using different currency). This creates a problem for using the derivative $f'(p)$ to describe the sensitivity of the demand y for a certain product to the price p charged for the product: $y = f(p)$.

Instead, economists use the **(arc) elasticity of demand** to describe price sensitivity, which is defined as the percentage change of quantity demanded divided by the percentage change in prices. To avoid denominator choice affecting percentages, economists use the midpoint:

$$\epsilon_d(p_1, p_2) = \frac{\frac{|\Delta y|}{\overline{y}}}{\frac{|\Delta p|}{\overline{p}}} = \frac{\frac{|y_2 - y_1|}{(y_2 + y_1)/2}}{\frac{|p_2 - p_1|}{(p_2 + p_1)/2}}$$

where $y_2 = f(p_2)$, $y_1 = f(p_1)$, $\Delta y = y_2 - y_1$, $\Delta p = p_2 - p_1$, $\overline{y} = \frac{y_2+y_1}{2}$, and $\overline{p} = \frac{p_2+p_1}{2}$.

When the units are divisible, we can let $p_2 = p + \Delta p$ approximate the starting point $p_1 = p$ and a continuous demand function would result in $y_2 = f(p+\Delta p)$ approximating $y_1 = f(p) = y$. Hence, we obtain the **point elasticity of demand**

$$\lim_{\Delta p \to 0} \epsilon_d(p, p + \Delta p) = \left| \frac{p}{y} \cdot \frac{dy}{dp} \right|$$

Example: Mayor Hua-Ma dropped KRT's rush hour ticket price from NT\$25 to NT\$0 during winter time (December 1, 2017 to February 28, 2018). On the first Monday, Liberty Times reported that the number of rush hour KRT rides increased by 12,109 to reach 69,559.

1. What is the arc elasticity of demand for rush hour KRT rides with $p_1 = 25$ and $p_2 = 0$? Does your answer change if you use US\$ instead of NT\$? Why or why not? (*Hint:* What happens to elasticity when you set $\hat{y} = Cy$ and $\hat{p} = Dp$, C, D constants?)

2. Find the linear approximation $y = A + Bp$ for the demand for KRT rides that passes through the above two data points. What is its point elasticity of demand at $p_1 = 25$ and $p_2 = 0$? Does the slope or point elasticity vary when you choose different units of measurement, such as "thousands of passengers"?

3. What is the point elasticity of demand when demand takes the functional form $y = Kp^{-r}$, where K and r are positive constants? Does it depend on price p?