Lowest Unique Bid Auctions with Population Uncertainty*

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Abstract

We characterize the unique Poisson-Nash equilibrium of the *l*owest *u*nique *b*id *a*uction (LUBA) when the number of bidders is uncertain and follows a Poisson distribution.

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1 Introduction

In the *l*owest *u*nique *b*id *a*uction (LUBA), players submit bids for an object, and the one who submits the lowest unique bid wins the object and pays her bid. In recent years this auction format has gained considerable popularity in online markets.[1] LUBA has also spurred academic interest and has been studied both theoretically and empirically (De Wachter and Norman, 2006; Rapoport, Otsubo, Kim and Stein, 2013; Eichberger and Vinogradov, 2008; Raviv and Virag, 2009; Houba, Laan and Veldhuizen, 2011; Scarsini, Solan and Vieille, 2010; Gallice, 2013; Pigolotti, Bernhardsson, Juul, Galster and Vivo, 2012; Radicchi, Baronchelli and Amaral, 2012; Costa-Gomes and Shimoji, 2014; Chakraborty, Tammana, Ganguly and Mukherjee, 2014). Although substantial progress has been made, analyzing LUBA has proven to be complicated. For example, many papers have focused on numerical computations of symmetric equilibria of LUBA. However, it has not yet been shown that there is a unique equilibrium. In this paper, we show that it is straightforward to show uniqueness if one is willing to assume that the number of players is uncertain. This is indeed the case in many online LUBAs—there is often a set deadline and everybody that submits a bid before the deadline gets to participate—which implies that there is uncertainty about the total number of bidders.

Specifically, following Östling, Wang, Chou and Camerer's (2011) study of the *l*owest *u*nique *p*ositive *i*nteger (LUPI) game, we assume that the number of players is Poisson distributed with mean n, which allows us to use results derived by Myerson (1998, 2000). We show that the Poisson-Nash equilibrium of the Poisson LUBA is unique and satisfy a number of theoretical properties, some of which has previously been shown to hold for the fixed-N equilibrium. Furthermore, the Poisson LUBA equilibrium can easily be computed and is practically indistinguishable from previously reported numerical computations of the fixed-N equilibrium. In addition, we show that the Poisson LUBA can be extended to take entry and risk preferences as well as private values of bidders into account.[2]

2 Poisson LUBA

In the lowest unique bid auction studied here, N players simultaneously submit one bid $k \in \mathbb{N}$ each.[3] The player who submits the lowest unique bid wins a prize. If there is no unique bid, there is no winner. Each participant pays a fixed fee $f \geq 0$. The winner wins

[1] See Eichberger and Vinogradov (2008) and Gallice (2013) for some information about online LUBAs.

[2] All papers on LUBA except Gallice (2013) and Costa-Gomes and Shimoji (2014) work exclusively with known common values.

[3] Scarsini et al. (2010) provide some results for the case of multiple bids, under a severe restriction on the strategy space. Gallice (2013) allows for multiple rounds with one bid per round. He assumes that the probability of winning is given by a Tullock contest.

a prize with common value $V > 1$ and pays her bid k. We assume that bidders maximize expected utility and that their utility is strictly increasing in the monetary payoff. We may without loss of generality restrict attention to the pure strategy set $S = \{1, 2, ..., K\}$ for some K. Consequently the mixed strategy space is the $(K-1)$-dimensional simplex $\Delta = \left\{p \in \mathbb{R}_+^K : \sum_{i=1}^K p_i = 0\right\}$.

Let $k^*(s)$ denote the winning number under pure strategy profile $s = (s_1, ..., s_N) \in S^N$:

$$k^*(s) = \min_{j \in \{k \in \{1,...,N\} : s_k \neq s_l, \forall l \neq k\}} s_j.$$

The payoff to a player choosing the number $k \in S$ under pure strategy profile s is

$$u_k(s) = \begin{cases} u(V - k - f) & \text{if } k = k^*(s), \\ u(-f) & \text{otherwise.} \end{cases}$$

Note that bidding $k > V$ is weakly dominated by bidding 1.[4] Moreover, in equilibrium there is a positive chance of winning by bidding 1 and hence no player would bid $k > V$. Without loss of generality we therefore set $K = V$.

We imagine a large population of potential bidders. A number of individuals are drawn from this population to play the game. The number of players N is Poisson distributed with mean $n > 0$.[5] Let p denote the population average strategy, i.e. p_k is the probability that a randomly chosen player will pick pure strategy k. Let $X(k) \sim Poisson(np_k)$ be the *total* number of players who are drawn to participate and choose strategy k. As shown by Myerson (1998), from the point of view of a player that is drawn to play the game, the number of *other* individuals who are drawn to play is $Poisson(n)$, and the number of other individuals that are drawn and plays k is $Poisson(np_k)$.

Let $w_k(p)$ be the probability that an individual who is drawn to play and picks number k will win, when the population average strategy is p. This win probability is the same as in the LUPI game, and from Östling et al. (2011) we consequently know that

$$w_k(p) = \Pr(X(k) = 0) \prod_{i=1}^{k-1} \Pr(X(i) \neq 1) = e^{-np_k} \prod_{i=1}^{k-1} \left(1 - np_i e^{-np_i}\right). \tag{1}$$

The expected utility to a player putting all probability on strategy k given the population

[4] Bidding $k > V$ yields $u(V - k - f)$ when winning and $u(-f)$ when losing, so it is strictly better to lose by bidding 1 than to win by bidding $k > V$.

[5] The Poisson assumption can be motivated in different ways, for example by supposing there is a large population of m potential bidders, each of which participates with probability $r \in (0, 1)$. The probability r may depend on the process by which potential players become aware of the game, say, by surfing the internet, and may also depend on the entry cost f relative to the expected payoff of participating. The sum of participants has a binomial distribution with parameters m and r, which approximates a Poisson distribution with mean mr for large m and small r.

average strategy p is

$$\pi_k(p) = w_k(p)\left[u\left(V-k-f\right)-u\left(-f\right)\right]+u\left(-f\right).$$

A Poisson-Nash equilibrium p requires $\pi(p,p,n) \geq \pi(p',p,n)$ where $\pi(p',p,n)$ is the expected utility of strategy p' when the population plays according to strategy p.[6] In a mixed strategy equilibrium that has both k and $k+1$ in its support, $\pi_{k+1}(p) = \pi_k(p)$. Applying (1), the mixed equilibrium condition becomes:

$$e^{np_{k+1}} = \left(e^{np_k}-np_k\right)\frac{u\left(V-k-1-f\right)-u\left(-f\right)}{u\left(V-k-f\right)-u\left(-f\right)} \equiv \left(e^{np_k}-np_k\right)\frac{\Delta u_{k+1}}{\Delta u_k}. \tag{2}$$

We may now prove that there is a unique equilibrium with convex support, characterize conditions for the equilibrium to be mixed, and show that it is continuous in n.

Proposition 1 *A Poisson-Nash equilibrium $p^* = (p_1, p_2, ..., p_K)$ of the Poisson LUBA game exists and is unique. The equilibrium p^* is continuous in n, and it is mixed with convex support $\{1, 2, ..., \kappa\}$ for some $\kappa \leq K$ if n is large enough so that*

$$e^n - n > \frac{u\left(V-1-f\right)-u\left(-f\right)}{u\left(V-2-f\right)-u\left(-f\right)} = \frac{\Delta u_1}{\Delta u_2}. \tag{3}$$

Otherwise, the equilibrium p^* is for everyone to bid 1.

Proof. Existence of equilibrium follows directly from Myerson (1998). In order to prove convexity of the equilibrium support, note first that $p_1 > 0$ in any equilibrium since picking 1 would otherwise guarantee a win with the lowest possible bid. Suppose now by contradiction that the support is non-convex, i.e. that there is a k such that $p_k = 0$ and $p_{k+1} > 0$. Then, since $e^{-np_{k+1}} < 1$ for $p_{k+1} > 0$,

$$\pi_{k+1}(p) - \pi_k(p) = \left(e^{-np_{k+1}}\Delta u_{k+1} - \Delta u_k\right)\cdot\prod_{i=1}^{k-1}\left(1-np_ie^{-np_i}\right) < 0.$$

But this cannot hold in an equilibrium with $p_k = 0$ and $p_{k+1} > 0$, so we must have $p_k > 0$. By induction we have a convex support $\{1, 2, ..., \kappa\}$ for some $\kappa \leq K$.

If the Poisson-Nash equilibrium is in pure strategies, it must be that everyone bids 1 since $p_1 > 0$ in any equilibrium. The pure strategy equilibrium $p_1 = 1$ requires $\pi_2(p) - \pi_1(p) = \left[(1-ne^{-n})\Delta u_2 - e^{-n}\Delta u_1\right] \leq 0$, which gives condition (3). In order to prove that the pure-strategy equilibrium is unique, it remains to show that there is no other mixed-strategy equilibrium. Suppose to the contrary that there is another mixed-strategy

[6] Note that we cannot define asymmetric equilibria in a game with population uncertainty since participation of each player is uncertain.

equilibrium p' such that $p'_1 < 1$. The mixed-strategy equilibrium requirement $\pi_1(p') = \pi_2(p')$ implies $\left(e^{np'_1} - np'_1\right)e^{-np'_2} = \Delta u_1/\Delta u_2 \geq e^n - n$ by condition (3). Since $e^{-np'_2} < 1$ and $e^{np'_1} - np'_1 < e^n - n$, this inequality cannot hold.

In order to prove uniqueness of a mixed-strategy equilibrium, note that we can rewrite the recursive equilibrium condition (2) as

$$p_{k+1} = \frac{1}{n}\ln\left(e^{np_k} - np_k\right) + \frac{1}{n}\ln\frac{\Delta u_{k+1}}{\Delta u_k}. \tag{4}$$

The partial derivative $\partial p_{k+1}/\partial p_k = \left(e^{np_k} - 1\right)/\left(e^{np_k} - np_k\right)$ is positive, so p_{k+1} is an increasing function of p_k. Suppose there are two equilibria p and p', and their supports have possibly different end-points $\kappa \geq \kappa'$. Since p_{k+1} is uniquely determined by p_k, we must have $p_1 \neq p'_1$ for p and p' to differ. There are two cases to consider. If $\kappa = \kappa'$, we can assume without loss of generality $p_1 > p'_1$. Then, $\sum_{i=1}^{\kappa} p_i > \sum_{i=1}^{\kappa'} p'_i$ since p_{k+1} is uniquely determined by p_k and strictly increasing in p_k. This is a contradiction since probabilities must sum to one in both equilibria. Suppose instead $\kappa > \kappa'$, so that $p_k > p'_k = 0$ for all $\kappa \geq k > \kappa'$. For both p and p' to be equilibria, we must have $\pi_{\kappa'}(p') \geq \pi_{\kappa'+1}(p')$ and $\pi_{\kappa'}(p) = \pi_{\kappa'+1}(p)$. By condition (2), we have

$$e^{np_{\kappa'}} - np_{\kappa'} = e^{np_{\kappa'+1}} \cdot \frac{\Delta u_{\kappa'}}{\Delta u_{\kappa'+1}} \geq e^{np_{\kappa'+1}} \cdot \left(e^{np'_{\kappa'}} - np'_{\kappa'}\right) > e^{np'_{\kappa'}} - np'_{\kappa'}$$

since $e^{np_{\kappa'+1}} > 1$. Since $(e^{np} - np)$ is increasing in p, this implies $p_{\kappa'} > p'_{\kappa'}$. Since p_{k+1} is an increasing function of p_k, this further implies that $p_k > p'_k$ for all $k \leq \kappa'$. However, this is a contradiction, as $\sum_{k=1}^{\kappa'} p_k > \sum_{k=1}^{\kappa'} p'_k = 1$.

We now show that the equilibrium p is continuous in n. Since the equilibrium is unique, we define $p^* : \mathbb{R}_+ \to \Delta$ to be the mapping from the expected number of players, n, to the corresponding Poisson-Nash equilibrium $p^*(n)$. Suppose p^* is not continuous, so there exists a sequence $n^t \to \tilde{n}$ and $p^t \to \tilde{p}$ such that $p^t = p^*(n^t)$ for all t but $\tilde{p} \neq p^*(\tilde{n})$. (That is, p^t is a Poisson-Nash equilibrium for n^t for all t, but $\tilde{p}$ is not a Poisson-Nash equilibrium for $\tilde{n}$.) This means that there is some p' and $\varepsilon > 0$ such that $\pi(p', \tilde{p}, \tilde{n}) > \pi(\tilde{p}, \tilde{p}, \tilde{n}) + 2\varepsilon$. Since π is continuous in n and p, it follows that there is some T such that if $t > T$, $|\pi(p', \tilde{p}, \tilde{n}) - \pi(p', p^t, n^t)| < \frac{\varepsilon}{2}$ and $|\pi(\tilde{p}, \tilde{p}, \tilde{n}) - \pi(p^t, p^t, n^t)| < \frac{\varepsilon}{2}$. Hence, $\pi(p', p^t, n^t) > \pi(p^t, p^t, n^t) + \varepsilon$. This contradicts the assumption that $p^t = p^*(n^t)$. Q.E.D.

Remark 1 *The pure-strategy equilibrium condition (3) shows that all bidders would submit the lowest possible bid if and only if they expect very few opponents. In fact, for risk-neutral bidders with* $u(x) = x$, $\Delta u_1/\Delta u_2 = 1 + 1/(V-2)$ *and* $V \geq 3.4$, $n = 1$ *satisfies condition (3).*

Proposition 1 together with equilibrium condition (2) implies that it is straightforward to compute the mixed-strategy equilibrium by guessing p_1, calculating $p_2, ..., p_\kappa$ using the recursive condition and then verifying that the equilibrium probabilities sum to one. In the Online Appendix, we provide some illustrative numerical computations of the LUBA equilibrium, and compare the Poisson LUBA equilibrium with the fixed-N computations provided by Houba et al. (2011) and Costa-Gomes and Shimoji (2014). The Poisson LUBA and fixed-N Nash equilibrium are practically indistinguishable, in particular with many players (which is not entirely surprising given the possibility to approximate the multinomial distribution with a Poisson distribution).

Building on the characterization in Proposition 1 we are now able to show that the equilibrium probabilities are decreasing over the convex support, while the probability of winning, $w_k(p)$, is increasing. Also, the equilibrium p^* converges to a uniform distribution with many players, converges to the LUPI equlibrium as $V \to \infty$, and is independent of f for risk-neutral players. For the following result and the remainder of the paper, we assume that the utility function is continuously differentiable.

Proposition 2 *The Poisson-Nash equilibrium $p^* = (p_1, p_2, ..., p_K)$ of the Poisson LUBA game has the following properties for all k and $k+1$ in the support of the equilibrium:*

1. *Equilibrium probabilities are decreasing in k, $p_k > p_{k+1}$, and win probabilities are increasing in k, $w_k(p) < w_{k+1}(p)$.*
2. *As $n \to \infty$, $p_k - p_{k+1} \to 0$.*
3. *The equilibrium p^* converges to the LUPI equilibrium when $V \to \infty$.*
4. *If $u'' = 0$, the equilibrium p^* is independent of f.*

Proof. It is easy to verify that a pure-strategy equilibrium trivially satisfies these properties, so we assume condition (3) holds so that the equilibrium is mixed.

We first show that $w_k(p) < w_{k+1}(p)$ and $p_k > p_{k+1}$. The mixed-strategy equilibrium requires $\pi_{k+1}(p) = \pi_k(p)$. Since the utility of winning on k, $u(V - k - f)$, is larger than that of winning on $k+1$, $u(V - k - 1 - f)$, the probability of winning on k must be strictly lower than the probability of winning on $k+1$, i.e. $w_k(p) < w_{k+1}(p)$. It is easy to verify that $w_k(p) < w_{k+1}(p)$ is equivalent to $e^{np_{k+1}} - e^{np_k} < -np_k$. The right hand side is negative (since $p_k > 0$), so for the left hand side to be negative we need $p_k > p_{k+1}$.

To show that $p_k - p_{k+1} \to 0$ as $n \to \infty$, we rewrite the equilibrium condition (2) as

$$p_k - p_{k+1} = -\frac{1}{n} \ln\left(1 - np_k e^{-np_k}\right) - \frac{1}{n} \ln \frac{\Delta u_{k+1}}{\Delta u_k}. \qquad (5)$$

Since $p_k > p_{k+1}$, we have $1 > p_1 \geq 1/K$ (since probabilities must sum up to 1). Hence, $np_1 e^{-np_1} \leq ne^{-n/K} \to 0$ as $n \to \infty$. Thus, $p_1 - p_2 = -(1/n)\ln(1 - np_1 e^{-np_1}) - (1/n)\ln(\Delta u_{k+1}/\Delta u_k) \to 0$ as $n \to \infty$. This implies that for any $\delta_1 \in (0, 1/K)$, there is some n_1 such that $p_2 \geq p_1 - \delta_1 \geq 1/K - \delta_1$ for $n > n_1$. The argument can be iterated to show that for each k, there is some $\delta_k \in (0, 1/K - \delta_{k-1})$ such that $p_{k+1} \geq 1/K - \delta_k$, and thus $p_k - p_{k+1} \to 0$ as $n \to \infty$.

To see that the LUBA equilibrium converges to the LUPI equilibrium as $V \to \infty$, note that $\lim_{V\to\infty} \ln(\Delta u_{k+1}/\Delta u_k) = 0$, so the second term on the right hand side of (5) vanishes. The remainder of the expression is identical to condition (2) in Östling et al. (2011). Similarly, to examine the effect of f on the equilibrium, note that if $u'' = 0$, then $\Delta u_{k+1}/\Delta u_k = (V - k - 1)/(V - k)$, which is independent of f. Q.E.D.

2.1 Risk Preferences and Entry

Organizers of LUBA primarily raise revenue from the fixed fees that bidders pay and the sum of all fixed fees often vastly exceeds the value of the prize.[7] This raises the question why bidders might participate in such auctions. One potential explanation is a love for risk, i.e. a convex utility function over total wealth.[8] To illustrate that risk loving preferences can potentially explain participation in LUBA auctions, we prove two additional results.

Suppose the initial wealth level is given by $W > 0$ and that $V - 1 > f > 0$. Risk-neutral bidders will not want to enter if their expected payoff of bidding $k = 1$ is negative, which is equivalent to the condition $e^{-np_1^*(V,f,n)} < f/(V-1)$. In general, bidders would want to enter if and only if the expected utility for participating is higher than $u(W)$, where $u(W)$ is the utility of not participating, or if

$$\mathbb{E}\left[u|p^*(V,f,n)\right] = e^{-np_1^*(V,f,n)}\left[u(W + V - 1 - f) - u(W - f)\right] + u(W - f) > u(W),$$

which is equivalent to

$$e^{-np_1^*(V,f,n)} > \frac{u(W) - u(W - f)}{u(W + V - 1 - f) - u(W - f)} \equiv Q(W, V, f). \tag{6}$$

Note that if u is convex, $0 < Q < f/(V - 1) < 1$. The following result establishes

[7]In fact, organizers paid back less than 50% of the fixed fees in the closely related LUPI game (Östling et al., 2011).

[8]Other potential explanations include utility from gambling (Conlisk, 1993, Diecidue, Schmidt and Wakker, 2004) or incorrect beliefs due to "non-belief in the law of large numbers" (Benjamin, Rabin and Raymond, 2014). In the Online Appendix, we show that both these explanations are consistent with the equilibrium derived above under the assumption that players are rational, risk-neutral and the number of players (n) is interpreted as the perceived number of players rather than the actual number of players.

that it is possible to vary the expected number of players n (which auctioneers might be able to do via advertising) so that the expected utility from bidding $k = 1$ falls in this range, i.e. so that a risk-loving bidder prefers to bid, but a risk-neutral bidder prefers not to bid.

Proposition 3 *Suppose $V - 1 > f > 0$ and u is convex. For any $\delta \in [Q, f/(V-1)]$, there is an $n^* > 0$ such that $e^{-np_1^*(V,f,n)} = \delta$.*

Proof. In equilibrium, $p_k > p_{k+1}$ so that $p_1^* > 1/K$ for all finite n and $\lim_{n\to\infty} p_1^* = 1/K$. It follows that $\lim_{n\to\infty} e^{-np_1^*(V,f,n)} = \lim_{n\to\infty} e^{-nK} = 0$. Moreover, since e^{-np_1} is decreasing in p_1 we have $1 = e^0 \geq e^{-np_1^*(V,f,n)} \geq e^{-n}$, and $\lim_{n\to 0} e^{-n} = 1$, so $\lim_{n\to 0} e^{-np_1^*(V,f,n)} = 1$. Thus, there is some n^0 such that for $n < n^0$, $0 \leq e^{-np_1^*(V,f,n)} < Q$, and some n^1 such that for $n > n^1$, $f/(V-1) < e^{-np_1^*(V,f,n)} \leq 1$. Since $p_1^*(V,f,n)$ is continuous in n, $e^{-np_1^*(V,f,n)}$ is also continuous in n. The result then follows from the intermediate value theorem. Q.E.D.

Proposition 3 shows that for given risk preferences there is some n that makes risk-loving buyers want to participate in the auction, despite the fact that at least one of the bids they put positive probability on (number one) is associated with a negative expected payoff. Conversely the next proposition shows that for a fixed n we can vary risk preferences in order to induce buyers to participate.[9] Suppose buyers have utility function $u(x) = x^{1+\rho}/(1+\rho)$ with Arrow-Pratt-De Finetti measure of relative risk aversion $-xu''(x)/u'(x) = -\rho$ (larger ρ means more risk loving).

Proposition 4 *Suppose $V - 1 > f > 0$ and $u(x) = x^{1+\rho}/(1+\rho)$. For any n there is a $\tilde{\rho} > 0$ such that if $\rho > \tilde{\rho}$ then $\mathbb{E}[u|p^*(V,f,n)] > u(W)$.*

Proof. The limit of the right hand side of the participation condition (6) is

$$\lim_{\rho\to\infty} \frac{(W)^{1+\rho} - (W-f)^{1+\rho}}{(W+V-1-f)^{1+\rho} - (W-f)^{1+\rho}} = \lim_{\rho\to\infty} \frac{\left(\frac{W}{W+V-1-f}\right)^{1+\rho} - \left(\frac{W-f}{W+V-1-f}\right)^{1+\rho}}{1^{1+\rho} - \left(\frac{W-f}{W+V-1-f}\right)^{1+\rho}} = 0,$$

since $W + V - 1 - f > W > W - f$. Since $e^{-np_1^*(V,f,n)} > e^{-n} > 0$, condition (6) holds for sufficiently high ρ. Q.E.D.

[9] Proposition 3 shows that it is possible for the expected payoff to bid 1 to be negative while the expected utility is positive. This does not rule out, however, that the expected payoff from making higher bids might be positive.

2.2 Independent Private Values

The theory of Poisson games (Myerson 1998, 2000) makes it straightforward to analyze LUBA with private values. Suppose each individual's value is drawn independently from $\Theta = [0, \bar{\theta}] \subseteq \mathbb{R}$ according to a continuous distribution μ. Conditional on being of type $\theta \in \Theta$ an individual plays action k with probability $p_k(\theta)$. The probability that a randomly drawn player plays strategy k is

$$p_k = \int_{\theta \in \Theta} \mu(\theta) \, p_k(\theta) \, d\theta.$$

The number of players choosing action k, still denoted $X(k)$, is Poisson distributed with mean np_k. Thus the expected utility to a player of type θ that bids k is

$$\pi_k^\theta(p) = w_k(p) \left[u(\theta - k - f) - u(-f) \right] + u(-f),$$

where $w_k(p)$ is the same expression as in the known common-value case analyzed above. This allows us to use the same argument as in Proposition 1 to show that the (aggregate) equilibrium strategy p has a convex support and includes the lowest bid. We are also able to say something about how the supports of the strategies of different types, denoted $C(p^\theta)$, relate to each other provided that the players are risk averse or risk neutral.[10]

Proposition 5 *There exists an equilibrium of the private-values Poisson LUBA game and any such equilibrium has convex support $\{1, 2, ..., \kappa\}$ for some $\kappa \leq K$. Furthermore, if ($u'' \leq 0$), the following holds: Let p^θ and $p^{\theta'}$ be the equilibrium strategies of types θ and θ'. If $\theta < \theta'$ then the highest (lowest) number that p^θ puts positive probability on is weakly lower than the highest (lowest) number that $p^{\theta'}$ puts positive probability on, i.e.,* $\max C\left(p^\theta\right) \leq \max C\left(p^{\theta'}\right)$ *and* $\min C\left(p^\theta\right) \leq \min C\left(p^{\theta'}\right)$.

Proof. Existence follows from Myerson (2000). The proof that the support is $\{1, 2, ..., \kappa\}$ for some $\kappa \leq K$, is the same as in the known common value case.

Suppose $\theta < \theta'$. We define $\Delta u_k^\theta = u(\theta - k - f) - u(-f)$. To obtain a contradiction, assume that in equilibrium $k = \max C(p^\theta) > \max C(p^{\theta'}) = k'$. Hence, $\theta - k - f < \theta - k' - f$,

[10] As Proposition 5 shows, the supports of the equilibrium strategies of high and low types overlap, which implies that the revenue equivalence theorem (e.g. Klemperer, 1999) cannot be used to compare revenues from LUBA with revenues from other auction formats.

and $\Delta u_k^\theta < \Delta u_{k'}^\theta$. Since $k \in C(p^\theta)$ in equilibrium, it must be that $0 \leq \pi_k^\theta(p) - \pi_{k'}^\theta(p)$, or

$$\begin{aligned}
0 &\leq e^{-np_k}\prod_{i=1}^{k-1}\left(1-np_ie^{-np_i}\right)\cdot\Delta u_k^\theta - e^{-np_{k'}}\prod_{i=1}^{k'-1}\left(1-np_ie^{-np_i}\right)\cdot\Delta u_{k'}^\theta \\
&< \left[e^{-np_k}\prod_{i=1}^{k-1}\left(1-np_ie^{-np_i}\right) - e^{-np_{k'}}\prod_{i=1}^{k'-1}\left(1-np_ie^{-np_i}\right)\right]\cdot\Delta u_{k'}^\theta = \Delta P\cdot\Delta u_{k'}^\theta.
\end{aligned}$$

Since $\theta - k - f > -f$ and $k > k'$, we have $\theta - k' > 0$, or $\Delta u_{k'}^\theta > 0$. Thus, $\Delta P > 0$. Using $u'' \leq 0$ one finds $\frac{\partial}{\partial\theta}u(\theta-k-f) \geq \frac{\partial}{\partial\theta}u(\theta-k'-f)$. Since $\Delta P > 0$ and $u' > 0$, we have

$$\frac{\partial}{\partial\theta}\left(\pi_k^\theta(p) - \pi_{k'}^\theta(p)\right) > \Delta P\cdot\frac{\partial}{\partial\theta}u\left(\theta-k'-f\right) > 0.$$

This together with $\pi_k^\theta(p) \geq \pi_{k'}^\theta(p)$ implies $\pi_k^{\theta'}(p) > \pi_{k'}^{\theta'}(p)$, which is inconsistent with θ' putting positive probability on k' and zero probability on k.

To prove $\min C(p^\theta) \leq \min C(p^{\theta'})$, assume the contrary that in equilibrium $k = \min C(p^\theta) > \min C(p^{\theta'}) = k'$. As before we can find $\frac{\partial}{\partial\theta}\left(\pi_k^\theta(p) - \pi_{k'}^\theta(p(p)\right) > 0$. Together with $\pi_k^{\theta'}(p) \leq \pi_{k'}^{\theta'}(p)$ (which follows from $k' \in C(p^{\theta'})$) this implies $\pi_k^\theta(p) < \pi_{k'}^\theta(p)$. This is not consistent with θ putting positive probability on k and zero probability on k'. Q.E.D.

3 Conclusion

The results in this paper suggest that the assumption that the number of players follows a Poisson distribution is both useful and innocuous when studying LUBA. This assumption allows us to prove that the equilibrium is unique, and to characterize its basic properties.

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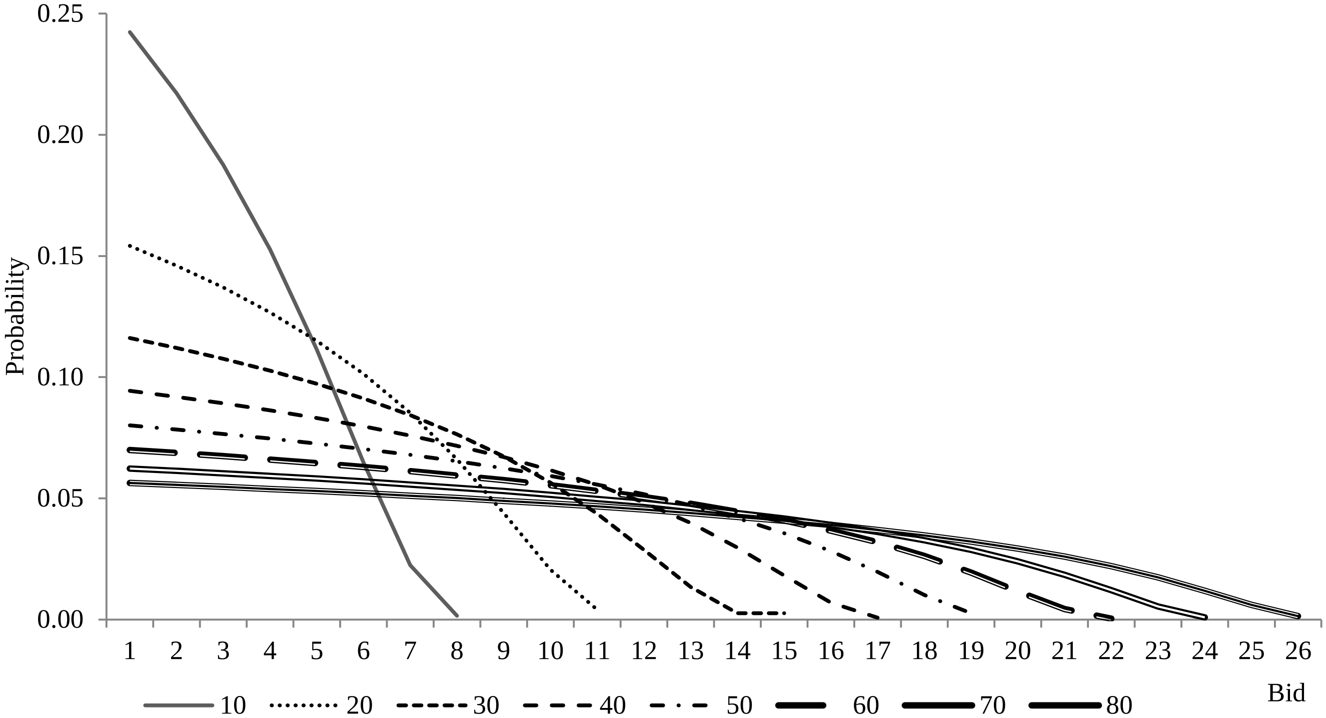
Online Appendix

Numerical Equilibrium Computations

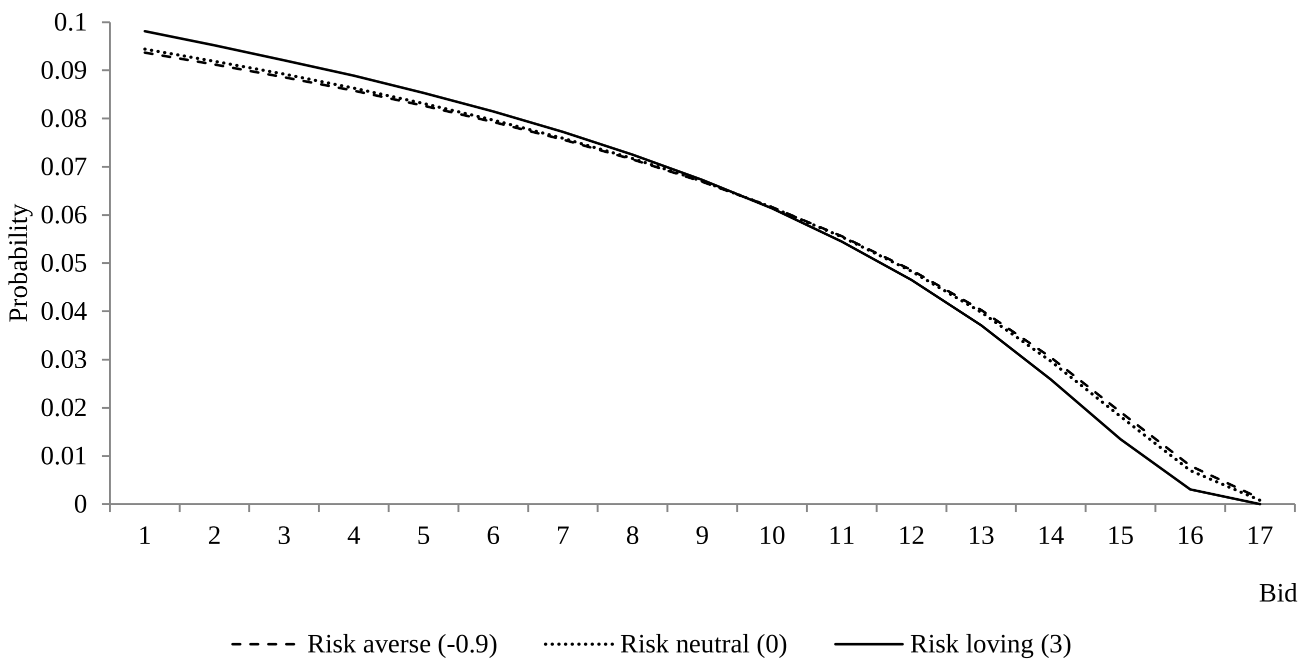
Figure A1 shows the equilibrium prediction with risk neutral bidders, $V = 100$ and $n = \{10, 20, 30, 40, ..., 80\}$. Figure A2 shows the equilibrium prediction for agents with utility function $u(x) = x^{1+\rho}/\left(1+\rho\right)$ for initial wealth $W = 20$, $V = 100$, $n = 40$ and $f = 0$. The equilibrium prediction is shown for risk averse bidders ($\rho = -0.9$), risk neutral bidders ($\rho = 0$) and risk-loving bidders ($\rho = 3$).

Figure A1. Poisson LUBA Equilibrium with Risk Neutral Bidders



Notes: This figure shows the symmetric Nash equilibrium of the LUBA game when the number of risk neutral bidders is Poisson distributed with mean n = 10, 20,…, 80 players and value of the prize is V = 100.

Figure A2. Poisson LUBA Equilibrium with Different Risk Attitudes



Notes: This figure shows the symmetric Nash equilibrium of the LUBA game for $\rho = -0.9$, 0 and 3, initial wealth W = 20, the number of bidders is Poisson distributed with mean n = 40, there is no cost of bidding (f = 0) and the value of the prize is V = 100.

Several papers have provided theoretical analyses of symmetric Nash equilibria with a fixed number of players. Houba et al. (2011) prove that (i) a symmetric Nash equilibrium has a convex support that includes the lowest bid, (ii) the probability put on each number is strictly decreasing over the support, and (iii) the probability of a bid winning is strictly increasing over the support. These three results are also proved by Gallice (2013) and Costa-Gomes and Shimoji (2014).[11] As shown by Proposition 1 and 2, these properties are shared by the Poisson equilibrium. Several papers also report numerical computations of the LUBA equilibrium with a fixed number of players. We have compared the Poisson LUBA equilibrium with the fixed-N computations provided by Houba et al. (2011) and Costa-Gomes and Shimoji (2014). The Poisson LUBA and fixed-N Nash equilibrium is practically indistinguishable. Table A1 shows some illustrative computations for up to $N = 20$ players.

Table A1. Poisson vs. fixed-N LUBA

	Houba et al. (2010)				Costa-Gomes & Shimoji (2014)			
	Fixed	Poisson	Fixed	Poisson	Fixed	Poisson	Fixed	Poisson
V	50	50	100	100	100	100	100	100
n	5	5	8	8	10	10	20	20
p_1	0.364	0.366	0.273	0.277	0.237	0.242	0.150	0.154
p_2	0.320	0.291	0.253	0.242	0.222	0.217	0.145	0.146
p_3	0.193	0.205	0.219	0.199	0.201	0.188	0.138	0.137
p_4	0.095	0.109	0.157	0.149	0.166	0.153	0.130	0.127
p_5	0.028	0.029	0.080	0.092	0.113	0.111	0.120	0.115
p_6			0.017	0.036	0.052	0.064	0.107	0.101
p_7				0.004	0.009	0.022	0.090	0.085
p_8							0.067	0.066
p_9							0.039	0.044
p_{10}							0.012	0.021
p_{11}							0.001	0.004

The table shows the fixed-N equilibrium probabilities reported in Houba et al. (2010) and Costa-Gomes & Shimoji (2014) and the Poisson LUBA equilibrium for some combinations of n and V.

The difference between the Poisson and fixed-N Nash equilibrium appears to be decreasing in N; for the parameter values used in Figure A1, the absolute difference summed

[11] Raviv and Virag (2009) prove the first of these two results under the strong assumption that players strive to maximize probability of win, disregarding the payoff consequences. They also assume that LUBA is repeated or that the fee is returned in case of a tie (thus implying constant payoffs).

over all strategies is lower the higher is N.[12]

MATLAB Code

The MATLAB code used to generate the numerical computations above consists of the two programs below called PoissonLUBAEQ and iter_luba.

```
function F = PoissonLUBAEQ(W,rho,n,V,f)
p1_in_l = 0;
p1_in_h = 1;
p1_in_diff = 0.1;
p1_diff = p1_in_diff;
p1_in = p1_in_l:p1_in_diff:p1_in_h;
[junk, range] = size(p1_in);
p = ones(V,range);
error = 1;
while error > 1/10^6
    p(1,:)  = p1_in;
    for kkk = 1:(V-1)
        p(kkk+1,:)  = iter_luba(W, rho, n, V, f, p(kkk,:), kkk);
    end
    sum_p = nansum(p);
    turn_point = 0;
    p1_new_l = p1_in(1,1);
    p1_new_h = p1_in(1,range);
    for i=1:(range-1)
        if (sum_p(1,i) < 1) & (sum_p(1,i+1) > 1)
            turn_point = i;
            p1_new_l = p1_in(1,i);
            p1_new_h = p1_in(1,i+1);
        end
    end
```

[12] We have also visually compared the Poisson LUBA equilibrium to the fixed-N computations provided by Rapoport et al. (2013) and the distributions appear similar. The numerical computations by Eichberger and Vinogradov (2009) have a different shape compared to the Poisson equilibrium as well as equilibria reported in other papers. Pigolotti et al. (2012) assume like this paper that the number of bidders is Poisson distributed. They only provide solutions for the case when $V \to \infty$, i.e. the LUPI game, and they do not prove uniqueness. Visual comparison of their reported equilibrium analysis for the LUPI game in their Figure 1 and our computations of the corresponding Poisson LUBA equilibrium suggests that the two cases are practically indistinguishable (which is due to the fact that they only report equilibria with high-value objects).

```
    p1_diff = p1_diff*p1_in_diff;
    p1_in = p1_new_l:p1_diff:p1_new_h;
    error = abs(sum_p(turn_point) - sum_p(turn_point+1));
end
[r,c] = size(p);
F = sum(p,2)/c;

function p_next = iter_luba(W, rho, n, V, f, p, k)
[size_r, size_c] = size(p);
I = ones(size_r, size_c);
np = n*p;
exp_np = exp(np);
exp_np_next = (exp_np - np)*((((W+V-k-1-f)^(1+rho))/(1+rho) - ((W-f)^
(1+rho))/(1+rho)) / (((W+V-k-f)^(1+rho))/(1+rho) - ((W-f)^(1+rho))
/(1+rho)));
np_next = log(exp_np_next);
p_next = np_next / n;
for i = 1:size_c
    if p_next(1,i) < 0
        p_next(1,i) = NaN;
    end
end
```

Alternative Explanations for LUBA Participation

Utility from Gambling. The possibility that people derive utility from gambling in a way that is not captured by concerns for total wealth has been suggested by many people, but rarely been modelled explicitly. One exception is Diecidue et al. (2004).[13] Consider a gamble (μ, ω), where $\omega \in \mathbb{R}^n$ is a vector of monetary outcomes and μ is the vector of probabilities of the different outcomes. Suppose a subject has initial wealth $w \in \mathbb{R}$. So far we have assume that the subjective utility is equal to expected total wealth; $u = w + \mu \cdot \omega$. Following Diecidue et al. (2004) we may try to capture the (dis)utility from gambling by positing a utility function $v = u(\mu, \omega) - c(w) = w + \mu \cdot \omega - c(w)$, where $c(w)$ is negative for all relevant w. This model (intentionally) creates a discontinuity in the transition between risky and riskless alternatives. Once a subject has decided to enter the auction the term $c(w)$ does not matter and the individual behaves just like a standard risk-neutral expected payoff maximizer. Thus we obtain the same risk-neutral equilibrium as above.

[13] For an interesting discussion of the concept, see Conlisk (1993).

However, before the subject has entered the term $c(w)$ matters. If $c(w)$ is negative the subject may choose to enter even though the expected monetary payoff is negative.

Non-Belief in the Law of Large Numbers. Kahneman and Tversky (1972) present evidence that people neglect sample size, and instead seem to believe that sample proportions will reflect some "universal sampling distribution." In large samples this means that people overestimate the probability that empirical distribution deviates from the true distribution. For example, the event of finding more than 600 boys in a sample of 1000 babies, is judged to be roughly the same as the probability of the event of finding more than 60 boys in a sample of 100 babies, even though the latter even is much more likely (given that the probability of any given baby being a boy is below 0.6). Recently Benjamin et al. (2014) has dubbed this the *non-belief in the law of large numbers* (NBLLN), and modelled this phenomenon in a non-strategic context. Here we suggest another way of thinking about NBLLN which we think is more suitable for our strategic context.

It might be that non-belief in the law of large numbers causes non-Nash behaviour even after players have had time to learn and adapt their behaviour.[14] However, in our studies of the LUPI game (Östling et al., 2011 and Mohlin, Östling and Wang, 2014) we find that aggregate play eventually comes remarkably close the Nash equilibrium. Still players continue to participate even though the expected monetary payoff of doing so is negative in the Nash equilibrium. Thus if players are motivated solely by expected monetary gain their decision to enter a LUBA (or LUPI) must be due to an over-estimation of the resulting equilibrium payoff, at the time of making the entry decision. A non-believer in the law of large numbers may fall prey to such an illusion because she predicts that the equilibrium monetary payoff will be as if there is a smaller number of participants than there actually is.

[14] It might also be interesting to model an "equilibrium" p^* in which everyone best responds given a correct belief about p^* and an incorrect belief about n.