Lowest Unique Bid Auctions with Population Uncertainty

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September 9, 2014

Abstract

We characterize the unique symmetric Nash equilibrium of the lowest unique bid auction (LUBA) when the number of bidders is uncertain and follows a Poisson distribution.

JEL classification: C72, D44, L83.

Keywords: Lowest Unique Bid Auction, Least Unmatched Price Auction; Minbid game; Poisson game; Congestion game; Mixed equilibrium; Guessing game.

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*We thank Alexandre de Cornière, Péter Eső, Paul Klemperer, and Makoto Shimoji for helpful comments. Erik Mohlin acknowledges financial support from the European Research Council, Grant no. 230251, Robert Östling acknowledges financial support from the Jan Wallander and Tom Hedelius Foundation, and Joseph Tao-yi Wang acknowledges financial support from the NSC of Taiwan.

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1 Introduction

In the lowest unique bid auction (LUBA), players submit bids for an object, and the one who submits the lowest unique bid wins the object and pays her bid.\footnote{This family of auctions is also known as least unmatched price auctions (Eichberger and Vinogradov, 2009), minibid games (De Wachter and Norman, 2006) and gambling auctions (Raviv and Virag, 2009).} In recent years this auction format has gained considerable popularity in online markets.\footnote{A Google search for “lowest unique bid auction” produced about 968k hits in August 2014. See Eichberger and Vinogradov (2009) and Gallice (2013) for some information about the kind of goods sold and the volumes of trade at different online auction sites.} LUBA has also spurred academic interest and has been studied both theoretically and empirically (De Wachter and Norman, 2006; Rapoport, Otsubo, Kim and Stein, 2009; Eichberger and Vinogradov, 2009; Raviv and Virag, 2009; Houba, Laan and Veldhuizen, 2011; Scarsini, Solan and Vieille, 2010; Gallice, 2013; Pigolotti, Bernhardsson, Juul, Galster and Vivo, 2012; Radicchi, Baronchelli and Amaral, 2012; Costa-Gomes and Shimoji, 2014; Chakraborty, Tammana, Ganguly and Mukherjee, 2014). Although substantial progress has been made, analyzing LUBA has proved to be complicated. For example, many papers have focused on numerical computations of symmetric equilibria of LUBA. However, it has not yet been shown that there is a unique symmetric equilibrium. In this paper, we show that it is straightforward to show uniqueness if one is willing to assume that the number of players is uncertain. This is indeed the case in many online LUBAs – there is often a set deadline and everybody that submits a bid before the deadline gets to participate, which implies that there is uncertainty about the total number of bidders.

Specifically, following Östling, Wang, Chou and Camerer’s (2011) study of the lowest unique positive integer (LUPI) game, we assume that the number of players is Poisson distributed, which allows us to use results by Myerson (1998, 2000). We show that the equilibrium of the Poisson LUBA inherits the previously shown theoretical properties of the fixed-\(N\) equilibrium, and that the Poisson LUBA has a unique symmetric equilibrium. Furthermore, we also show that the Poisson LUBA equilibrium can easily be computed, that it easily lends itself to comparative statics analysis and that it is practically indistinguishable from previously reported numerical computations of the fixed-\(N\) equilibrium. In addition, we show that the Poisson LUBA can easily be extended to take risk preferences of bidders as well as private values into account.\footnote{All papers on LUBA except Gallice (2013) and Costa-Gomes and Shimoji (2014) work exclusively with known common values.}
2 Poisson LUBA

In the lowest unique bid auction studied here, \( N \) players simultaneously submit one bid \( k \in \mathbb{N} \) each. The player who submits the lowest unique bid wins a prize. If there is no unique bid then there is no winner. Each participant pays a fixed fee \( f \geq 0 \). The winner wins a prize with common value \( V > 1 \) and pays her bid \( k \). We assume that bidders maximize expected utility and that their utility is strictly increasing in the monetary payoff. Although the pure strategy space is in principle the set of natural numbers, we may without loss of generality restrict attention to \( S = \{1, 2, \ldots, K\} \) for some \( K \). Consequently the mixed strategy space is the \((K - 1)\)-dimensional simplex \( \Delta \).

Let \( k^*(s) \) denote the winning number under strategy profile \( s = (s_1, \ldots, s_N) \in S^N \):

\[
    k^*(s) = \min_{j \in \{k \in \{1, \ldots, N\}: s_k \neq s_i, \forall i \neq k\}} s_j.
\]

The payoff to a player playing strategy \( k \in S \) under strategy profile \( s \) is

\[
    u_k(s) = \begin{cases} 
        u(V - k - f) & \text{if } k = k^*(s), \\
        u(-f) & \text{otherwise}. 
    \end{cases}
\]

Note that bidding \( k > V \) is weakly dominated by bidding 1. Bidding \( k > V \) results in a payoff of \( u(V - k - f) \) when winning and \( u(-f) \) when losing, so it is better to lose by bidding 1 than to win by bidding \( k > V \). In a symmetric mixed equilibrium every player has some positive probability of winning by bidding 1, and as we will see below, the only candidate for a symmetric pure equilibrium is for everyone to bid 1. Hence in a symmetric Nash equilibrium players will not bid \( k > V \). Without loss of generality we therefore set \( K = V \).

We imagine a large (technically infinite) population of potential bidders. A number of individuals are drawn from this population to play the game. The number of players \( N \) is Poisson distributed with mean \( n \). Let \( p \) denote the population average strategy, i.e. \( p_i \) is the probability that a randomly chosen player will pick pure strategy \( i \). Let \( X(k) \sim \text{Poisson}(np_k) \) be the total number of players who are drawn to participate and choose strategy \( k \). As shown by Myerson (1998), Poisson games have an independent actions property: The numbers of players picking two different actions are independent of one another. Furthermore Poisson games display an environmental equivalence property: An individual that is drawn to play perceives the uncertainty the same way an outsider does. More precisely, from the point of view of a player that is drawn to play the game, the number of other individuals who are drawn to play is \( \text{Poisson}(n) \), and the number of other individuals that are drawn and plays \( k \) is \( \text{Poisson}(np_k) \).
Let $w_k(p)$ be the probability that an individual who is drawn to play and picks number $k$ will win, when the population average strategy is $p$. This win probability is the same as in the LUPI game and from Östling et al. (2011) we consequently know that

$$w_k(p) = \Pr(X(k) = 0) \prod_{i=1}^{k-1} \Pr(X(i) \neq 1) = e^{-np_k} \prod_{i=1}^{k-1} (1 - np_i e^{-np_i}).$$

The simplicity of this expression is due to the special properties of Poisson games, in particular the 'independent actions property'. The expected utility to a player putting all probability on strategy $k$ given the population average strategy $p$ is

$$\pi_k(p) = w_k(p) [u(V - k - f) - u(-f)] + u(-f).$$

In a symmetric mixed strategy equilibrium that has both $k$ and $k+1$ in its support, $\pi_{k+1}(p) = \pi_k(p)$. Using the expression for the win probabilities and simplifying we get the following mixed equilibrium condition:

$$e^{np_{k+1}} = (e^{np_k} - np_k) \frac{u(V - k - 1 - f) - u(-f)}{u(V - k - f) - u(-f)}.$$  \hspace{1cm} (1)

We now prove that there is a unique symmetric equilibrium and that the equilibrium probabilities are decreasing over the convex support.

**Proposition 1** For any $V > 1$, $f \geq 0$ and $n > 0$, a symmetric Nash equilibrium $p = (p_1, p_2, ..., p_K)$ of the Poisson LUBA game has the following properties:

1. A symmetric equilibrium exists and is unique. If

$$e^n - n > \frac{u(V - 1 - f) - u(-f)}{u(V - 2 - f) - u(-f)}.$$  \hspace{1cm} (2)

then the equilibrium is mixed.\(^4\)

2. The equilibrium has convex support $\{1, 2, ..., \kappa\}$ for some $\kappa \leq K$.

3. The probability of winning, $w_k(p)$, is increasing in $k$ over the support, and the probability $p_k$ is decreasing over the support.

4. If bidders are risk neutral, the following holds:

\(^4\)For example, with risk-neutral bidders and $V \geq 3$, the right hand side of (2) is below 2 so that the condition is satisfied for $n \gtrsim 1.15$. 

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(i) The equilibrium distribution converges to a uniform distribution with many players: \( n \to \infty \) implies that \( p_{k+1} \to p_k \) for all strategies in the support of the equilibrium.

(ii) The equilibrium distribution flattens with higher \( V \) and converges to the LUPI equilibrium distribution when \( V \to \infty \).

(iii) The equilibrium distribution is independent of \( f \).

**Proof.** Existence of a symmetric equilibrium follows directly from Myerson (1998). We first prove the second property since it is needed for the proof of the other properties.

In order to prove the second property, first note that a symmetric equilibrium must have \( p_1 > 0 \). If this was not the case, then picking 1 would guarantee a win with the lowest possible bid (and hence the highest possible payoff). We prove convexity of the support by contradiction. Suppose that there is an equilibrium with non-convex support. Then there is a \( k \) such that \( p_k = 0 \) and \( p_{k+1} > 0 \). In this case

\[
\pi_{k+1}(p) - \pi_k(p) = e^{-np_{k+1}} \prod_{i=1}^{k} (1 - np_i e^{-np_i}) u(V - k - 1 - f) \\
- \prod_{i=1}^{k-1} (1 - np_i e^{-np_i}) u(V - k - f) \\
= \prod_{i=1}^{k-1} (1 - np_i e^{-np_i}) (e^{-np_{k+1}} u(V - k - 1 - f) - u(V - k - f))
\]

Since \( np_{k+1} > 0 \) we have \( e^{-np_{k+1}} < 1 \), so it follows that \( \pi_{k+1}(p) < \pi_k(p) \). This cannot hold in an equilibrium with \( p_k = 0 \) and \( p_{k+1} > 0 \). Hence, we must have \( p_k > 0 \). By induction we have a support of the form \( \{1, 2, \ldots, k \} \) for some \( \kappa \leq K \).

In order to prove the remaining properties, we first consider the case when the symmetric equilibrium is in pure strategies. The only candidate for a pure symmetric equilibrium is one in which everyone plays number 1, i.e. \( p_1 = 1 \), since otherwise a player could deviate to number 1 and be guaranteed to win. The pure strategy equilibrium \( p_1 = 1 \) requires that \( \pi_1(p) \geq \pi_2(p) \), but if condition (2) holds, then \( \pi_1(p) < \pi_2(p) \) so there is no pure equilibrium. In order to prove that the pure equilibrium is unique when it exists it remains to show that there is no mixed equilibrium when (2) does not hold. To show this, suppose that there is a pure symmetric equilibrium \( p' \) with \( p'_1 = 1 \) and a mixed symmetric equilibrium \( p \) such that \( p_1 < 1 \). The equilibrium requirement \( \pi_1(p) = \pi_2(p) \) implies, by (1),

\[
e^{np_1} - np_1 = e^{np_2} \frac{u(V - 1 - f) - u(-f)}{u(V - 2 - f) - u(-f)}.
\]
The condition $\pi_1(p') \geq \pi_2(p')$ requires

$$e^n - n \leq \frac{u(V - 1 - f) - u(-f)}{u(V - 2 - f) - u(-f)}.$$  

Since $e^{np_2} > 1$ this implies $e^{np_1} - np_1 > e^n - n$, but this cannot hold since $e^{np_1} - np_1$ is increasing in $p_1$ whenever $np_1 > 0$, and we have $p_1 < 1$.

It is easy to verify that a pure equilibrium trivially satisfies the other properties stated in the proposition (2, 3, and 4(i)-(iii)), so in the remainder of the proof we assume that the symmetric equilibrium is mixed.

In order to prove uniqueness of a mixed symmetric equilibrium, note that the second property implies that $\pi_{k+1}(p) = \pi_k(p)$ for all $k < \kappa$. The recursive equilibrium condition (1) can be rewritten as

$$p_{k+1} = \frac{1}{n} \ln (e^{np_k} - np_k) + \frac{1}{n} \ln \frac{u(V - k - 1 - f) - u(-f)}{u(V - k - f) - u(-f)}. \quad (3)$$

The partial derivative of the right hand side with respect to $p_k$ is $(e^{np_k} - 1)/(e^{np_k} - np_k)$, which is strictly positive since $np_k > 0$. Hence, $p_{k+1}$ is an increasing function of $p_k$. To see that this implies that there can be at most one equilibrium, suppose to the contrary that $p_k = p_{k}'$ in order for $p$ and $p'$ to differ. There are two cases to consider. Case I: suppose that $\kappa = \kappa'$ and assume without loss of generality that $p_1 > p'_1$. Since $p_{k+1}$ is uniquely determined by $p_k$, and strictly increasing in $p_k$, this would imply that $\sum_{i=1}^{\kappa} p_i > \sum_{i=1}^{\kappa'} p'_i$, which is a contradiction because probabilities must sum to one in both equilibria. Case II: suppose that $\kappa > \kappa'$ so that $p_k > p'_k = 0$ for all $\kappa \geq k > \kappa'$. The fact that $p_{k+1}$ is an increasing function of $p_k$ further implies that $p_k < p'_k$ for all $k \leq \kappa'$. In particular, for both $p$ and $p'$ to be equilibria, we must have that $p_\kappa' < p'_{\kappa'}$, $\pi_{\kappa'}(p') \geq \pi_{\kappa'+1}(p')$ and $\pi_{\kappa'}(p) = \pi_{\kappa'+1}(p)$. The equilibrium condition (1) is equivalent to

$$e^{np_\kappa'} - np_\kappa' = e^{np_{\kappa'+1}} \frac{u(V - \kappa' - f) - u(-f)}{u(V - \kappa' - 1 - f) - u(-f)},$$

whereas $\pi_{\kappa'}(p') \geq \pi_{\kappa'+1}(p')$ implies

$$e^{np_\kappa'} - np'_{\kappa'} \leq \frac{u(V - \kappa' - f) - u(-f)}{u(V - \kappa' - 1 - f) - u(-f)}.$$

The left hand sides are increasing in $p_{\kappa'}$ and $p'_{\kappa'}$, respectively, and $e^{np_{\kappa'+1}} > 1$, implying
that
\[ e^{np_{k'}'} - np_{k'}' > e^{np_{k'}'} - np_{k'}' > \frac{u(V - \kappa' - f) - u(-f)}{u(V - \kappa' - 1 - f) - u(-f)}, \]
which results in a contradiction.

In a mixed equilibrium, the expected utility is the same to each action in the support. The utility of winning on \( k \) is \( u(V\_k) \) and the utility of winning on \( k + 1 \) is \( u(V\_k + 1) \). In order to have \( \pi_{k+1}(p) = \pi_k(p) \), the probability of winning on \( k \) must be strictly lower than the probability of winning on \( k + 1 \), i.e. \( w_k(p) < w_{k+1}(p) \).

It is easy to verify that \( w_k(p) < w_{k+1}(p) \) is equivalent to \( e^{np_{k+1}} - e^{np_k} < -np_k \). Since \( p_k > 0 \), the right hand side of this inequality is negative. In order for the left hand side to be negative we need \( p_k > p_{k+1} \).

With risk neutral bidders, the recursive equilibrium condition can be rewritten as

\[ p_k - p_{k+1} = -\frac{1}{n} \ln (1 - np_k e^{-np_k}) - \frac{1}{n} \ln \frac{(V - k - 1)}{(V - k)}. \]

Taking the limit of the right hand side as \( n \to \infty \) implies that \( p_k = p_{k+1} \) in the support of the equilibrium. Furthermore, the recursive condition is independent of \( f \) and decreasing in \( V \), implying that the equilibrium distribution becomes flatter for higher \( V \). The LUBA equilibrium converges of the LUPI equilibrium when \( V \to \infty \). To see this note that

\[ \lim_{V \to \infty} \ln \left( \frac{(V - k - 1)}{(V - k)} \right) = 0 \]

so that the second term on the right hand side of the condition above vanishes. The remainder of the expression is identical to condition (2) in Östling et al. (2011).

In order to prove that \( p_1 \) is decreasing in \( n \), let \( p_1^*(n) \) denote the equilibrium value of \( p_1 \) given \( n \). To obtain a contradiction assume that \( n < n' \) and \( p_1^*(n) < p_1^*(n') \). Q.E.D.

Proposition 1 together with equilibrium condition (1) implies that it is straightforward to compute the equilibrium by guessing \( p_1 \), calculating \( p_2, \ldots, p_k \) using the recursive condition and then verifying that the equilibrium probabilities sum to one. Figure 1 shows the equilibrium prediction with risk neutral bidders, \( f = 0 \), \( V = 100 \) and \( n = \{10, 20, 30, 40, \ldots, 90\} \).

2.1 Endogenous Entry

The assumption that players are Poisson distributed can be motivated by a Poisson process resulting from a constant arrival rate of bidders to the auction. This might be a reasonable

\[ \text{INSERT FIGURE 1 HERE} \]

\[ \text{Östling et al. (2011) show that the LUPI equilibrium distribution is concave for low } k \text{ and convex for high } k. \text{ We have not shown such a result for LUBA, but Figure 1 suggests that it holds for LUBA as well.} \]
model of LUBA if, for example, there is a set deadline, the number of bidders is unknown before the deadline and bidders largely consists of new customers that are surfing the web randomly and arrive to the LUBA web site at a constant rate. However such an explanation is silent regarding the potential bidders’ choices whether or not to enter the LUBA. In order to model the entry decision more explicitly one may assume that there is a large population of size $m$ of potential bidders. Each individual in the population participates with probability $r \in (0, 1)$. For each individual $i$, let $Z_i$ be a random variable that is equal to 1 if $i$ participates and 0 otherwise. Since $Z_i$ is a Bernoulli variable, the sum of participants $Z = \sum_{i=1}^{m} Z_i$ is binomial with parameters $m$ and $r$. For large $m$ and small $r$ the binomial distribution is approximated by a Poisson distribution with mean $mr$. Thus, for a large population of agents, each of which participates with a small probability, the number of participants is approximately Poisson distributed.

2.2 Risk Preferences

Organizers of LUBA primarily raise revenue from the fixed fees that bidders pay and the sum of all fixed fees often vastly exceeds the value of the prize. For example, in the closely related LUPI game studied by Östling et al. (2011), organizers paid back less than 50% of the fixed fees. This raises the question why bidders might participate in such auctions. There are a number of different biases, or unconventional preferences, which might explain why people participate in LUBA despite the negative expected monetary payoff. A more conventional candidate explanation is a love for risk, i.e. a convex utility function over total wealth, which could easily explain why bidders would prefer to play LUBA. Risk loving preferences will, however, also affect the equilibrium prediction. The curvature of the utility function determines the slope of the equilibrium distribution by equilibrium condition (1): bidding a higher number on the one hand increases the chance of winning the auction, but results in lower profits when winning. Intuitively, risk-loving bidders will be more willing to take a chance and bid low to get a higher payoff. Figure 2 confirms this intuition by plotting the equilibrium prediction for agents with utility function $u(x) = (20 + x)^\alpha$ for $V = 100$, $n = 20$ and $f = 0$. Figure 2 shows the equilibrium distribution for risk averse bidders ($\alpha = 0.5$), risk neutral bidders ($\alpha = 1$) and

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6 For example, Diecidue, Schmidt and Wakker (2004) (see also Conlisk, 1993) capture the utility from gambling by adding a constant to the utility function which is positive for risky alternatives and zero for riskless alternatives. Alternatively it might be the case that participants have systematically incorrect expectations about payoffs. Specifically they may fail to take into account the effect of a large number of players, e.g. due to “non-belief in the law of large numbers” (Benjamin, Rabin and Raymond, 2014). Both explanations are consistent with the equilibrium derived above under the assumption that players are rational, risk-neutral and the number of players ($n$) is interpreted as the perceived number of players rather than the actual number of players. But one of the explanations assumes irrationality, in the form of the inability to predict the equilibrium payoffs, while the other explanation assumes rationality. Hence the two explanations have very different welfare explanations.
risk-loving bidders ($\alpha = 2$ vs. $\alpha = 4$). In line with intuition, the equilibrium distribution is steeper the higher $\alpha$ is, although the difference is very small.

2.3 Independent Private Values

The theory of Poisson games (Myerson 1998, 2000) makes it relatively straightforward to analyse LUBA with private values. Suppose each individual’s value is drawn independently from $\Theta = [0, \theta] \subseteq \mathbb{R}$ according to a continuous distribution $\mu$. Conditional on being of type $\theta \in \Theta$ an individual plays action $k$ with probability $p_k(\theta)$. The probability that a randomly drawn player plays strategy $k$ is

$$p_k = \int_{\theta \in \Theta} \mu(\theta) p_k(\theta) d\theta.$$

The number of players choosing action $k$, still denoted $X(k)$, is Poisson distributed with mean $np_k$. Thus the expected utility to a player of type $\theta$ that bids $k$ is

$$\pi_k^\theta(p) = w_k(p) [u(\theta - k - f) - u(f)] + u(f),$$

where $w_k(p)$ is the same expression as in the common-value case analysed above. This allows us to use the same argument as in Proposition 1 to show that the (aggregate) equilibrium strategy $p$ has a convex support and includes the lowest bid. We are also able to say something about how the supports of the strategies of different types, denoted $C(p^\theta)$, relate to each other provided that the players are risk averse or risk neutral.

**Proposition 2** There is at least one symmetric equilibrium of the private-values Poisson LUBA game and any such equilibrium has a convex support $\{1, 2, \ldots, \kappa\}$ for some $\kappa \leq K$. Furthermore, if players are risk neutral or risk averse, the following holds: Let $p^\theta$ and $p^{\theta'}$ be the equilibrium strategies of types $\theta$ and $\theta'$. If $\theta < \theta'$ then the highest (lowest) number that $p^\theta$ puts positive probability on is weakly lower than the highest (lowest) number that $p^{\theta'}$ puts positive probability on, i.e., $\max C(p^\theta) \leq \max C(p^{\theta'})$ and $\min C(p^\theta) \leq \min C(p^{\theta'})$.

**Proof.** Existence follows from Myerson (2000). The proof that the support is $\{1, 2, \ldots, \kappa\}$ for some $\kappa \leq K$, is the same as in the common value case.

Suppose $\theta < \theta'$. To obtain a contradiction assume that in equilibrium $k = \max C(p^\theta) >
max \( C(p^\theta) = k' \). Using \( u'' \leq 0 \) one finds

\[
\frac{\partial}{\partial \theta} \left( \pi_k^\theta (p) - \pi_{k'}^\theta (p) \right) > \left( e^{-n_p k - 1} \prod_{i=1}^{k-1} (1 - np_i e^{-np_i}) - e^{-n_p k' - 1} \prod_{i=1}^{k'-1} (1 - np_i e^{-np_i}) \right) \times \frac{\partial u(\theta - k' - f)}{\partial \theta}.
\]

Next, note that if \( k \in C(p^\theta) \) in equilibrium, then it must be that \( \pi_k^\theta (p) \geq \pi_{k'}^\theta (p) \). Thus

\[
e^{-n_p k} \prod_{i=1}^{k-1} (1 - np_i e^{-np_i}) [u(\theta - k - f) - u(f)] \geq e^{-n_p k'} \prod_{i=1}^{k'-1} (1 - np_i e^{-np_i}) [u(\theta - k' - f) - u(f)].
\]

This implies

\[
\left( e^{-n_p k} \prod_{i=1}^{k-1} (1 - np_i e^{-np_i}) - e^{-n_p k'} \prod_{i=1}^{k'-1} (1 - np_i e^{-np_i}) \right) [u(\theta - k' - f) - u(f)] > 0.
\]

Since \( k \in C(p^\theta) \) in equilibrium, and a player can always guarantee herself a payoff of \(-1\), it must be the case that \( \theta - k \geq -1 \), and hence \( \theta - k' \geq 0 \). Thus \( u(\theta - k' - f) \geq u(f) \), so

\[
e^{-n_p k} \prod_{i=1}^{k-1} (1 - np_i e^{-np_i}) - e^{-n_p k'} \prod_{i=1}^{k'-1} (1 - np_i e^{-np_i}) > 0.
\]

Since \( u' > 0 \) it follows that \( \frac{\partial}{\partial \theta} \left( \pi_k^\theta (p) - \pi_{k'}^\theta (p) \right) > 0 \). Together with \( \pi_k^\theta (p) \geq \pi_{k'}^\theta (p) \) this implies \( \pi_k^\theta (p) > \pi_{k'}^\theta (p) \). This is not consistent with \( \theta' \) putting positive probability on \( k' \) and zero probability on \( k \).

To prove \( \min C(p^\theta) \leq \min C(p^{\theta'}) \), assume (in order to obtain a contradiction) that in equilibrium \( k = \min C(p^{\theta'}) > \min C(p^\theta) = k' \). As before we find \( \frac{\partial}{\partial \theta} \left( \pi_k^\theta (p) - \pi_{k'}^\theta (p) \right) > 0 \). Together with \( \pi_k^\theta (p) \leq \pi_{k'}^\theta (p) \) (which follows from \( k' \in C(p^{\theta'}) \)) this implies \( \pi_k^\theta (p) < \pi_{k'}^\theta (p) \). This is not consistent with \( \theta \) putting positive probability on \( k \) and zero probability on \( k' \). Q.E.D.

We have assumed that each agent’s private value is drawn independently from \( \Theta = [0, \theta] \subseteq \mathbb{R} \) according to a common distribution \( \mu \). The revenue equivalence theorem is formulated for such a setting. One of the conditions of the theorem is that the auction mechanism is such that the objects always go to the buyers with the highest valuation (e.g. Klemperer, 1999). This is not the case in LUBA. If \( n \) is large enough then the equilibrium is mixed, hence there is a positive probability that an individual of a lower
type wins over an individual with a higher type. For example this could happen if three individuals are drawn to play and the two individuals with highest valuation pick the same number.

3 Comparison with Previous Results

Several papers have provided theoretical analyses of symmetric Nash equilibria with a fixed number of players. Houba et al. (2011) prove that (i) a symmetric Nash equilibrium has a convex support that includes the lowest bid, (ii) the probability put on each number is strictly decreasing over the support, and (iii) the probability of a bid winning is strictly increasing over the support. These three results are also proved by Gallice (2013) and Costa-Gomes and Shimoji (2014).\footnote{Raviv and Virag (2009) prove the first of these two results under the strong assumption that players strive to maximize probability of win, disregarding the payoff consequences. They also assume that LUBA is repeated or that the fee is returned in case of a tie (thus implying constant payoffs).} As shown by Proposition 1, these properties are shared by the Poisson equilibrium. Several papers also report numerical computations of the LUBA equilibrium with a fixed number of players. We have compared the Poisson LUBA equilibrium with the fixed-\(N\) computations provided by Houba et al. (2011) and Costa-Gomes and Shimoji (2014). The Poisson LUBA and fixed-\(N\) Nash equilibrium is practically indistinguishable, especially for a large number of players. Table 1 shows some illustrative computations.

We have also visually compared the Poisson LUBA equilibrium to the fixed-\(N\) computations provided by Rapoport et al. (2009) and the distributions appear very similar with one exception.\footnote{This is probably due to an error in their paper – the probabilities presented in the top panel in Figure 3 does not appear to sum up to one.} The numerical computations by Eichberger and Vinogradov (2009) have a different shape compared to the Poisson equilibrium as well as equilibria reported in other papers. Pigolotti et al. (2012) assume like this paper that the number bidders is Poisson distributed. They only provide solutions for the case when \(V \to \infty\), i.e. the LUPI game, and they do not prove uniqueness. Visual comparison of their reported equilibrium analysis for the LUPI game in their Figure 1 and our computations of the corresponding Poisson LUBA equilibrium suggests that the two cases are practically indistinguishable (which is due to the fact that they only report equilibria with high-value objects).
Table 1. Poisson vs. fixed-\( N \) LUBA

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<tbody>
<tr>
<td></td>
<td>Fixed Poisson</td>
<td>Fixed Poisson</td>
</tr>
<tr>
<td></td>
<td>50 50</td>
<td>100 100</td>
</tr>
<tr>
<td>( n )</td>
<td>5 5</td>
<td>10 10</td>
</tr>
<tr>
<td>( p_1 )</td>
<td>0.364 0.366</td>
<td>0.237 0.242</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>0.320 0.291</td>
<td>0.222 0.217</td>
</tr>
<tr>
<td>( p_3 )</td>
<td>0.193 0.205</td>
<td>0.201 0.188</td>
</tr>
<tr>
<td>( p_4 )</td>
<td>0.095 0.109</td>
<td>0.166 0.153</td>
</tr>
<tr>
<td>( p_5 )</td>
<td>0.028 0.029</td>
<td>0.113 0.111</td>
</tr>
<tr>
<td>( p_6 )</td>
<td>0.017 0.036</td>
<td>0.052 0.064</td>
</tr>
<tr>
<td>( p_7 )</td>
<td>0.004</td>
<td>0.009 0.022</td>
</tr>
<tr>
<td>( p_8 )</td>
<td></td>
<td>0.067 0.066</td>
</tr>
<tr>
<td>( p_9 )</td>
<td></td>
<td>0.039 0.044</td>
</tr>
<tr>
<td>( p_{10} )</td>
<td></td>
<td>0.012 0.021</td>
</tr>
<tr>
<td>( p_{11} )</td>
<td></td>
<td>0.001 0.004</td>
</tr>
</tbody>
</table>

The table shows the fixed-\( N \) equilibrium probabilities reported in Houba et al. (2010) and Costa-Gomes & Shimoji (2014) and the Poisson LUBA equilibrium for some combinations of \( n \) and \( V \).

4 Conclusion

The results in this paper suggest that the assumption that the number of players follows a Poisson distribution is useful and innocuous when studying LUBA. This assumption allows us to prove that the symmetric equilibrium is unique, and to characterize its basic properties. However, there is one simplifying assumption our analysis share with most previous papers that may not be as innocuous: we assume that each player submits a single bid.\(^9\) We leave it to future work to study this case.

References


\(^9\)Scarsini et al. (2010) provide some preliminary results for this case and show that if \( N > 2 \) then there is no symmetric equilibrium in monotone strategies. A mixed strategy is called monotone if all pure strategies in its support are of the form \( \{1, 2, \ldots, k\} \). Gallice (2013) studies a setting in which there are many rounds and players may submit one bid per round. To simplify the analysis he assumes that the probability of winning is given by a Tullock contest.


