Expected utility theory exercises

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Solutions

1. Consider the expected utility theory presented in class, but with $C = \{c_1, c_2, c_3\}$. We assume that $c_1 \prec c_3$.

1.1 Show that if there is a function $u : C \to \mathbb{R}$ such that

$$\mu \precsim \mu' \text{ if and only if } \sum_{i=1}^{n} \mu(c_i)u(c_i) \leq \sum_{i=1}^{n} \mu'(c_i)u(c_i), \tag{1}$$

then $\precsim$ satisfies (EU1)-(EU3).

Solution. I only show (EU2) and (EU3). For (EU3), since u represents $\succ$, we have

$$\sum_{i=1}^{3} \mu_1(c_i)u(c_i) < \sum_{i=1}^{3} \mu_2(c_i)u(c_i) < \sum_{i=1}^{3} \mu_3(c_i)u(c_i).$$

If we let

$$\alpha = \frac{\sum_{i=1}^{3} \mu_3(c_i)u(c_i) - \sum_{i=1}^{3} \mu_2(c_i)u(c_i)}{\sum_{i=1}^{3} \mu_3(c_i)u(c_i) - \sum_{i=1}^{3} \mu_1(c_i)u(c_i)},$$

then it is straightforward to verify that

$$\sum_{i=1}^{3} \mu_2(c_i)u(c_i) = \alpha \left(\sum_{i=1}^{3} \mu_1(c_i)u(c_i) \right) + (1-\alpha) \left(\sum_{i=1}^{3} \mu_3(c_i)u(c_i) \right),$$

and hence (EU3) follows from representation.

Now consider (EU2). Suppose that $\mu_1 \prec \mu_2$. Then,

$$\sum_{i=1}^{3} \mu_1(c_i)u(c_i) < \sum_{i=1}^{3} \mu_2(c_i)u(c_i).$$

Hence,

$$\alpha\sum_{i=1}^{3}\mu_1(c_i)u(c_i)+(1-\alpha)\sum_{i=1}^{3}\mu_3(c_i)u(c_i)<\alpha\sum_{i=1}^{3}\mu_2(c_i)u(c_i)+(1-\alpha)\sum_{i=1}^{3}\mu_3(c_i)u(c_i).$$

Similarly, if $\alpha\mu_1+(1-\alpha)mu_3\prec\alpha\mu_2+(1-\alpha)\mu_3$ and $\alpha<1$, then

$$\alpha\sum_{i=1}^{3}\mu_1(c_i)u(c_i)+(1-\alpha)\sum_{i=1}^{3}\mu_3(c_i)u(c_i)<\alpha\sum_{i=1}^{3}\mu_2(c_i)u(c_i)+(1-\alpha)\sum_{i=1}^{3}\mu_3(c_i)u(c_i)$$

which implies that

$$\alpha\sum_{i=1}^{3}\mu_1(c_i)u(c_i)<\alpha\sum_{i=1}^{3}\mu_2(c_i)u(c_i),$$

and, dividing both sides by α, this implies that $\mu_1\prec\mu_2$.

1.2 Suppose that $\precsim$ satisfies (EU1)-(EU3). Construct u as in class with $u(c_3)=1$ and $u(c_1)=0$. Show that (1) holds with the following steps.

(a) Show that $\alpha c_3+(1-\alpha)c_1\prec\alpha' c_3+(1-\alpha')c_1$ if and only if $\alpha<\alpha'$.

Solution. We use δ_{c_i} to denote the lottery that concentrates on c_i. Suppose that $\alpha<\alpha'$. Then,

$$\delta_{c_3}=\alpha\delta_{c_3}+(1-\alpha)\delta_{c_3}\succ\alpha\delta_{c_3}+(1-\alpha)\delta_{c_1},\tag{2}$$

where the preference follows from (EU2) and $c_3\succ c_1$. Now,

$$\begin{aligned}
& \alpha'\delta_{c_3}+(1-\alpha')\delta_{c_1}\\
= \;& \frac{\alpha'-\alpha}{1-\alpha}\delta_{c_3}+\left(1-\frac{\alpha'-\alpha}{1-\alpha}\right)[\alpha\delta_{c_3}+(1-\alpha)\delta_{c_1}]\\
\succ \;& \frac{\alpha'-\alpha}{1-\alpha}[\alpha\delta_{c_3}+(1-\alpha)\delta_{c_1}]+\left(1-\frac{\alpha'-\alpha}{1-\alpha}\right)[\alpha\delta_{c_3}+(1-\alpha)\delta_{c_1}]\\
= \;& \alpha\delta_{c_3}+(1-\alpha)\delta_{c_1},
\end{aligned}$$

where the strict preference follows from (EU2) and (2). The other direction is similar.

(b) For any $\mu\in\Delta(C)$, show that

$$\mu\sim[u(c_2)\mu(c_2)+\mu(c_3)]c_3+[(1-u(c_2))\mu(c_2)+\mu(c_1)]c_1.$$

Solution.

$$\begin{aligned}
\mu &= \mu(c_2)\delta_{c_2} + (1-\mu(c_2))\left[\frac{\mu(c_1)}{1-\mu(c_2)}\delta_{c_1} + \frac{\mu(c_3)}{1-\mu(c_2)}\delta_{c_3}\right] \\
&\sim \mu(c_2)[u(c_2)\delta_{c_3} + (1-u(c_2))\delta_{c_1}] + (1-\mu(c_2))\left[\frac{\mu(c_1)}{1-\mu(c_2)}\delta_{c_1} + \frac{\mu(c_3)}{1-\mu(c_2)}\delta_{c_3}\right] \\
&= [u(c_2)\mu(c_2) + \mu(c_3)]\delta_{c_3} + [(1-u(c_2))\mu(c_2) + \mu(c_1)]\delta_{c_1},
\end{aligned}$$

where the indifference follows from (EU2).

(c) Show that the result follows from (a) and (b) and that $\mathbb{E}_\mu(u) = u(c_2)\mu(c_2) + \mu(c_3)$.

Solution. The result is immediate.

2. Show that the set of simple lotteries, $\Delta(\mathbb{R}_+)$, is closed under compound lottery, that is, if μ_1 and μ_2 are simple lotteries and $\alpha \in (0,1)$, then $\alpha\mu_1 + (1-\alpha)\mu_2$ is well-defined and is itself a simple lottery.

Solution. Let μ_1 and μ_2 be two simple lotteries, and let

$$C = \{c \in \mathbb{R}_+ : \mu_1(c) > 0 \text{ or } \mu_2(c) > 0\}.$$

Clearly, C is also a finite set. This shows that $\alpha\mu_1 + (1-\alpha)\mu_2$ is also a simple lottery

3. Show that if $\precsim$ is a relation over $\Delta(\mathbb{R}_+)$ satisfying (EU1)-(EU3) represented by u, then

1. $\precsim$ satisfies MC iff u is strictly increasing;

   Solution. If u is strictly increasing, then $c_1 > c_2$ implies tat $u(c_1) > u(c_2)$ and hence, $c_1 \succ c_2$. Thus, MC is satisfied. Similarly, if MC is satisfied and if $c_1 > c_2$, then $c_1 \succ c_2$ and hence $u(c_1) > u(c_2)$.

2. $\precsim$ satisfies (strict) risk aversion iff u is (strictly) concave.

   Solution. If u is concave, then for any μ, $\mathbf{E}_\mu[u(c)] \leq u[\mathbf{E}_\mu(c)]$ by Jensen's inequality, which implies $\mu \precsim \mathbb{E}_\mu(c)$, and hence $\precsim$ satisfies risk aversion. Conversely, if $\precsim$ satisfies risk aversion, then for any c_1, c_2 and any $\alpha \in (0,1)$,

$$\alpha\delta_{c_1} + (1-\alpha)\delta_{c_2} \precsim \delta_{\alpha c_1 + (1-\alpha)c_2},$$

and hence

$$\alpha u(c_1) + (1-\alpha)u(c_2) \leq u[\alpha c_1 + (1-\alpha)c_2].$$

Thus, u is concave.

4. Consider the insurance problem presented in class. There are two states of the world: high (h) and low (ℓ), and the probability of ℓ is μ. Without insurance, consumption at h is w_h and at ℓ is w_ℓ with $w_\ell < w_h$. One unit of insurance pays 1 at ℓ but charges premium p. The agent chooses how much insurance to buy, and, with x units of insurance, consumption levels are

$$c_h = w_h - px \text{ and } c_\ell = w_\ell + (1-p)x.$$

The agent maximizes expected utility with utility function u and is strictly risk averse.

4.1 Suppose that $p = \mu$. Find the optimal x.

Solution. The maximization problem is

$$\max_{x \geq 0} (1-\mu)u(w_h - px) + \mu u[w_\ell + (1-p)x].$$

The FOC then implies

$$-(1-\mu)pu'(w_h - px) + \mu(1-p)u'[w_\ell + (1-p)x] \leq 0, \tag{3}$$

with equality whenever $x > 0$.

Thus, when $p = \mu$, this implies

$$-u'(w_h - px) + u'[w_\ell + (1-p)x] \leq 0. \tag{4}$$

Now, since $w_h > w_\ell$,

$$x^* = w_h - w_\ell > 0$$

solve (4) and is unique.

4.2 Show that there exists an upper bound $\bar{p} < 1$ on the premium such that for all $p \geq \bar{p}$, optimal $x = 0$.

Solution. Let $\bar{p}$ be determined by

$$\frac{\bar{p}}{1-\bar{p}} = \frac{\mu u'(w_\ell)}{(1-\mu)u'(w_h)}.$$

Then, for any $p \geq \bar{p}$, (3) is satisfied with $x = 0$.