Exercise for Expected Utility Theory

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1. Consider the expected utility theory presented in class, but with $C = \{c_1, c_2, c_3\}$. We assume that $c_1 \prec c_3$.

1.1 Show that if there is a function $u : C \to \mathbb{R}$ such that

$$\mu \precsim \mu' \text{ if and only if } \sum_{i=1}^{n} \mu(c_i)u(c_i) \leq \sum_{i=1}^{n} \mu'(c_i)u(c_i), \tag{1}$$

then $\precsim$ satisfies (EU1)-(EU3).

1.2 Suppose that $\precsim$ satisfies (EU1)-(EU3). Construct u as in class with $u(c_3) = 1$ and $u(c_1) = 0$. Show that (1) holds with the following steps.

(a) Show that $\alpha\delta_{c_3} + (1-\alpha)\delta_{c_1} \prec \alpha'\delta_{c_3} + (1-\alpha')\delta_{c_1}$ if and only if $\alpha < \alpha'$.

(b) For any $\mu \in \Delta(C)$, show that

$$\mu \sim [u(c_2)\mu(c_2) + \mu(c_3)]c_3 + [(1 - u(c_2))\mu(c_2) + \mu(c_1)]c_1.$$

(c) Show that the result follows from (a) and (b) and that $\mathbb{E}_\mu(u) = u(c_2)\mu(c_2) + \mu(c_3)$.

2. Show that the set of simple lotteries, $\Delta(\mathbb{R}_+)$, is closed under compound lottery, that is, if μ_1 and μ_2 are simple lotteries and $\alpha \in (0, 1)$, then $\alpha\mu_1 + (1-\alpha)\mu_2$ is well-defined and is itself a simple lottery.

3. Show that if $\precsim$ is a relation over $\Delta(\mathbb{R}_+)$ satisfying (EU1)-(EU3) represented by u, then

1. $\precsim$ satisfies MC iff u is strictly increasing;

2. $\precsim$ satisfies (strict) risk aversion iff u is (strictly) concave.

4. Consider the insurance problem presented in class. There are two states of the world: high (h) and low (ℓ), and the probability of ℓ is μ. Without insurance, consumption at h is w_h and at ℓ is w_ℓ with $w_\ell < w_h$. One unit of insurance pays 1 at ℓ but charges premium p. The agent chooses how much insurance to buy, and, with x units of insurance, consumption levels are

$$c_h = w_h - px \text{ and } c_\ell = w_\ell + (1-p)x.$$

The agent maximizes expected utility with utility function u and is strictly risk averse.

4.1 Suppose that $p = \mu$. Find the optimal x.

4.2 Show that there exists an upper bound $\bar{p} < 1$ on the premium such that for all $p \geq \bar{p}$, optimal $x = 0$.