Epistemic Logic and Applications

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Logical system

Logical inferences are crucial in game theoretical arguments

- derivation of best responses
- derivation of others' best responses and then equilibrium

Symmetry is an important assumption in social sciences

- the analyst assumes the subjects are symmetric to himself in many ways
- including the logical abilities

Logical inferences

Mathematical logic treats logical inferences as objects of study

- an inference is simply a sequence of symbols
- but follows a certain rules

Connection between *provability* and *validity*

- a statement is provable if there is a proof for it
- a statement is valid if it is true in all states of the world

Propositional logic

Simplest setting to study logical inferences

- begins with a set of *elementary* or atomic propositions
- each statement consists of elementary statements connected by logical connectives

Belief operators to distinguish different players' scopes of thinking

- discuss logical inferences within each player's scope
- the analyst makes inference about the objective world

Propositions

A set of elementary propositions, $\mathcal{P}_0$

- typical element denoted by p, q, r
- interpreted as "indecomposable" propositions

A set of logical connectives

- $\vee$, or
- $\wedge$, and
- $\neg$, negation
- $\Rightarrow$, implication

Propositions (cont.)

The set of all (well-formulated) propositions is defined by induction

- base is $\mathcal{P}_0$
- induction step: construct new propositions from previous layers by connection by logical connectives

Formally, the set $\mathcal{P}$ is generated by finite applications of:

- if $A \in \mathbf{P}_0$, then $p \in \mathcal{P}$
- if $A, B \in \mathcal{P}$, then $A \vee B,\ A \wedge B, \neg A, A \Rightarrow B \in \mathcal{P}$

Example: $p \Rightarrow (q \Rightarrow p)$ is constructed from

- first, $A = (q \Rightarrow p)$ from p and q
- then, $p \Rightarrow A$

Syntax vs Semantics

Syntax is concerned with the “form” of a proposition

- $p \wedge q$ and $q \wedge p$ are syntactically different
- but they seem to have the same meaning

Semantics is concerned with the “meaning” of a proposition

- formalized by *truth assignment*

Truth assignment

A truth assignment is a function $\tau : \mathcal{P}_0 \to \{\top, \bot\}$

- $\top$ means “true, $\bot$ means “false”

τ can then be extended to $\mathcal{P}$ by induction

- if $\tau(A) = \top$, or $\tau(B) = \top$, then $\tau(A \vee B) = \top$; o/w, $\tau(A \vee B) = \bot$
- if $\tau(A) = \top = \tau(B)$, then $\tau(A \wedge B) = \top$; o/w, $\tau(A \wedge B) = \bot$
- if $\tau(A) = \top$, then $\tau(\neg A) = \bot$; o/w, $\tau(A \vee B) = \top$
- if $\tau(A) = \bot$, or $\tau(B) = \top$, then $\tau(A \Rightarrow B) = \top$; o/w, $\tau(A \Rightarrow B) = \bot$

Validity

A proposition A is a *tautology* if

$$\tau(A) = \top \text{ under any truth assignment } \tau$$

Examples:

- $p \Rightarrow (q \Rightarrow p)$
- $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$
- $((\neg p) \Rightarrow (\neg q)) \Rightarrow (((\neg p) \Rightarrow q) \Rightarrow p)$

If $A \in \mathcal{P}$ contains n elementary propositions, how many verifications do you need to check validity of A?

Some (meta)propositions

Proposition

1. *Let A be a tautology, p a elementary proposition, and B a proposition. Then, A', obtained from A by replacing all occurrences of p by B, is also a tautology.*
2. *Suppose that A and $A \Rightarrow B$ are both tautologies. Then, B is a tautology.*

- these are *meta*-propositions, propositions about propositional logic, *not* propositions within propositional logic

Examples

Let $A, B, C \in \mathcal{P}$; show that the followings are tautologies:

- $A \Rightarrow (B \Rightarrow A)$
- $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
- $((\neg A) \Rightarrow (\neg B)) \Rightarrow (((\neg A) \Rightarrow B) \Rightarrow A)$

Proof Theory

Formally, a proof is a sequence of propositions

- each item is either an *axiom*
- or follows from previous items a according to a *inference rule*

Axioms: let $A, B, C \in \mathcal{P}$

L1 $A \Rightarrow (B \Rightarrow A)$

L2 $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

L3 $((\neg A) \Rightarrow (\neg B)) \Rightarrow (((\neg A) \Rightarrow B) \Rightarrow A)$

Inference rule: from $A \Rightarrow B$ and A infer B (MP, *modus ponens*)

Proof

A sequence of propositions, $\{A_1, A_2, ..., A_n\}$ is a *proof* of B if

- $A_n = B$
- for each $i = 1, ..., n$, either
  - A_i is an axiom, or
  - A_i is obtained from $A_{i'}$ and $A_{i''}$ using MP, $i', i'' < i$
- We use $\vdash B$ to denote the fact that B is provable

Let $\Gamma \subset \mathcal{P}$; we use $\Gamma \vdash B$ to denote the fact that

- there is proof for B, in which
- propositions in Γ can be used as axioms

Examples

For any $A \in \mathcal{P}$, the proposition $A \Rightarrow A$ is provable

S1 let $B = (A \Rightarrow A)$; then, L2 implies
$(A \Rightarrow (B \Rightarrow A)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow A))$

S2 but L1 implies $A \Rightarrow ((A \Rightarrow A) \Rightarrow A)$, that is, $A \Rightarrow (B \Rightarrow A)$

S3 then, from S1 and S2, MP implies $(A \Rightarrow B) \Rightarrow (A \Rightarrow A)$

S4 L1 implies $A \Rightarrow (A \Rightarrow A)$, that is, $A \Rightarrow B$

S5 from S3 and S4, MP implies $A \Rightarrow A$

Deduction Theorem

Theorem

Let $\Gamma \subset \mathcal{P}$ and $A, B \in \mathcal{P}$. If $\Gamma, A \vdash B$, then $\Gamma \vdash A \Rightarrow B$.

- as a corollary, $A \vdash B$ if and only if $\vdash A \Rightarrow B$

Corollary

(a) $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow C$

(b) $A \Rightarrow (B \Rightarrow C), B \vdash A \Rightarrow C$

Soundness Theorem

Theorem

If B is provable, then B is a tautology.

Completeness Theorem

Theorem

If B is a tautology, then B is provable.

The proof uses the following lemma

Lemma

Let $A \in \mathcal{P}$ and let $B_1, \dots, B_n$ be the elementary propositions that occur in A. For any truth assignment τ, define

$$B_i' = B_i \text{ if } \tau(B_i) = \top, \text{ and } B_i' = \neg B_i \text{ if } \tau(B_i) = \bot.$$

Similarly, define $A' = A$ if $\tau(A) = \top$ and $A' = \neg A$ otherwise. Then,

$$B_1', \dots, B_n' \vdash A'.$$

Consistency

The propositional logic is *consistent* in the sense that for any $B \in \mathcal{P}$, it cannot be the case that

$$\vdash B \text{ and } \vdash \neg B$$

A set of propositions, Γ, is *consistent* if, for any B, it is not the case that

$$\Gamma \vdash B \text{ and } \Gamma \vdash \neg B$$

Epistemic logic

A framework for formal epistemology

- different thinking scopes for different individuals
- describing different individuals' beliefs
- describe different individuals' inferences

Potential applications to game theory and economics

- formalize the notion "common knowledge" or the lack of it
- formalize bounded interpersonal reasoning, e.g., level-k theory

Belief operators

Set of individuals (players): $i = 1, ..., N$

- each player is capable of logical inferences
- we use $\mathbf{B}_i$, the belief operator, to describe the scope of i's thinking

Set of propositions, $\mathcal{L}$:

- if A and B are propositions, so are $A \Rightarrow B$, $A \Rightarrow B$, $\neg A$, and $A \wedge B$
- if A is a proposition, so is $\mathbf{B}_i(A)$

For example, $\mathbf{B}_1(\mathbf{B}_2(A))$ is a proposition

- it describes player 1's belief about player 2's belief

This is a finite language, but includes higher order beliefs of arbitrary orders

Epistemic proof theory

Axioms for propositional logic, (L1)-(L3), and MP

- all tautologies are provable

Epistemic axioms: for all $i = 1, ..., N$

K $\mathbf{B}_i(A \Rightarrow B) \Rightarrow (\mathbf{B}_i(A) \Rightarrow \mathbf{B}_i(B))$

D $\neg\mathbf{B}_i(A \wedge \neg A)$

Epistemic inference rule:

Nec from A infer $\mathbf{B}_i(A)$

- axiom K and rule Nec ensure that the player has perfect logical ability
- axiom D ensures that player i's beliefs are consistent

Kripke semantics

Extends the truth assignment τ to belief operators

- to do so, need to have a scope for each player
- Kripke semantics uses connection between different possible worlds to model the scopes

A Kripke model is a list $M = (W, P_1, ..., P_N, \tau)$

- set of possible worlds, W (set of states)
- accessibility relation for each i, P_i (possibility relation)
- truth valuation: $\tau : W \times \mathcal{P}_0 \rightarrow \{\top, \bot\}$

Truth evaluation

The function τ is extended to $W \times \mathcal{L}$ as follows:

- $\tau(w, A) = \top$ iff $\tau(w, \neg A) = \bot$
- $\tau(w, A \wedge B) = \top$ iff $\tau(w, A) = \top = \tau(w, B)$
- $\tau(w, A \Rightarrow B) = \top$ iff $\tau(w, A) = \bot$ or $\tau(w, B) = \top$
- $\tau(w, \mathbf{B}_i(A)) = \top$ iff $\tau(v, A) = \top$ for all v such that $(w, v) \in P_i$

A is valid (denoted $\models_M A$) under M iff $\tau(w, A) = \top$ for all w

Some meta-theorems

A proposition A is *non-epistemic* if it does not contain any belief operator

Theorem

(a) *Let A be a non-epistemic proposition. Then, A is valid under any M if and only if A is a tautology.*

(b) *Let $A \in \mathcal{L}$ and let M be a model. If A is valid under M, so is $\mathbf{B}_i(A)$.*

(c) *Let $A, B \in \mathcal{L}$ and let M be a model. Then, $\mathbf{B}_i(A \Rightarrow B) \Rightarrow (\mathbf{B}_i(A) \Rightarrow \mathbf{B}_i(B))$ is valid under M.*

Construct a model M such that $\mathbf{B}_i(p \wedge \neg p)$ is valid under M

Completeness theorem

A model M is *serial* if for all i and for all $w \in W$, there exists v such that $(w, v) \in P_i$

Theorem

For any proposition $A \in \mathcal{L}$, $\vdash A$ if and only if $\models_M A$ for any M that is serial.

Other epistemic axioms

- T $\mathbf{B}_i(A) \Rightarrow A$
- 4 $\mathbf{B}_i(A) \Rightarrow \mathbf{B}_i(\mathbf{B}_i(A))$
- 5 $\neg\mathbf{B}_i(A) \Rightarrow \mathbf{B}_i(\neg\mathbf{B}_i(A))$

Interpretations

- axiom T connects player$i$'s belief to the outer world
- axioms 4 and 5 impose introspection, positive and negative

In the literature

- system with K+T+4 is called S4 system
- system with K+T+5 is called S5 system, or *partition model*

Completeness theorem for S4 and S5

A model M is called

- *transitive* if for all i and $u, v, w \in W$, $(u, v) \in P_i$ and $(v, w) \in P_i$ imply $(u, w) \in P_i$
- *reflexive* if for all i and $w \in W$, $(w, w) \in P_i$
- *euclidean* if for all i and $u, v, w \in W$, $(u, v) \in P_i$ and $(u, w) \in P_i$ imply $(v, w) \in P_i$

Note that P_i is an equivalence relation if and only if it is transitive, reflexive, and euclidean

Completeness theorems

Theorem

For any proposition $A \in \mathcal{L}$,

(1) $\vdash A$ *under S4 if and only if* $\models_M A$ *for any* M *that is transitive and reflexive;*

(1) $\vdash A$ *under S45 if and only if* $\models_M A$ *for any* M *that is transitive, euclidean and reflexive.*