General Equilibrium for the Exchange Economy

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(Lecture 9, Micro Theory I)
What We Learned from the 2x2 Economy?

- **Pareto Efficient Allocation (PEA)**
  - Cannot make one better off without hurting others
- **Walrasian Equilibrium (WE)**
  - When Supply Meets Demand
  - Focus on Exchange Economy First
- **1st Welfare Theorem**: WE is Efficient
- **2nd Welfare Theorem**: Any PEA can be supported as a WE
- **These also apply to the general case as well!**
General Exchange Economy

- **$n$ Commodities**: 1, 2, ..., $n$
- **$H$ Consumers**: $h = 1, 2, \ldots, H$
  - Consumption Set: $X^h \subset \mathbb{R}^n$
  - Endowment: $\omega^h = (\omega_1^h, \ldots, \omega_n^h) \in X^h$
  - Consumption Vector: $x^h = (x_1^h, \ldots, x_n^h) \in X^h$
  - Utility Function: $U^h(x^h) = U^h(x_1^h, \ldots, x_n^h)$
  - Aggregate Consumption and Endowment:
    \[ x = \sum_{h=1}^{H} x^h \text{ and } \omega = \sum_{h=1}^{H} \omega^h \]
- **Edgeworth Cube (Hyperbox)**)
Feasible Allocation

- A allocation is feasible if
- The sum of all consumers’ demand doesn’t exceed aggregate endowment: \( x - \omega \leq 0 \)
- A feasible allocation \( \bar{x} \) is Pareto efficient if
- there is no other feasible allocation \( x \) that is
  - strictly preferred by at least one: \( U^i(x^i) > U^i(\bar{x}^i) \)
  - and is weakly preferred by all: \( U^h(x^h) \geq U^h(\bar{x}^h) \)
Walrasian Equilibrium

- **Price-taking:** Prices $p \geq 0$
- **Consumers:** $h=1, 2, \ldots, H$
  - **Endowment:** $\omega^h = (\omega_1^h, \ldots, \omega_n^h)$ \quad $\omega = \sum_h \omega^h$
  - **Wealth:** $W^h = p \cdot \omega^h$
  - **Budget Set:** $\{x^h \in X^h \mid p \cdot x^h \leq W^h\}$
  - **Consumption Set:** $\bar{x}^h = (\bar{x}_1^h, \ldots, \bar{x}_n^h) \in X^h$
- **Most Preferred Consumption:**
  \[ U^h(\bar{x}^h) \geq U^h(x^h) \text{ for all } x^h \text{ such that } p \cdot x^h \leq W^h \]
- **Vector of Excess Demand:** $\bar{e} = \bar{x} - \omega$
Definition: Walrasian Equilibrium Prices

- The price vector $p \geq 0$ is a Walrasian Equilibrium price vector if
- there is no market in excess demand ($\bar{e} \leq 0$),
- and $p_j = 0$ for any market that is in excess supply ($\bar{e}_j < 0$).

- We are now ready to state and prove the “Adam Smith Theorem” (WE $\Rightarrow$ PEA)…
Proposition 3.2-1: First Welfare Theorem

- If preferences of each consumer satisfies LNS, then the Walrasian Equilibrium allocation is Pareto efficient.

- Proof:
  1. Since \( U^h(x^h) > U^h(\bar{x}^h) \) \( \Rightarrow p \cdot x^h > p \cdot \omega^h \)
  2. By LNS, \( U^h(x^h) \geq U^h(\bar{x}^h) \) \( \Rightarrow p \cdot x^h \geq p \cdot \omega^h \)
  3. Then, \( \sum_{h} (p \cdot x^h - p \cdot \omega^h) = p \cdot (x - \omega) > 0 \)

- Which is not feasible \( (x - \omega > 0) \), since \( p \geq 0 \)
First Welfare Theorem: \( WE \rightarrow PE \)

1. Why \( U^h(x^h) > U^h(x^h) \Rightarrow p \cdot x^h > p \cdot \omega^h \)?

   \( x^h \) solves \( \max \{ U^h(x^h) \mid p \cdot x^h \leq p \cdot \omega^h \} \)

2. Why \( U^h(x^h) \geq U^h(x^h) \Rightarrow p \cdot x^h \geq p \cdot \omega^h \)?
   - Suppose not, then \( p \cdot x^h < p \cdot x^h \)
   - All bundles in sufficiently small neighborhood of \( x^h \) is in budget set \( \{ x^h \in X^h \mid p \cdot x^h \leq W^h \} \)
   - LNS requires a \( \hat{x}^h \) in this neighborhood to have \( U^h(\hat{x}^h) > U^h(x^h) \), a contradiction.
Lemma 3.2-2: Quasi-concavity of $V$

- If $U^h, h = 1, \cdots, H$ is quasi-concave,
- Then so is the indirect utility function

\[ V^i(x) = \max_{x^h} \left\{ U^i(x^i) \left| \sum_{h=1}^{H} x^h \leq x, \right. \right\} \]

\[ U^h(x^h) \geq U^h(\hat{x}^h), h \neq i \]
Lemma 3.2-2: Quasi-concavity of $V$

- Proof: Consider $V^i(b) \geq V^i(a)$, for any $c = (1 - \lambda)a + \lambda b$, need to show $V^i(c) \geq V^i(a)$

  Assume $\{a^h\}_{h=1}^H$ solves $V^i(a)$, $\{b^h\}_{h=1}^H$ solves $V^i(b)$, $\{c^h\}_{h=1}^H$ is feasible since $c^h = (1 - \lambda)a^h + \lambda b^h$

  $\Rightarrow V^i(c) \geq U^i(c^i)$

  Now we only need to prove $U^i(c^i) \geq V^i(a)$. 
Lemma 3.2-2: Quasi-concavity of $V$

- Since $\{a^h\}_{h=1}^H$ solves $V^i(a)$, $\{b^h\}_{h=1}^H$ solves $V^i(b)$, $U^i(a^i) = V^i(a)$ and $U^i(b^i) = V^i(b) \geq V^i(a)$
- $\Rightarrow U^i(c^i) \geq V^i(a)$ by quasi-concavity of $U^i$
- $\Rightarrow V^i(c) \geq U^i(c^i) \geq V^i(a)$

- Note: (By quasi-concavity of $U^h$) $U^h(a^h) \geq U^h(\hat{x}^h)$ for all $h \neq i$ $\Rightarrow U^h(c^h) \geq U^h(\hat{x}^h)$
- $U^h(b^h) \geq U^h(\hat{x}^h)$ for all $h \neq i$
Proposition 3.2-3: Second Welfare Theorem

- Suppose $X^h = \mathbb{R}_+^n$, and utility functions $U^h(.)$
- continuous, quasi-concave, strictly monotonic.
- If $\{\hat{x}^h\}_{h=1}^H$ is Pareto efficient, then there exist a price vector $p \geq 0$ such that
  $$U^h(x^h) > U^h(\hat{x}^h) \implies p \cdot x^h > p \cdot \hat{x}^h$$
- Proof:
Proposition 3.2-3: Second Welfare Theorem

- Proof: Assume nobody has zero allocation
  - Relaxing this is easily done…
- By Lemma 3.2-2, \( V^i(x) \) is quasi-concave
- \( V^i(x) \) is strictly increasing since \( U^i(\cdot) \) is also
  - (and any increment could be given to consumer \( i \))
- Since \( \{\hat{x}^h\}_{h=1}^H \) is Pareto efficient, \( V^i(\omega) = U^i(\hat{x}^i) \)
- Since \( U^i(\cdot) \) is strictly increasing,

\[
\sum_{h=1}^H \hat{x}^h = \omega
\]
Proposition 3.2-3: Second Welfare Theorem

- Proof (Continued):
- Since $\omega$ is on the boundary of $\{x | V^i(x) \geq V^i(\omega)\}$
- By the Supporting Hyperplane Theorem, there exists a vector $p \neq 0$ such that
  $$V^i(x) > V^i(\omega) \Rightarrow p \cdot x > p \cdot \omega$$
  and $V^i(x) \geq V^i(\omega) \Rightarrow p \cdot x \geq p \cdot \omega$

- Claim: $p > 0$, then,
  $$U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot \sum_{h=1}^{H} x^h \geq p \cdot \omega = p \cdot \sum_{h=1}^{H} \hat{x}^h$$
Proposition 3.2-3: Second Welfare Theorem

- Proof (Continued):
- Why \( p > 0 \)? If not, define \( \delta = (\delta_1, \cdots, \delta_n) > 0 \) such that \( \delta_j > 0 \) iff \( p_j < 0 \) (others = 0)
- Then, \( V^i(\omega + \delta) > V^i(\omega) \) and \( p \cdot (\omega + \delta) < p \cdot \omega \)
- Contradicting (result from the Supporting Hyperplane Theorem)

\[
U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot \sum_{h=1}^{H} x^h \geq p \cdot \omega
\]
Proposition 3.2-3: Second Welfare Theorem

- Since $U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot \sum_{h=1}^{H} x^h \geq p \cdot \sum_{h=1}^{H} \hat{x}^h$

- Set $x^k = \hat{x}^k, \ k \neq h$, then for consumer $h$

  $U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \hat{x}^h$

- Need to show strict inequality implies strict…

- If not, then $U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h = p \cdot \hat{x}^h$

- Hence, $p \cdot \lambda x^h < p \cdot \hat{x}^h$ for all $\lambda \in (0, 1)$

- $U^h$ continuous $\Rightarrow U^h(\lambda x^h) > U^h(\hat{x}^h)$ for large $\lambda$

- Contradiction!
Summary of 3.2

- Pareto Efficiency:
  - Cannot make one better off without hurting others
- Walrasian Equilibrium: market clearing prices
- Welfare Theorems:
  - First: Walrasian Equilibrium is Pareto Efficient
  - Second: Pareto Efficient allocations can be supported as Walrasian Equilibria (with transfer)
- Homework: Read “Thinking Outside the Box”
  http://essentialmicroeconomics.com/08R3/OutsideTheBox.pdf
- Do Exercise 3.2-1~3