Budget Constrained Choice with Two Commodities

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(Lecture 5, Micro Theory I)

The Consumer Problem

- We have some powerful tools:
  - Constrained Maximization (Shadow Prices)
  - Envelope Theorem (Changing Environment)
- How can they help us understand behavior of a consumer?
  - Either “maximizing utility while facing a budget constraint”, or “minimizing cost while maintaining a certain welfare level”…

Key Problems to Consider

- Consumer Problem: How can consumer's Utility Maximization result in demand?
- Income Effect: How does an increase (or decrease) in income (budget) affect demand?
- Dual Problem: How is Minimizing Expenditure related to Maximizing Utility?
- Substitution Effect: How does an increase in commodity price affect compensated demand?
- Total Price Effect = S. E. + I. E.

Why do we care about this? An Example in Public Policy

- Taiwan’s ministry of defense has to decide whether to buy more fighter jets, or more submarines given a tight budget
- How does the military rank each combination?
- How do they choose which combination to buy?
- How would a price change affect their decision?
- How would a boycott in defense budget affect their decision?

Continuous Demand Function

A Consumer with income I, facing prices p_1, p_2

$$\max_x \{U(x) | p \cdot x \leq I, x \in \mathbb{R}^2_+\}$$

- Assume:
- LNS (local non-satiation)
  - Consumer spends all his/her income
- $U(x)$ is continuous, strictly quasi-concave on $\mathbb{R}^2_+$
  - There is a unique solution $x^0 = x(p, I)$
- Then, by Proposition 2.2-1,

$x(p, I)$ must be continuous.

Stronger Convenience Assumptions for this Lecture

- Assume:
- $U(x)$ is continuously differentiable on $\mathbb{R}^2_+$
  - FOC is gradient vectors of utility (+ constraint)
- LNS-plus: $$\frac{\partial U}{\partial x}(x) > 0 \text{ for all } x \in \mathbb{R}^2_+$$
  - At least one commodity has MU > 0
- No corners: $$\lim_{x_j \to 0} \frac{\partial U}{\partial x_j} = \infty, j = 1, 2$$
  - Always wants to consumer some of everything

Indifference Curve Analysis (Lagrangian Version)

A Consumer with income I, facing prices p_1, p_2

$$\max_x \left\{ U(x) | p \cdot x \leq I, x \in \mathbb{R}^2_+ \right\}$$

$$\text{Lagrangian is } \mathfrak{L} = U + \lambda(I - p \cdot x)$$

$$(FOC) \qquad \frac{\partial \mathfrak{L}}{\partial x_j} = \frac{\partial U}{\partial x_j}(x^*) - \lambda p_j, j = 1, 2$$

$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \lambda$$

Meaning of FOC

1. Same marginal value for last dollar spent on each commodity

$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \lambda$$

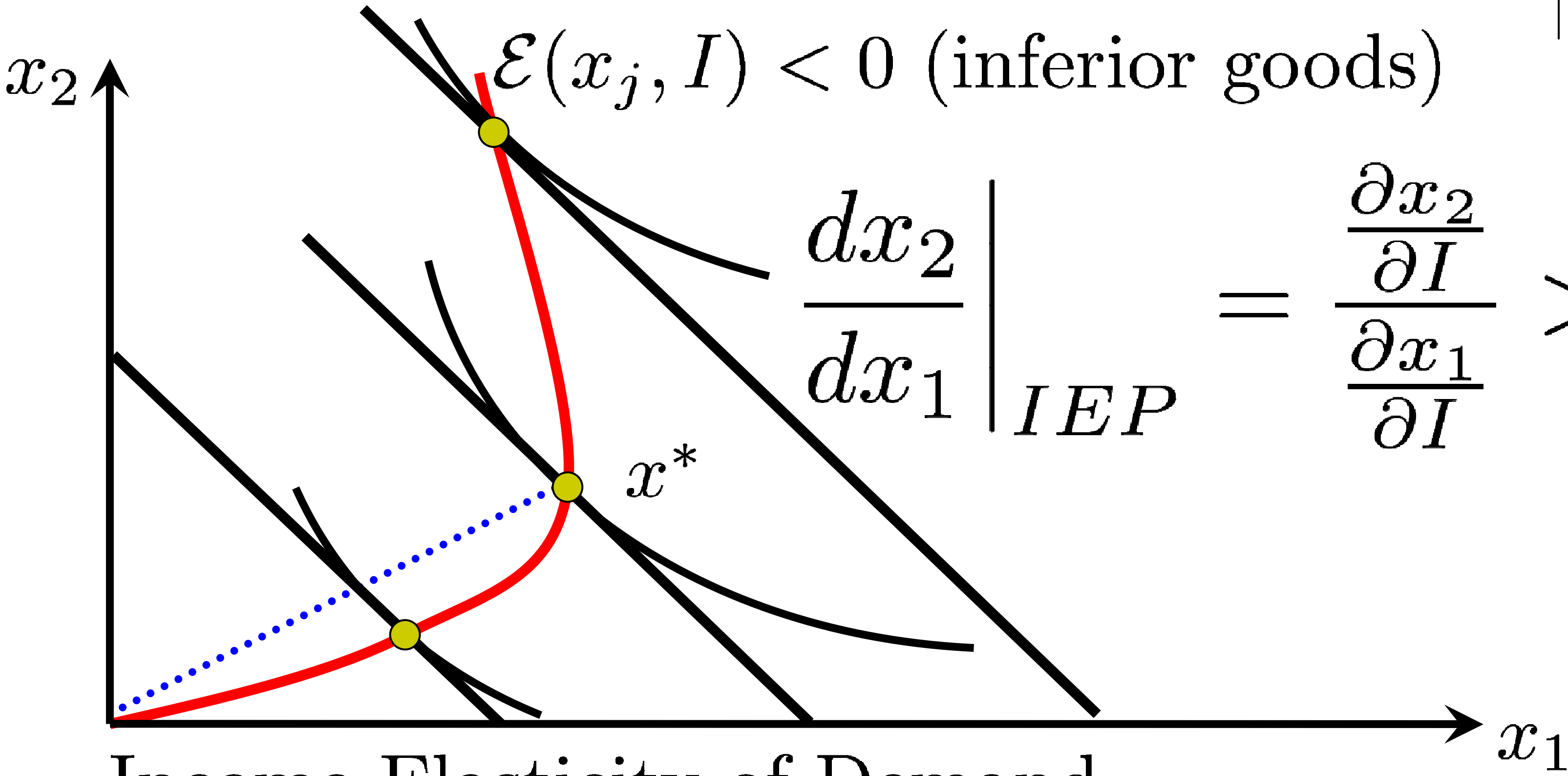
- Does Taiwan get same MU on fighter jets and submarines?

2. Indifference Curve tangent to Budget Line

$$MRS(x^*) = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{p_1}{p_2}$$

Income Effect

$\mathcal{E}(x_j, I) < 0$ (inferior goods)



$$\left.\frac{dx_2}{dx_1}\right|_{IEP} = \frac{\frac{\partial x_2}{\partial I}}{\frac{\partial x_1}{\partial I}} > \frac{x_2^*}{x_1^*}$$

Income Elasticity of Demand

$$\mathcal{E}(x_j, I) = \frac{I}{x_j}\frac{\partial x_j}{\partial I} > 0 \text{ (normal goods)}$$

Income Effect

- Slope of IEP steeper than line joining 0 and *x**

$$\left.\frac{dx_2}{dx_1}\right|_{IEP} = \frac{\frac{\partial x_2}{\partial I}}{\frac{\partial x_1}{\partial I}} > \frac{x_2^*}{x_1^*}$$

- Or,

$$\mathcal{E}(x_2, I) = \frac{I}{x_2}\frac{\partial x_2}{\partial I} > \mathcal{E}(x_1, I) = \frac{I}{x_1}\frac{\partial x_1}{\partial I}$$

- Lemma 2.2-2: Expenditure share weighted income elasticity add up to 1
- So, $\mathcal{E}(x_2, I) > 1 > \mathcal{E}(x_1, I)$

Three Examples

- Quasi-Linear Convex Preference

$$U(x) = v(x_1) + \alpha x_2$$

- Cobb-Douglas Preferences

$$U(x) = x_1^{\alpha_1} x_2^{\alpha_2}, \alpha_1, \alpha_2 > 0$$

- CES Utility Function

$$U(x) = \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}}$$

Dual Problem: Minimizing Expenditure

Consider the least costly way to achieve $\overline{U}$

$$M(p,\overline{U}) = \min_x \{p \cdot x | U(x) \geq \overline{U}\}$$

For $x(p,I)$ solving $\max_x\{U(x)|p \cdot x \leq I\}$

$U(x(p,I))$ is strictly increasing over I

For any $\overline{U}$, there is unique income M such that

$$\overline{U} = U(x(p,M))$$

Can solve $M(p,\overline{U})$ by inverting $\overline{U} = U(x(p,M))$

Substitution Effect for Compensated Demand

- Compensated Demand

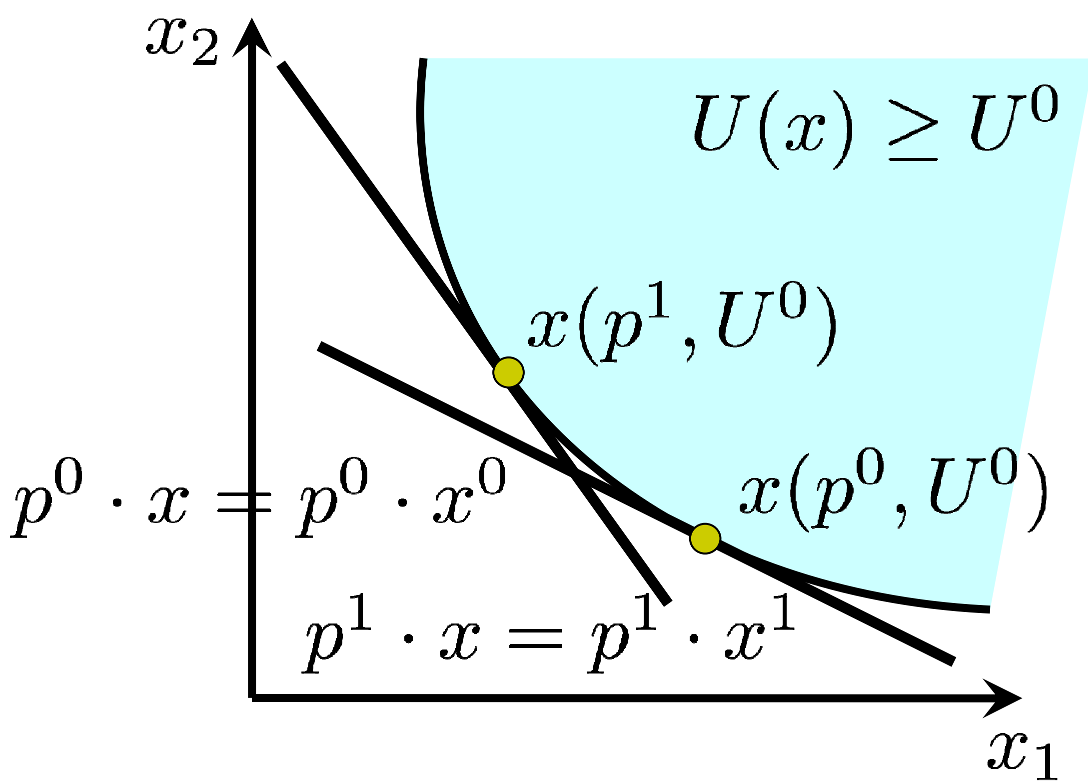
$$x^c(p,\overline{U}) \text{ solves } M(p,\overline{U}) = \min_x \{p \cdot x | U(x) \leq \overline{U}\}$$

- By Envelope Theorem:
- Effect of Price Change $\dfrac{\partial M}{\partial p_j} = x^c(p, U^0)$
  - How much more does Taiwan have to pay if the price of submarines increase (to maintain the same level of defense)?

Elasticity of Substitution (for Compensated Demand)

$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right) = \frac{\mathcal{E}(x_2^c, p_1)}{k_1} = -\frac{\mathcal{E}(x_1^c, p_1)}{1-k_1}$$

$$k_1 = \frac{p_1 x_1}{p \cdot x}$$



Verify that $\sigma = \theta$ for CES…

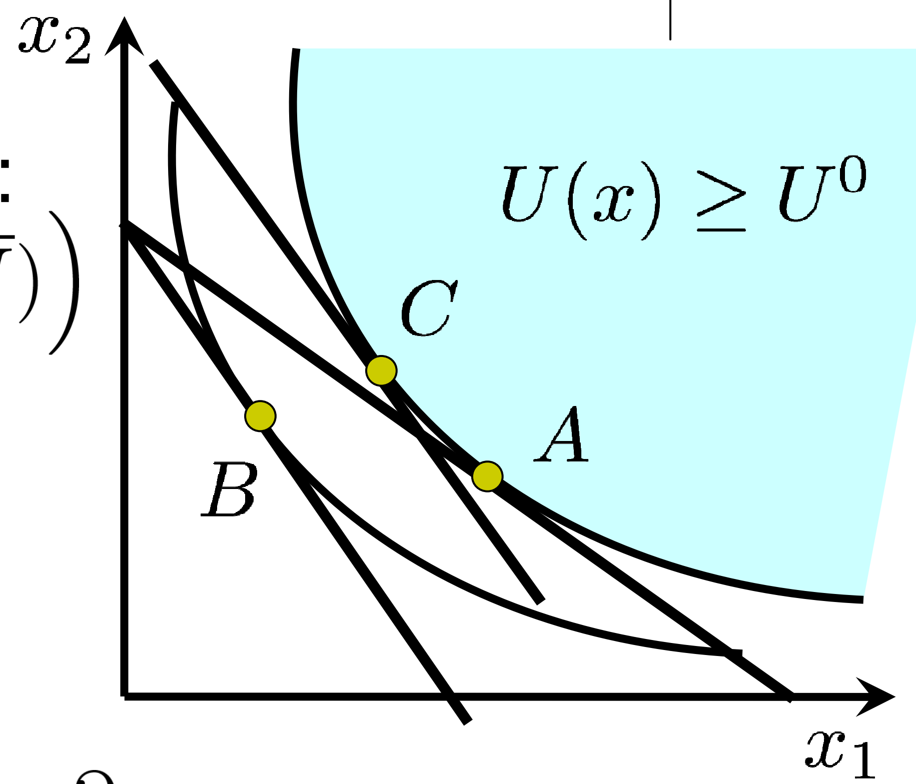
Total Price Effect = Income Effect + Substitution Effect

- For $M(p,\overline{U})$ & $x_1(p,I)$
- Compensated Demand:

$$x_1^c(p,\overline{U}) = x_1\left(p, M(p,\overline{U})\right)$$

$$\frac{\partial x_1^c}{\partial p_1} = \frac{\partial x_1}{\partial p_1} + \frac{\partial x_1}{\partial I}\cdot\frac{\partial M}{\partial p_1}$$

$$\left(\frac{\partial M}{\partial p_1} = x_1\right)$$



- Slutsky Equation:

$$\underbrace{\frac{\partial x_1}{\partial p_1}}_{A\to B} = \underbrace{\frac{\partial x_1^c}{\partial p_1}}_{A\to C} \underbrace{- x_1\cdot\frac{\partial x_1}{\partial I}}_{C\to B}$$

Total Price Effect = Income Effect + Substitution Effect

- Slutsky Equation:

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^c}{\partial p_1} - x_1 \cdot \frac{\partial x_1}{\partial I}$$

- Elasticity Version:

$$\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1} = \frac{p_1}{x_1}\frac{\partial x_1^c}{\partial p_1} - \frac{p_1 x_1}{I}\frac{I}{x_1} \cdot \frac{\partial x_1}{\partial I}$$

- Or,

$$\mathcal{E}(x_1, p_1) = \mathcal{E}(x_1^c, p_1) - k_1 \cdot \mathcal{E}(x_1, I)$$

Summary of 2.2

- Consumer Problem: Maximize Utility
- Income Effect
- Dual Problem: Minimize Expenditure
- Substitution Effect:
  - =Compensated Price Effect
  - Elasticity of Substitution
- Total Price Effect:
  - = Compensated Price Effect + Income Effect
- Homework: Exercise 2.2-1~7