Security Market Equilibrium

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2019/10/30

(Lecture 14, Micro Theory I)

Why do we care about this?

- Any asset can be reproduced via state-claims
- In reality, State-Claim Markets (aka Arrow-Debreu Markets) are rarely used
  - Except artificially created prediction markets!
- But we can construct all S state-claims using S linearly independent assets (securities)
  - And can in turn reproduce any asset using them!
- Security Market Equilibrium

  = Arrow-Debreu Equilibrium

Arrow-Debreu Equilibrium: 1-good, 1-period

- S States: State $s = 1, \ldots, S$ each w/ prob. π_s
- ~~F Firms: Firm $f = 1, 2, \ldots, F$ each with production vector $\vec{y}^f = (y_1^f, y_2^f, \cdots, y_S^f) \in \mathcal{Y}^f$~~ Production Set
- H Consumers: Consumer $h = 1, 2, \ldots, H$
  - Endowment: $\vec{\omega}^h = (\omega_1^h, \omega_2^h, \cdots, \omega_S^h)$, $\omega_i = \sum_h \omega_i^h$
  - ~~θ_f^h = Shareholding of firm f~~
  - Consumption: $\vec{x}^h = (x_1^h, x_2^h, \cdots, x_S^h) \in X^h \subset \mathbb{R}^S$
  - VNM Utility Function:

(Continuous, Strictly Increasing) $U^h(\vec{x}^h) = \sum_{s=1}^{S} \pi_s^h u^h(x_s^h)$

Arrow-Debreu Equilibrium (1-good, 1-period)

- Allocation $\{\vec{x}^h \in X^h\}_{h=1}^H, \{\vec{y}^f \in \mathcal{Y}^f\}_{f=1}^F$
- is a feasible allocation if

$$\sum_{h=1}^{H} \vec{x}^h \leq \sum_{h=1}^{H} \vec{\omega}^h + \sum_{h=1}^{H} \vec{y}^f$$

  - Exchange Economy: No production!

- Consider an economy with a market for each of the S state-claims...

Arrow-Debreu Equilibrium (1-good, 1-period)

$\{\vec{x}^h \in X^h\}_{h=1}^H, \{\vec{y}^f \in \mathcal{Y}^f\}_{f=1}^F$ is an Arrow-Debreu Equilibrium allocation if for some $\vec{p} \gg \vec{0}$

1. Consumption allocations are in budget sets:
$$\vec{p} \cdot \vec{x}^h \leq \vec{p} \cdot \vec{\omega}^h + \sum_{f=1}^{F} \theta_f^h \vec{p} \cdot \vec{y}^f$$
2. No strictly preferred allocation is in budget set:
$$U^h(\vec{z}) > U^h(\vec{x}^h) \Rightarrow \vec{p} \cdot \vec{z} > \vec{p} \cdot \vec{x}^h$$
3. Markets clear: $\sum_{h=1}^{H} \vec{x}^h = \sum_{h=1}^{H} \vec{\omega}^h + \sum_{h=1}^{H} \vec{y}^f$

Security Market Equilibrium: 1-good, 1-period

- For an economy without state-claim markets…
- A Assets: Asset $a = 1, \ldots, A$ has a price P_a
- and yields state-contingent dividend

$$\vec{d}_a = (d_{1a}, d_{2a}, \cdots, d_{Sa})$$

- Consumer h holds ξ_a^h units of asset a
- Portfolio $\vec{\xi}^h = (\xi_1^h, \xi_2^h, \cdots, \xi_S^h) \Rightarrow \sum_{a=1}^{A} P_a \xi_a^h = 0$
- Consumption: $\vec{x}^h = \vec{\omega}^h + \sum_{a=1}^{A} \vec{d}_a \xi_a^h$

≤ 0 if start with no asset

≥ 0 if strictly increasing

$(\vec{x}^h = \vec{\omega}^h + \sum_{a=1}^{A} \vec{d}_a \xi_a^h)$

Security Market Equilibrium: 1-good, 1-period

$\{\vec{x}^h \in X^h, \vec{\xi}^h \in \mathbb{R}^S\}_{h=1}^{H}$ is a Security Market Equilibrium allocation if for some $\vec{P} = (P_1, \cdots, P_A)$

1. Consumption allocations are in budget sets:

$$\vec{P} \cdot \vec{\xi}^h = 0 \ (h = 1, 2, \cdots, H) = \sum_{a=1}^{A} P_a \xi_a^h$$

2. No strictly preferred allocation is in budget set:

$$U^h\left(\vec{\omega}^h + \sum_{a=1}^{A} \vec{d}_a \psi_a^h\right) > U^h(\vec{x}^h) \Rightarrow \vec{P} \cdot \vec{\psi}^h > 0$$

3. Markets clear: $\sum_{h=1}^{H} \vec{x}^h = \sum_{h=1}^{H} \vec{\omega}^h, \ \sum_{h=1}^{H} \vec{\xi}^h = \vec{0}$

A-D Equilibrium is SM Equilibrium!

- Proposition 8.2-1:
- An Arrow-Debreu (A-D) Equilibrium is a Security Market (SM) Equilibrium if asset dividends span $\mathbb{R}^S$

- Proposition 8.2-2:
- An Security Market (SM) Equilibrium is an Arrow-Debreu (A-D) Equilibrium if asset dividends span $\mathbb{R}^S$

If Asset Dividends $\vec{d}_a$ Span $\mathbb{R}^S$

- Can re-label (or reconstruct) 1st S assets to get linearly independent dividend vectors $\{\vec{d}_a\}_{a=1}^A$
  - Any allocation in $\mathbb{R}^S$ can be reproduced by a linear combination of these S assets: $\sum_{a=1}^{A} \vec{d}_a \xi_a^h$
- For consumer h, there exists a portfolio $\vec{\xi}^h$ such that $\vec{x}^h - \vec{\omega}^h = \sum_{a=1}^{A} \vec{d}_a \xi_a^h$
- Let $P_a = \vec{p} \cdot \vec{d}_a \ (a = 1, \cdots, A)$
- Then,

$$\vec{p} \cdot \vec{x}^h - \vec{p} \cdot \vec{\omega}^h = \sum_{a=1}^{A} \vec{p} \cdot \vec{d}_a \xi_a^h = \sum_{a=1}^{A} P_a \xi_a^h$$

$P_a = \vec{p} \cdot \vec{d}_a \ (a = 1, \cdots, A)$ $\vec{x}^h - \vec{\omega}^h = \sum_{a=1}^{A} \vec{d}_a \xi_a^h$

Arrow-Debreu Equilibrium (1-good, 1-period)

$\{\vec{x}^h \in X^h\}_{h=1}^{H}$ is an Arrow-Debreu Equilibrium allocation if for some $\vec{p} \gg \vec{0}$

1. Consumption allocations are in budget sets:

$$\vec{p} \cdot \vec{x}^h - \vec{p} \cdot \vec{\omega}^h = \sum_{a=1}^{A} P_a \xi_a^h = 0 \text{ since } U^h \text{ strictly increasing!!}$$

2. No strictly preferred allocation is in budget set:

$$U^h\left(\vec{\omega}^h + \sum_{a=1}^{A} \vec{d}_a \psi_a^h\right) > U^h(\vec{x}^h) \Rightarrow \boxed{\vec{P} \cdot \vec{\psi}^h > 0}$$

$$\vec{p} \cdot \left(\vec{\omega}^h + \sum_{a=1}^{A} \vec{d}_a \psi_a^h\right) > \vec{p} \cdot \vec{x}^h = \vec{p} \cdot \vec{\omega}^h$$

Arrow-Debreu Equilibrium (1-good, 1-period)

$\{\vec{x}^h \in X^h\}_{h=1}^{H}$ is an Arrow-Debreu Equilibrium allocation if for some $\vec{p} \gg \vec{0}$

3. Markets clear:

$$\sum_{h=1}^{H} \vec{x}^h - \sum_{h=1}^{H} \vec{\omega}^h = \sum_{h=1}^{H} \sum_{a=1}^{A} \vec{d}_a \xi_a^h = \vec{0}$$

$$= \sum_{a=1}^{A} \vec{d}_a \left(\sum_{h=1}^{H} \xi_a^h \right)$$

- Since dividend vectors are linear independent,
  - The coefficients are all zero: $\sum_{h=1}^{H} \xi_a^h = 0$

A-D Equilibrium = SM Equilibrium!

- Proposition 8.2-1:
- An Arrow-Debreu (A-D) Equilibrium is a Security Market (SM) Equilibrium if asset dividends span $\mathbb{R}^S$
- Similarly, we have...
- Proposition 8.2-2:
- An Security Market (SM) Equilibrium is an Arrow-Debreu (A-D) Equilibrium if asset dividends span $\mathbb{R}^S$

Summary of 8.2

- Apply WE to Markets of Uncertainty
- State Claim Markets vs. Asset Markets
- I did not teach any thing new, just another (very important) application...
- Homework: Riley – 8.2-3, 4, 5; 2008 Final Q4, 2009 Final B, 2013 Final A, 2014 Final A