Equilibrium with Uncertainty

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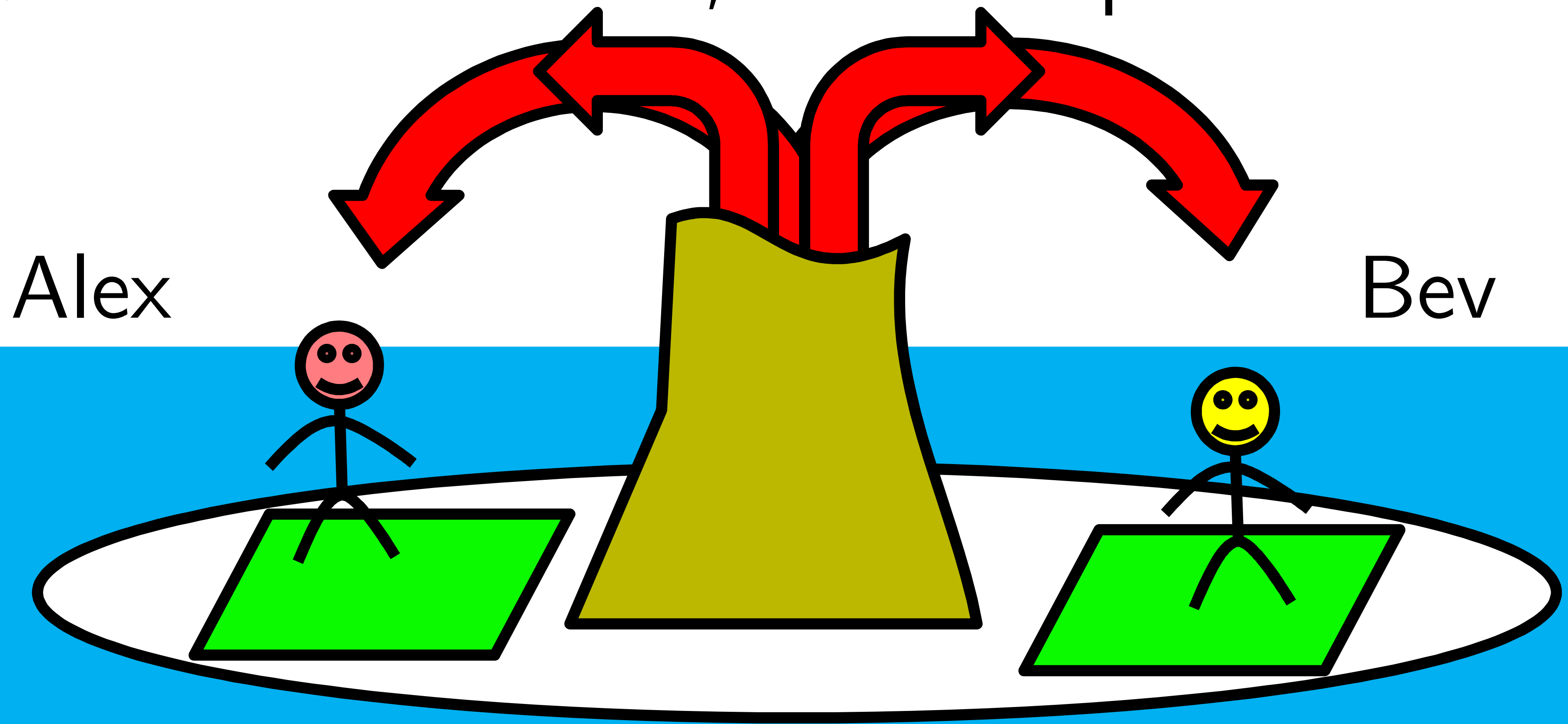
(Lecture 13, Micro Theory I)

Why do we care about this?

- Alex and Bev on Volcano Island...
- State 1: East wind; Alex's crops suffer a loss
- State 2: West wind; Bev's crops suffer a loss



Simple State Claim Economy

- 2 States: State 1 and 2, probability π_s
- 2 Consumers: Alex and Bev $h = A, B$
  - Endowment: $\vec{\omega}^h = (\omega_1^h, \omega_2^h)$, $\omega_i = \omega_i^A + \omega_i^B$
  - Consumption: $\vec{c}^h = (c_1^h, c_2^h)$
  - VNM Utility Function:

$$U^h(\vec{x}^h) = \sum_{s=1}^{2} \pi_s v^h(c_s^h)$$

- MRS:

$$MRS^h(c_1, c_2) = \frac{\pi_1 v_h'(c_1^h)}{\pi_2 v_h'(c_2^h)} = \frac{\pi_1}{\pi_2} \text{ at } 45^o$$

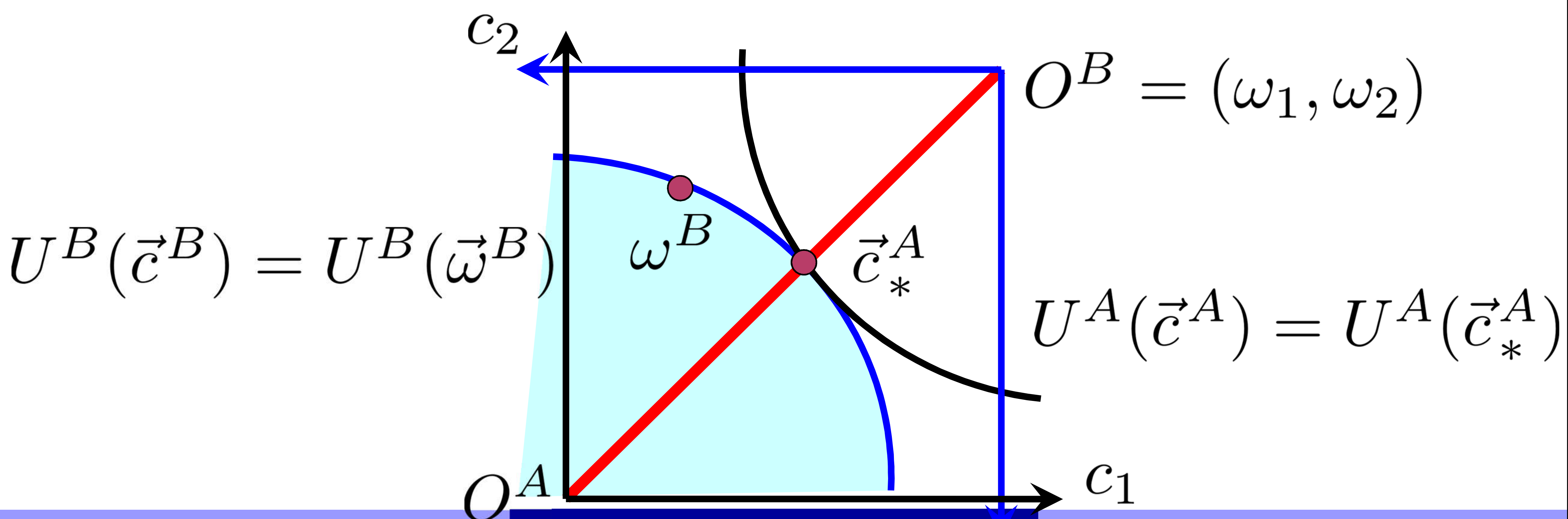
Why do we care about this?

- Learned about Walrasian Equilibrium (WE)
  - Closely related to Pareto Efficient Allocations (PEA)
- Apply to markets of uncertainty
  - Risky Investment, Futures, Sports Betting, etc.
- Few state claims in the real world?
  - Can create Prediction Markets...
- Not a problem if enough independent assets
  - Can replace state claim markets
- On-going research: Foundation of Asset Pricing

Case 1: No Aggregate Risk $\omega_1^A + \omega_1^B = \omega_2^A + \omega_2^B$

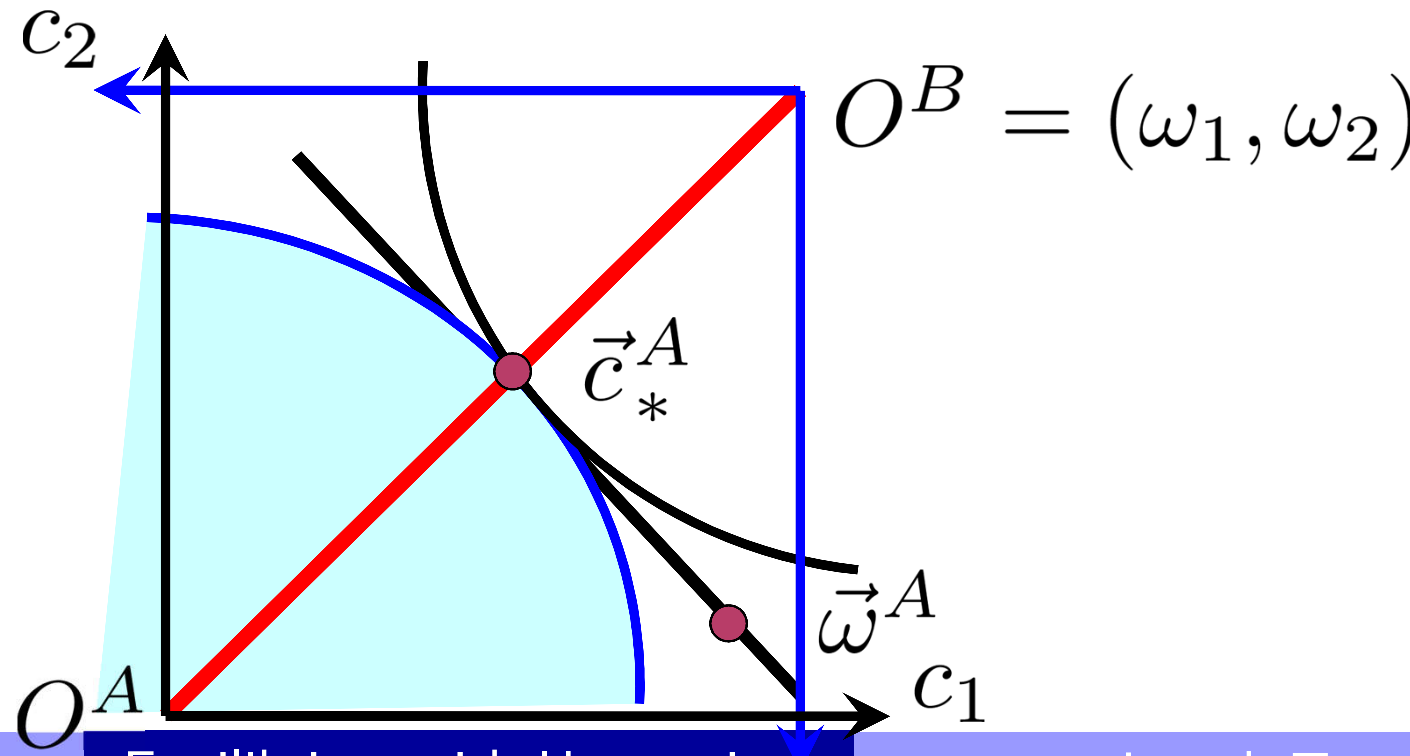
- Square Edgeworth Box $\omega_1^A = \omega_2^A - L, \omega_2^B = \omega_1^B - L$
- Pareto efficient allocation is the 45 degree line, since

$$MRS^A(c_1, c_2) = \frac{\pi_1 \cancel{v_h'(c_1^h)}}{\pi_2 \cancel{v_h'(c_2^h)}} = \frac{\pi_1}{\pi_2} = MRS^B(c_1, c_2)$$



Case 1: No Aggregate Risk

- Both want to buy insurance for the bad state
  - Buying insurance like trading in state claim market
- Standard Walrasian Equilibrium...
  - First Welfare Theorem says WE is PEA



Walrasian Equilibrium (Lecture 9 Revisited...)

- All Price-takers: Prices $\vec{p} \geq \vec{0}$
- 2 Consumers: Alex and Bev $h = A, B$
  - Endowment: $\vec{\omega}^h = (\omega_1^h, \omega_2^h)$, $\omega_i = \omega_i^A + \omega_i^B$
  - State Claim Purchase: $\vec{c}^h = (c_1^h, c_2^h) \in \mathbb{R}_+^2$
  - Wealth: $W^h = \vec{p} \cdot \vec{\omega}^h$
- Market Demand: $\vec{x}(p) = \sum_h \vec{x}^h(\vec{p}, \vec{p} \cdot \vec{\omega}^h)$
- Vector of Excess Demand: $\vec{e}(\vec{p}) = \vec{x}(\vec{p}) - \vec{\omega}$
  - Vector of total Endowment: $\vec{\omega} = \sum_h \vec{\omega}^h$

Market Clearing Prices (Lecture 9 Revisited...)

- Let excess demand for commodity j be $e_j(\vec{p})$
- The market for commodity j clears if

$$e_j(\vec{p}) \leq 0 \text{ and } p_j \cdot e_j(\vec{p}) = 0$$

- The price vector $\vec{p} \geq \vec{0}$ is a Walrasian Equilibrium price vector if all markets clear.
- With the Edgeworth Box, just need to find prices p_1/p_2 that make

$$\vec{c}^A + \vec{c}^B = \vec{\omega}^A + \vec{\omega}^B$$

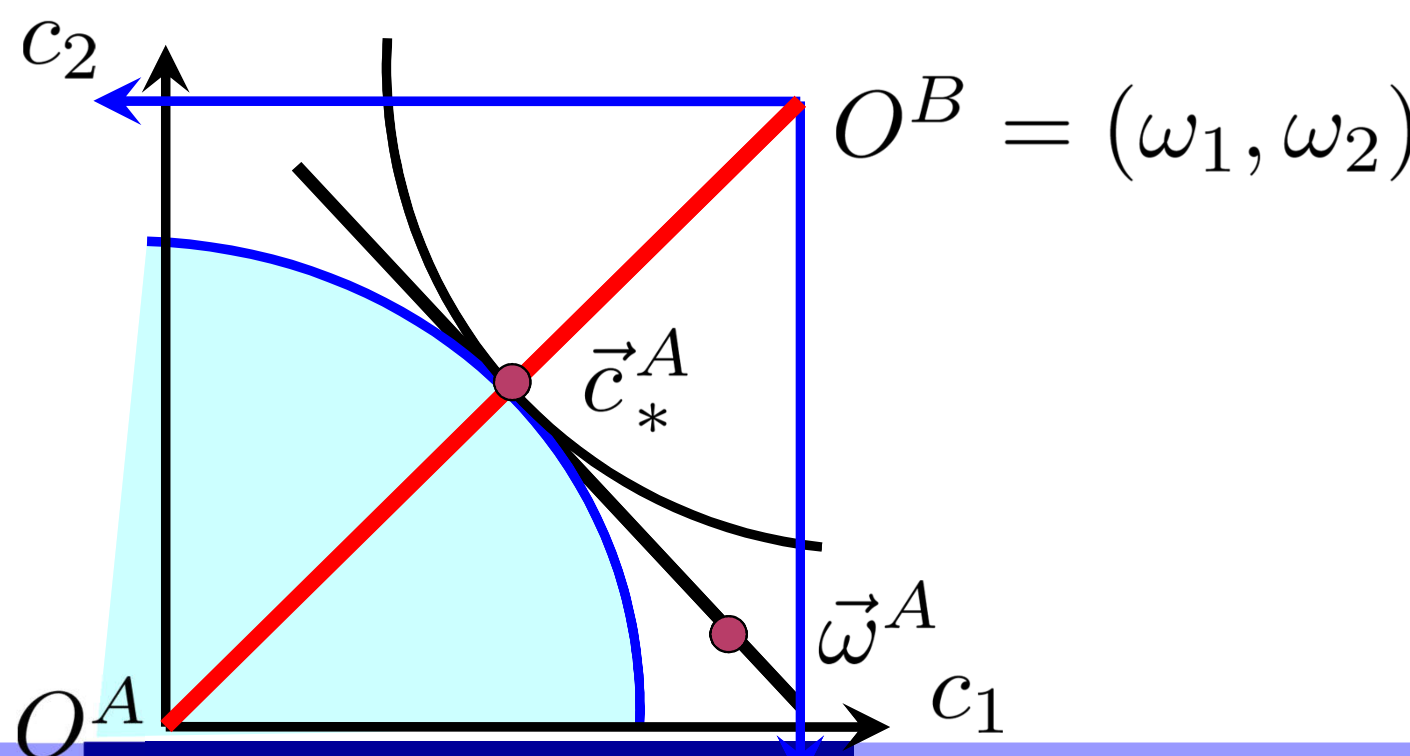
  - i.e. Being inside the box guarantees market clear

Case 1: No Aggregate Risk

- WE price ratio is

$$\frac{p_1}{p_2} = MRS_{12}^A = \frac{\pi_1 v_h'(c_1^h)}{\pi_2 v_h'(c_2^h)} = \frac{\pi_1}{\pi_2}$$

  - Equal to probability ratio ("odds")



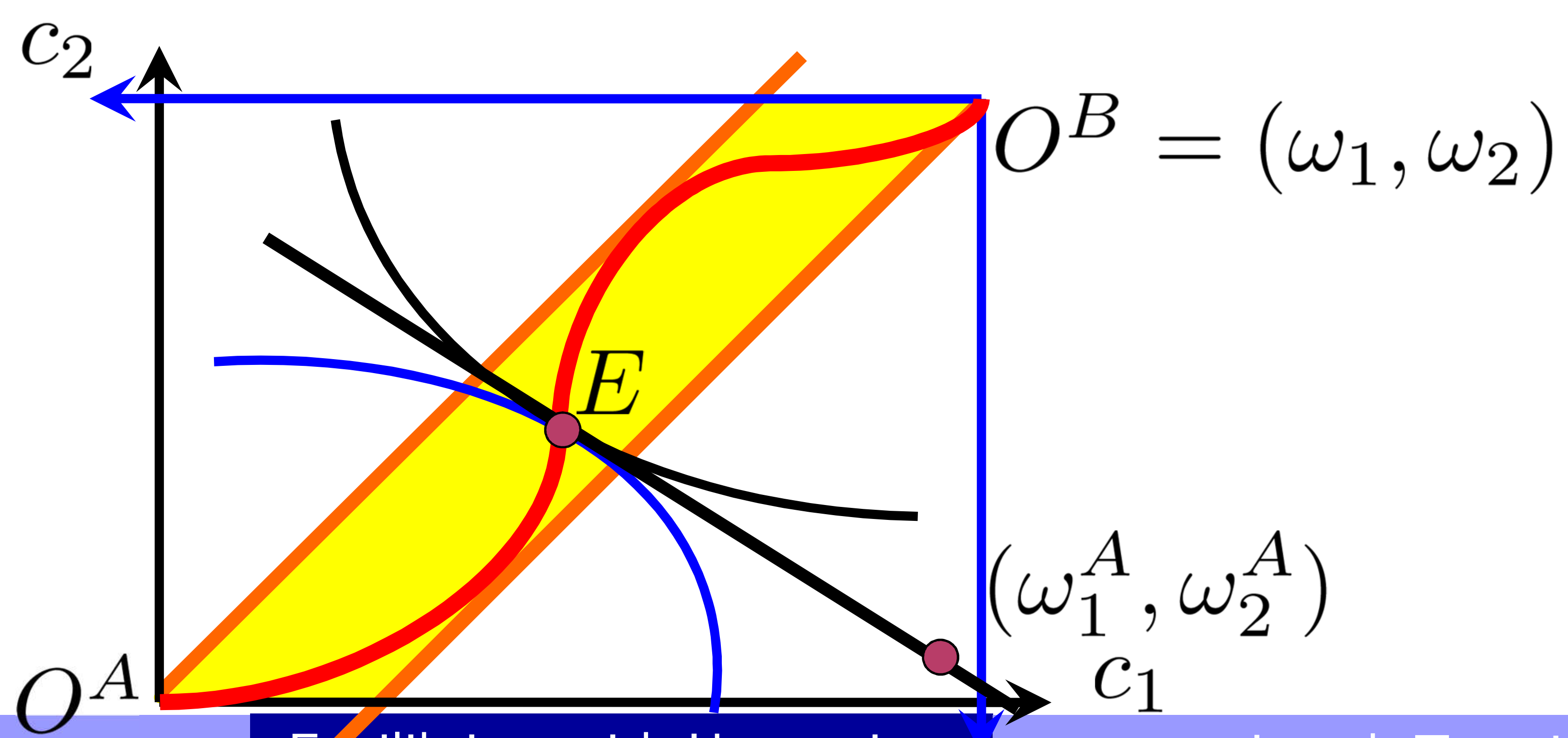
Case 2: Aggregate Risk (loss bigger in state 2)

- PEA is in the yellow area (between 45° lines)
  - Since in the upper triangle for Alex,

$$MRS_{12}^{A} > \frac{\pi_1}{\pi_2} > MRS_{12}^{B}$$

In the lower triangle for Bev

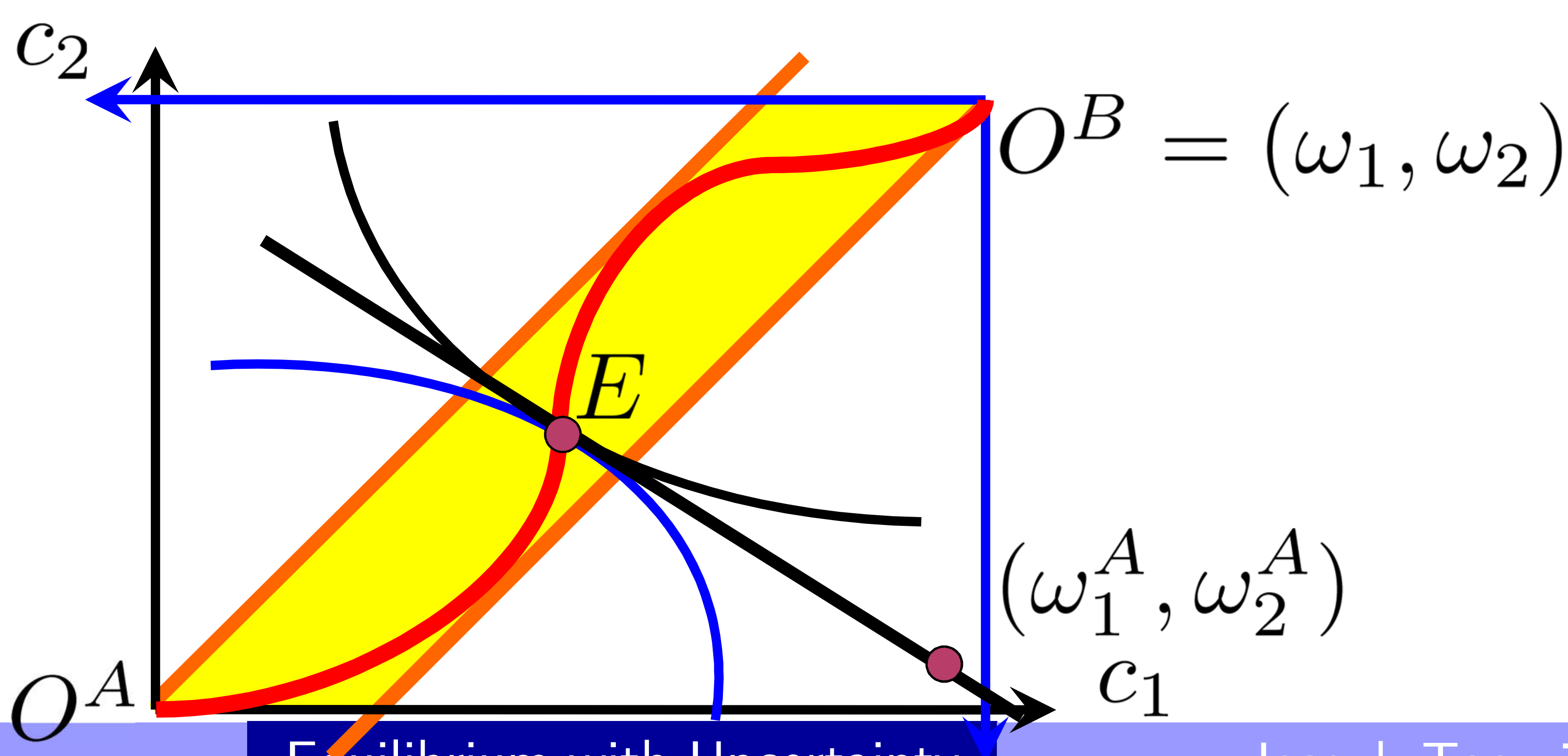
- W.E.
- Is PE!



Case 2: Aggregate Risk (loss bigger in state 2)

- Risk shared: More x_1 than x_2 allocated for both
- Prices reflect shortage of state 2 claims:

$$\frac{p_1}{p_2} = MRS_{12}^A < \frac{\pi_1}{\pi_2}$$



Case 3: Production - An Example

- Endowments:
- Alex owns a firm with uncertain output $(140, 80)$
- Bev owns a firm with out $(80 - \frac{z^2}{20}, z)$
- 2 states equally likely, Each has VNM utility:

$$U^h(c_1^h, c_2^h) = \frac{1}{2}\ln(c_1^h) + \frac{1}{2}\ln(c_2^h)$$

- Solve for WE prices such that
  - Given prices, firms Max. Π
  - Given prices, consumers max U
  - Markets Clear under these prices

Case 3: Production - Optimal Choice

$$U^h(c_1^h, c_2^h) = \tfrac{1}{2}\ln(c_1^h) + \tfrac{1}{2}\ln(c_2^h)$$

- Treat like Robinson Crusoe economy
  - Since preferences are homothetic and identical
- Aggregate supply is

$$(220 - \tfrac{z^2}{20}, 80 + z)$$

- RA solves

$$U^R = \frac{1}{2}\ln\left(220 - \tfrac{z^2}{20}\right) + \frac{1}{2}\ln(80 + z)$$

Case 3: Production – Optimal Choice

$$U^R = \frac{1}{2}\ln\left(220 - \tfrac{z^2}{20}\right) + \frac{1}{2}\ln(80 + z)$$

- FOC: (interior)

$$\frac{\partial U}{\partial z} = \frac{-\frac{z}{10}}{440 - \frac{z^2}{10}} + \frac{1}{160 + 2z} = 0$$

$$= \frac{-16z - \frac{z^2}{5} + 440 - \frac{z^2}{10}}{(160 + 2z)\left(440 - \frac{z^2}{10}\right)}$$

$$\Rightarrow 3z^2 + 160z - 4400 = 0 = (z - 20)(3z + 220)$$

- So, $z^* = 20$, and aggregate supply is $(200, 100)$

Case 3: Production - Walrasian Equilibrium

- Look for p_1/p_2 so $z^* = 20$ is indeed optimal
  - Iso-profit line and PPF tangent
- PPF: $y(z) = (80 - \frac{z^2}{20}, z)$
- Slope: $\frac{dy_2}{dy_1} = \frac{y_2'(z^*)}{y_1'(z^*)} = \frac{1}{-\frac{z^*}{10}} = -\frac{1}{2} = -\frac{p_1}{p_2}$
- Hence, setting $p_1 = 1$, we have $p_2 = 2$
- Firm values $W^A = (1, 2) \cdot (140, 80) = 300$

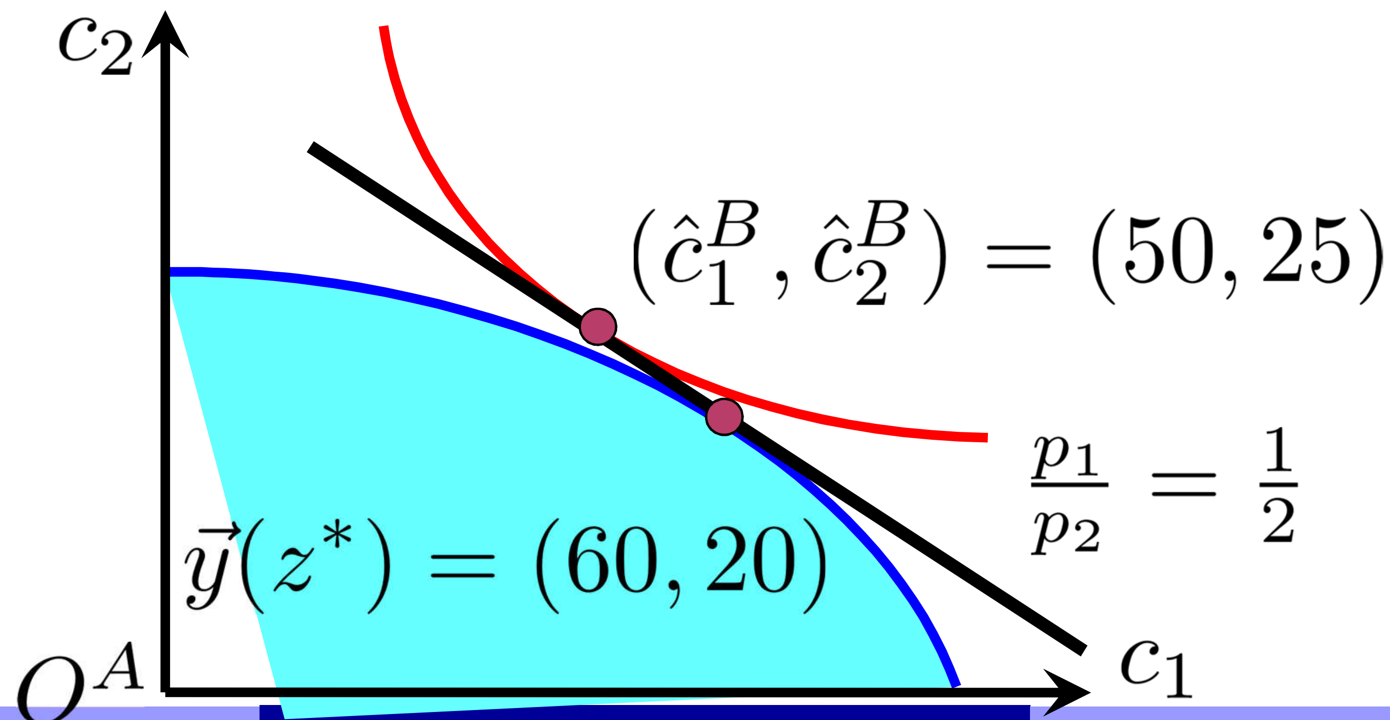
$$W^B = \vec{p} \cdot \vec{y}(z^*) = (1, 2) \cdot (60, 20) = 100$$

Case 3: Production - Bev's Equilibrium

- Budget Constraint: $c_1^B + 2c_2^B \leq W^B = 100$

$$\max U^B(c_1^B, c_2^B) = \tfrac{1}{2}\ln(c_1^B) + \tfrac{1}{2}\ln(c_2^B)$$

$$\frac{1}{c_1^B} = \frac{1}{2c_2^B} = \lambda \quad \Rightarrow \hat{c}_1^B = 50, \hat{c}_2^B = 25$$



Case 3: Production - State Claims vs. Assets

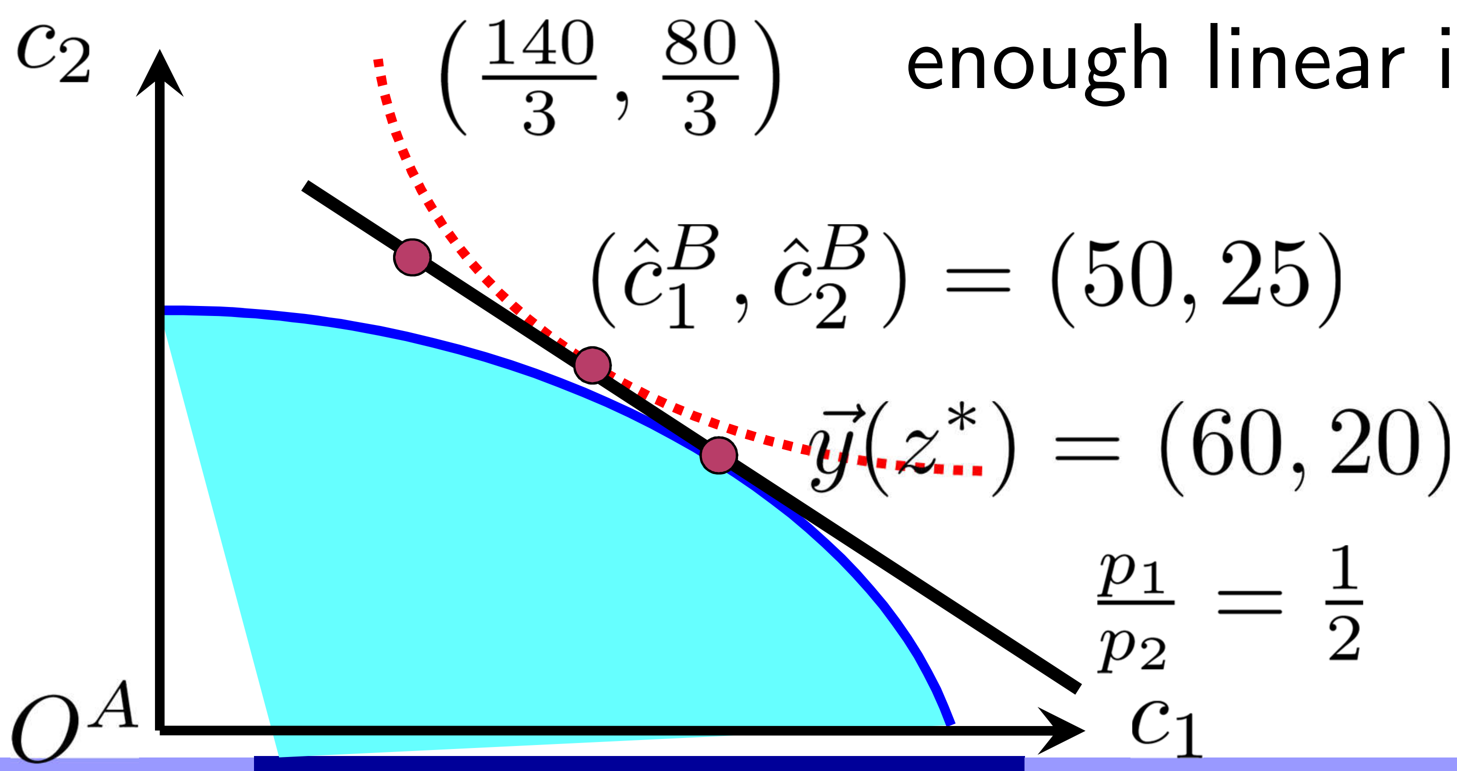
- Trade shares instead, consider Bev $(W^B = 100)$

- Autarky:
  - Hold 100% of Firm B; Consume $\vec{y}(z^*) = (60, 20)$
- Buy 1/3 of Firm A, paying $\frac{1}{3}W^A = 300/3 = 100$

  Hold 1/3 of Firm A, 0% of Firm B; Consume $\left(\frac{140}{3}, \frac{80}{3}\right)$
- Buy 1/4 of Firm A, paying $\frac{1}{4}W^A = 300/4 = 75$
  - Hold 25% of both Firm A and B; Consume $(50, 25)$

Case 3: Production - State Claims vs. Assets

- Autarky
- Buy 1/3 of Firm A
- Buy 1/4 of Firm A
- Trading assets mimic trading state claims if enough linear indep.



State Claims vs. Assets

Any allocation achievable by S state claim are also achievable by S linearly independent assets

- State claim equilibrium prices $\vec{p} = (p_1, \cdots, p_S)$
- z_{is}: Output of firm i at state s
- Equilibrium asset prices

$$\vec{P}^a = (P_1^a, \cdots, P_S^a) = \vec{p}'[z_{is}] = \vec{p}'\mathbf{Z}$$

  - Invertible if asset returns independent: $\vec{p}' = \vec{P}^{a\,\prime}\mathbf{Z}^{-1}$
- Budget Constraint: $\vec{p}'\vec{c}^h = (\vec{P}^{a\prime}\mathbf{Z}^{-1})\vec{c}^h \leq W^h$
  - Can obtain $\vec{c}^h$ by buying asset vector $\vec{q} = \mathbf{Z}^{-1}\vec{c}^h$

Case 3: Production - State Claims vs. Assets

- In the example (Case 3), $(\hat{c}_1^B, \hat{c}_2^B) = (50, 25)$
- Matrix of returns is

$$Z = \begin{bmatrix} 140 & 60 \\ 80 & 20 \end{bmatrix}$$

- Hence,

$$Z^{-1} = \frac{1}{\det \mathbf{Z}} \begin{bmatrix} 20 & -60 \\ -80 & 140 \end{bmatrix} = \begin{bmatrix} -1\% & 3\% \\ 4\% & -7\% \end{bmatrix}$$

- So Bev should hold:

$$\vec{q} = \mathbf{Z}^{-1} \vec{c}^h = \begin{bmatrix} -1\% & 3\% \\ 4\% & -7\% \end{bmatrix} \cdot \begin{bmatrix} 50 \\ 25 \end{bmatrix} = \begin{bmatrix} 25\% \\ 25\% \end{bmatrix}$$

Summary of 8.1

- Apply WE to Markets of Uncertainty
- State Claim Markets vs. Asset Markets
- I did not teach any thing new, just another (very important) application...
- Homework: Riley – 8.1-1, 3, 4, 2008 Final Q4, 2009 Final B, 2013 Final A, 2014 Final A