Principal-Agent Problem

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(Lecture 12, Micro Theory I)

Why Should We Care About This?

- Principal-Agent Relationships are Everywhere
  - Firm owner vs. manager
  - Insurance company vs. insurer
  - People vs. politician
  - Professor vs. student (or TA!)
  - Policymaker vs. people/firms
  - Planner vs. actor (even in your brain!)
  - Self-control: Your today-self vs. tomorrow-self

The Principal-Agent Problem

- Firm owner (Principal) hires manager (Agent)
- Revenue $y_1 < \cdots < y_S$ in state $s = 1 \sim S$, public
- Cost $C(x)$ for agent action $x \in X = \{x_1, \cdots, x_n\}$
  - Action only known to agent
- State s occurs with probability $\pi_s(x)$ given x
- Assume: Likelihood ratio increasing overs s

$$L(s, x, x') = \frac{\pi_s(x')}{\pi_s(x)}, x' > x$$

  - Greater output = more likely desirable action

Contracting under Full Information

- Principal's VNM utility function $u(\cdot)$
- Agent's utility is $Ev(\cdot) - C(x)$
- Contract: $w(x) = (w_1(x), w_2(x), \cdots, w_S(x))$
  – (Wage $w_s(x)$ depends on state and action x)
- Expected Utility of each party:

$$U_A(x, w) = \sum_{s=1}^{S} \pi_s(x) v(w_s(x)) - C(x)$$

$$U_P(x, w) = \sum_{s=1}^{S} \pi_s(x) u(y_s - w_s(x))$$

Contracting under Incomplete Information

- Contract: $w = (w_1, w_2, \cdots, w_S)$
  - Wage w_s depends on state, but not hidden action x
- Expected Utility of each party:

$$U_A(x, w) = \sum_{s=1}^{S} \pi_s(x) v(w_s) - C(x)$$

$$U_P(x, w) = \sum_{s=1}^{S} \pi_s(x) u(y_s - w_s)$$

- Note: Principal's Expected Utility still depends on hidden action x, but cannot contract on it!

Efficient Contract Under Full Information

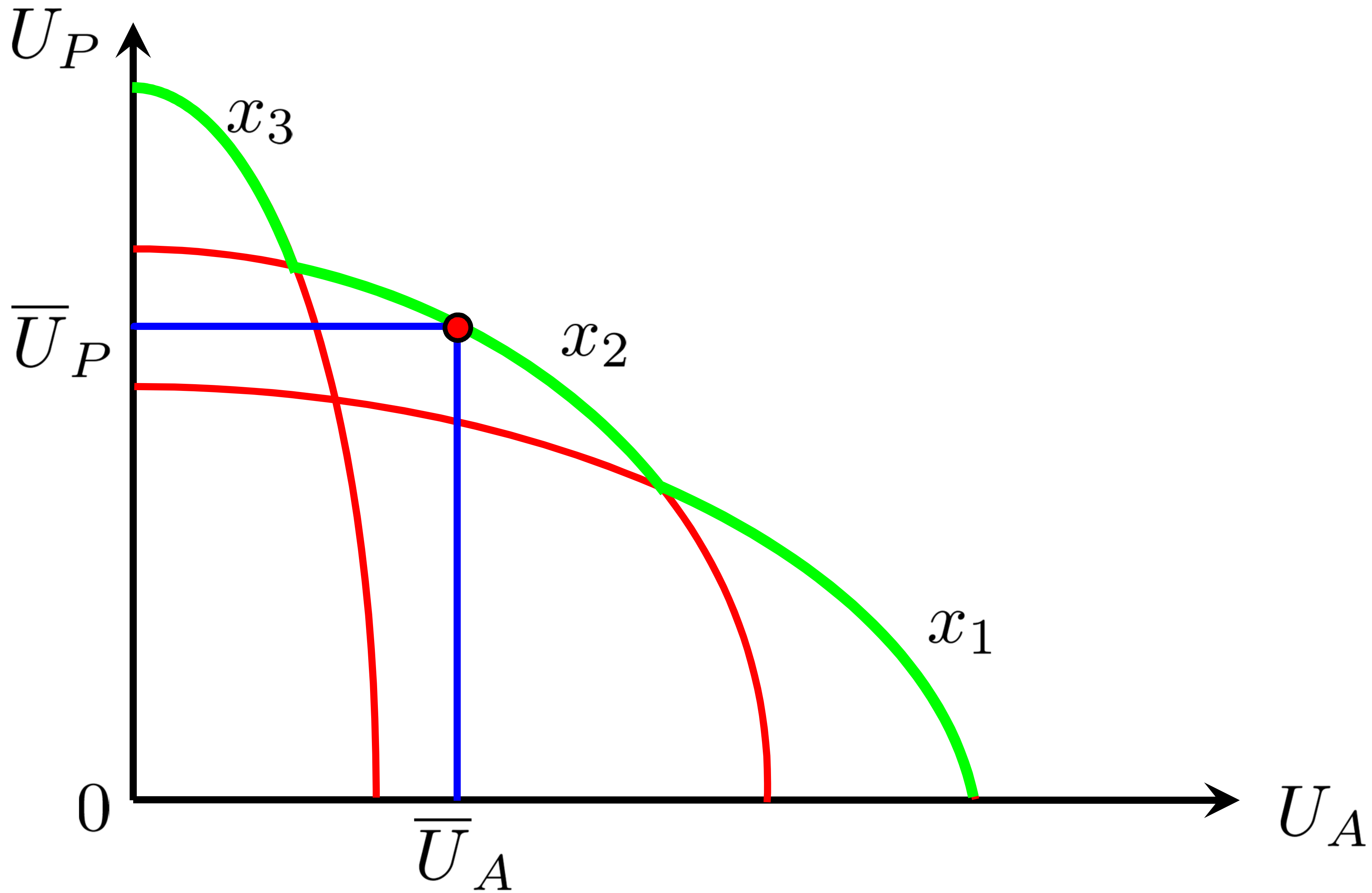
- If action is observable, solve Pareto problem:

$$\max_{x,w}\left\{\sum_{s=1}^{S}\pi_s(x)v(w_s(x))-C(x)\middle|\; x\in X=\{x_1,\cdots,x_n\},\sum_{s=1}^{S}\pi_s(x)u(y_s-w_s(x))\geq\overline{U}_P\right\}$$

- 2-step strategy:

1. Fix an action x, solve the Pareto problem
2. Find the envelope of PEAs under different x

Efficient Contract Under Full Information



Principal's Optimal Contract: Full Information

- Which efficient contract does Principal like?

$$\max_{x,w} \left\{ \sum_{s=1}^{S} \pi_s(x) u(y_s - w_s(x)) \,\middle|\, x \in \{x_1, \cdots, x_n\}, \sum_{s=1}^{S} \pi_s(x) v(w_s(x)) - C(x) \geq \overline{U}_A \right\}$$

- 2-step strategy:

1. Fix action x, solve the Pareto problem
2. Find the action x^* that maximizes U_P

Risk Neutral Principal vs. Risk Averse Agent

- Principal is risk neutral, solve Pareto problem:

$$\max_{x,w}\left\{\sum_{s=1}^{S}\pi_s(x)\underline{\underline{(y_s-w_s(x))}}\;\middle|\; x\in X, \sum_{s=1}^{S}\pi_s(x)v(w_s(x))-C(x)\geq \overline{U}_A\right\}$$

- Claim: Principal bears all risk and $w_s(x)=w(x)$
  - Agent is risk averse, can offer lower, but fixed wage and still make agent not worse off...

Why Fixed Wage Contract? Consider...

$$\overline{w}(x) = \sum_{s=1}^{S} \pi_s(x) w_s(x) \text{ for } w(x) = (w_1(x), \cdots, w_S(x))$$

- Agent is risk averse, so by Jensen's inequality:

$$v(\overline{w}(x)) - C(x) > \sum_{s=1}^{S} \pi_s(x) v(w_s(x)) - C(x) \geq \overline{U}_A$$

  – Inequality strict unless $w_1(x) = \cdots = w_S(x)$

- Principal can instead offer $w_s(x) = \overline{w}(x) - \epsilon$ to bear all risk (and agent still not worse off!)
- Not optimal unless wage is fixed: $w_s(x) = w$

Risk Neutral Principal vs. Risk Averse Agent

- Fix $\overline{x}$, the Pareto problem becomes:

$$\max_{w} \left\{ \sum_{s=1}^{S} \pi_s(\overline{x}) y_s - w \middle| v(w) - C(\overline{x}) \geq \overline{U}_A \right\}$$

$$\mathcal{L} = \sum_{s=1}^{S} \pi_s(\overline{x}) y_s - w + \lambda \left[v(w) - C(\overline{x}) - \overline{U}_A \right]$$

- FOC:

$$w : -1 + \lambda v'(w) \leq 0 \text{ with equality if } w > 0$$

$$\lambda : v(w) - C(\overline{x}) \geq \overline{U}_A \text{ with equality if } \lambda > 0$$

Risk Neutral Principal vs. Risk Averse Agent

$$w : -1 + \lambda v'(w) \le 0 \text{ with equality if } w > 0$$

$$\lambda : v(w) - C(\overline{x}) \ge \overline{U}_A \text{ with equality if } \lambda > 0$$

- Constraint must bind (or can decrease the fixed wage w and increase $\mathrm{U_P}$), Hence,
- $v(w) = \overline{U}_A + C(\overline{x})$, so optimal wage (for $\overline{x}$) is

$$w = v^{-1}(\overline{U}_A + C(\overline{x}))$$

- Find $x^* \in X = \{x_1, \cdots, x_n\}$ to:

$$\max_x \left\{ \sum_{s=1}^{S} \pi_s(x) y_s - v^{-1}(\overline{U}_A + C(x)) \right\}$$

Risk Averse Principal vs. Risk Neutral Agent

- Suppose instead: Agent is risk neutral, solve:

$$\max_{x,w}\left\{\sum_{s=1}^{S}\pi_s(x)u(y_s-w_s(x))\,\middle|\, x\in X, \sum_{s=1}^{S}\pi_s(x)(\underline{\underline{w_s(x)}})-C(x)\geq \overline{U}_A\right\}$$

- Claim: Agent bears all risk and $r_s = r$
  - Principal is risk averse, can offer lower, but fixed rent and still make principal not worse off...

Risk Averse Principal vs. Risk Neutral Agent

$$\overline{r}(x) = \sum_{s=1}^{S} \pi_s(x) r_s \text{ for } r(x) = (r_1, \cdots, r_S)$$
$$= (y_1 - w_1(x), \cdots, y_S - w_S(x))$$

- Principal is risk averse, so by Jensen's inequality:

$$u(\overline{r}(x)) > \sum_{s=1}^{S} \pi_s(x) u(y_s - w_s(x)) = \sum_{s=1}^{S} \pi_s(x) u(r_s)$$

  – Inequality strict unless $r_1(x) = \cdots = r_S(x)$

- Principal can keep $r_s(x) = \overline{r}(x)$ and have risk neutral agent bear all risk (and not be worse off!)
- Not optimal unless rent is fixed: $r = y_s - w_s(x)$

Risk Averse Principal vs. Risk Neutral Agent

- Fix $\overline{x}$, the Pareto problem becomes:

$$\max_{r} \left\{ u(r) \middle| \sum_{s=1}^{S} \pi_s(\overline{x})(y_s - r) - C(\overline{x}) \geq \overline{U}_A \right\}$$

$$\mathcal{L} = u(r) + \lambda \left[\sum_{s=1}^{S} \pi_s(\overline{x}) y_s - r - C(\overline{x}) - \overline{U}_A \right]$$

- FOC: $r : u'(r) - \lambda \leq 0 \text{ with equality if } r > 0$

$$\lambda : \sum_{s=1}^{S} \pi_s(\overline{x}) y_s - r - C(\overline{x}) \geq \overline{U}_A \text{ with equality if } \lambda > 0$$

Risk Averse Principal vs. Risk Neutral Agent

$$r : u'(r) - \lambda \le 0 \text{ with equality if } r > 0$$

$$\lambda : \sum_{s=1}^{S} \pi_s(\overline{x}) y_s - C(\overline{x}) \ge \overline{U}_A + r \text{ with equality if } \lambda > 0$$

- Constraint must bind (or can increase fixed rent r to raise $\mathrm{U_P}$), so optimal rent (for $\overline{x}$) is

$$r = \sum_{s=1}^{S} \pi_s(\overline{x}) y_s - C(\overline{x}) - \overline{U}_A$$

- Find x to:

$$\max_{x \in \{x_1, \cdots, x_n\}} \left\{ \sum_{s=1}^{S} \pi_s(x) y_s - C(x) \right\} - \overline{U}_A$$

Contracting under Incomplete Information

- Now consider Contract: $w = (w_1, w_2, \cdots, w_S)$
- EU:

$$U_A(x, w) = \sum_{s=1}^{S} \pi_s(x) v(w_s) - C(x)$$

$$U_P(x, w) = \sum_{s=1}^{S} \pi_s(x) u(y_s - w_s)$$

  - Principal's EU still depends on hidden action x, but cannot contract on it! Can only induce x by:
- Incentive Compatibility (IC) Constraint: Under w,

$$U_A(x, w) \leq U_A(x^*, w) \text{ for all } x \in X$$

Optimal Contract: Incomplete Information

- For hidden action, solve Pareto problem:

$$\max_{x,w}\left\{\sum_{s=1}^{S}\pi_s(x)u(y_s-w_s)\,\middle|\,U_A(\tilde{x},w)\le U_A(x,w),\ \forall\tilde{x},\ \sum_{s=1}^{S}\pi_s(x)v(w_s)-C(x)\ge\overline{U}_A\right\}$$

(IC constraint added)

- Not easy in general, except the case of…
- Risk Averse Principal vs. Risk Neutral Agent!!

Why is RA-Principal vs. RN-Agent Special?

- Optimal rent: $$r = \sum_{s=1}^{S} \pi_s(\overline{x}) y_s - C(\overline{x}) - \overline{U}_A$$
- and $\overline{x}$ solves $$\max_{x \in X} \left\{ \sum_{s=1}^{S} \pi_s(x) y_s - C(x) \right\} - \overline{U}_A$$
- So, under r, $$\sum_{s=1}^{S} \pi_s(x) y_s - r - C(x) = U_A(x, w)$$
- IC holds!

Can't do better than Full Info.

$$\leq \sum_{s=1}^{S} \pi_s(\overline{x}) y_s - r - C(\overline{x}) = U_A(\overline{x}, w)$$

Optimal Contract: Incomplete Information

- What if we are in the tough case solving...

$$\max_{x,w}\left\{\sum_{s=1}^{S}\pi_s(x)u(y_s - w_s)\,\middle|\, U_A(\tilde{x},w)\le U_A(x,w),\ \forall\tilde{x},\ \sum_{s=1}^{S}\pi_s(x)v(w_s)-C(x)\ge \overline{U}_A\right\}$$

  – EX: Risk Averse Agent vs. Risk Neutral Principal

1. Fix action x, solve the Pareto problem
2. Find the action x^* that maximizes U_P

Optimal Contract: Incomplete Information

- If only one IC binds
  - Lowest-cost action binds or only 2 actions (S = 2)

$$\max_{w} \{U_P(\overline{x}, w) | U_A(\underset{=}{\tilde{x}}, w) \leq U_A(\underset{=}{\overline{x}}, w), U_A(\overline{x}, w) \geq \overline{U}_A$$

$$\mathcal{L} = U_P(\overline{x}) + \lambda \left[U_A(\overline{x}, w) - \overline{U}_A\right] + \mu \left[U_A(\overline{x}, w) - U_A(\tilde{x}, w)\right]$$

$$\mathcal{L} = \sum_{s=1}^{S} \pi_s(\overline{x}) u(y_s - w_s) + \lambda \left[\sum_{s=1}^{S} \pi_s(\overline{x}) v(w_s) - C(\overline{x}) - \overline{U}_A\right]$$

$$+\mu \left[\sum_{s=1}^{S} \pi_s(\overline{x}) v(w_s) - C(\overline{x}) - \left(\sum_{s=1}^{S} \pi_s(\tilde{x}) v(w_s) - C(\tilde{x})\right)\right]$$

Optimal Contract: Incomplete Information

$$\mathcal{L} = \sum_{s=1}^{S} \pi_s(\overline{x})u(y_s - w_s) + (\lambda + \mu)\left[\sum_{s=1}^{S} \pi_s(\overline{x})v(w_s) - C(\overline{x})\right]$$

$$+\mu\left[\sum_{s=1}^{S} \pi_s(\overline{x})v(w_s) - \lambda\overline{U}_A - \mu\left[\sum_{s=1}^{S} \pi_s(\tilde{x})v(w_s) - C(\tilde{x})\right]\right]$$

$$w_s : -\pi_s(\overline{x})u'(y_s - w_s) + (\lambda + \mu)\pi_s(\overline{x})v'(w_s)$$

$$-\mu\pi_s(\tilde{x})v'(w_s) \leq 0 \text{ (w/ equality if } w_s > 0)$$

$$\lambda : \sum \pi_s(\overline{x})v(w_s) - C(\overline{x}) \geq \overline{U}_A \text{ (w/ equality if } \lambda > 0)$$

$$\mu : \text{(w/ equality if } \mu > 0) \geq \sum \pi_s(\tilde{x})v(w_s) - C(\tilde{x})$$

Risk Neutral Principal vs. Risk Averse Agent

$$w_s : -\pi_s(\overline{x})u'(y_s - w_s) + (\lambda+\mu)\pi_s(\overline{x})v'(w_s)$$
$$-\mu\pi_s(\tilde{x})v'(w_s) \le 0 \text{ (w/ equality if } w_s > 0)$$

- If $w_s > 0$, $\dfrac{u'(y_s - w_s)}{v'(w_s)} = (\lambda+\mu) - \mu\dfrac{\pi_s(\tilde{x})}{\pi_s(\overline{x})}, \tilde{x} < \overline{x}$
  - Risk Neutral Principal vs. Risk Averse Agent:

$$u'(y_s - w_s) = 1, v(w_s) \text{ concave}$$

- FOC:

$$\frac{1}{v'(w_s)} = (\lambda+\mu) - \mu\frac{\pi_s(\tilde{x})}{\pi_s(\overline{x})}$$

Increasing in s ?

Risk Neutral Principal vs. Risk Averse Agent

- FOC: $\dfrac{1}{v'(w_s)} = (\lambda + \mu) - \mu \dfrac{\pi_s(\tilde{x})}{\pi_s(\overline{x})}, \tilde{x} < \overline{x}$
- Monotone Likelihood Ratio Property required so w_s^* is increasing in s : $\dfrac{\pi_s(\overline{x})}{\pi_s(\tilde{x})} > 0, \text{ for } \overline{x} > \tilde{x}$
- IR/IC Constraints Bind:

$$\lambda : U_A(\overline{x}, w) = \sum \pi_s(\overline{x}) v(w_s) - C(\overline{x}) = \overline{U}_A$$

$$\mu : \sum \pi_s(\overline{x}) v(w_s) - C(\overline{x}) = \sum \pi_s(\tilde{x}) v(w_s) - C(\tilde{x})$$

$$U_A(\overline{x}, w) = U_A(\tilde{x}, w)$$

Summary of 7.4

- Principal-Agent Problem is an entire field!
  - Each problem is a research paper
- Complete Information – Simple Cases:
- P is Risk Neutral (A is Risk Averse)
  - Find Fixed Wage Contract for each possible action
  - Find Optimal Action to $\mathrm{Max}\, U_P$
- A is Risk Neutral (P is Risk Averse)
  - Find Fixed Rent Contract for each possible action
  - Find Optimal Action to $\mathrm{Max}\, U_A$

Summary of 7.4

- Hidden Action – Simple Cases:
- A is Risk Neutral (P is Risk Averse)
  - Use same Fixed Rent Contract w/o hidden action
  - (as in second case of complete information) since
  - It already satisfies IC (possible solution!), and
  - We cannot do better than the solution of the same problem with less constraints
- Homework: 2014 Final C, 2013 Final B7-B13