Theory of Risky Choice

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(Lecture 10, Micro Theory I)

Theory of Risky Choice

- We analyzed preferences, utility and choices
- Apply them to study risk and uncertainty
  - Preference for probabilities
  - Expected Utility
- Discuss Experimental Anomalies

1. Allais paradox and Ellsberg paradox
2. Bayes' Rule paradoxes: Soft vs. Hard prob., Game show paradox (Monty Hall problem)
3. Rabin paradox

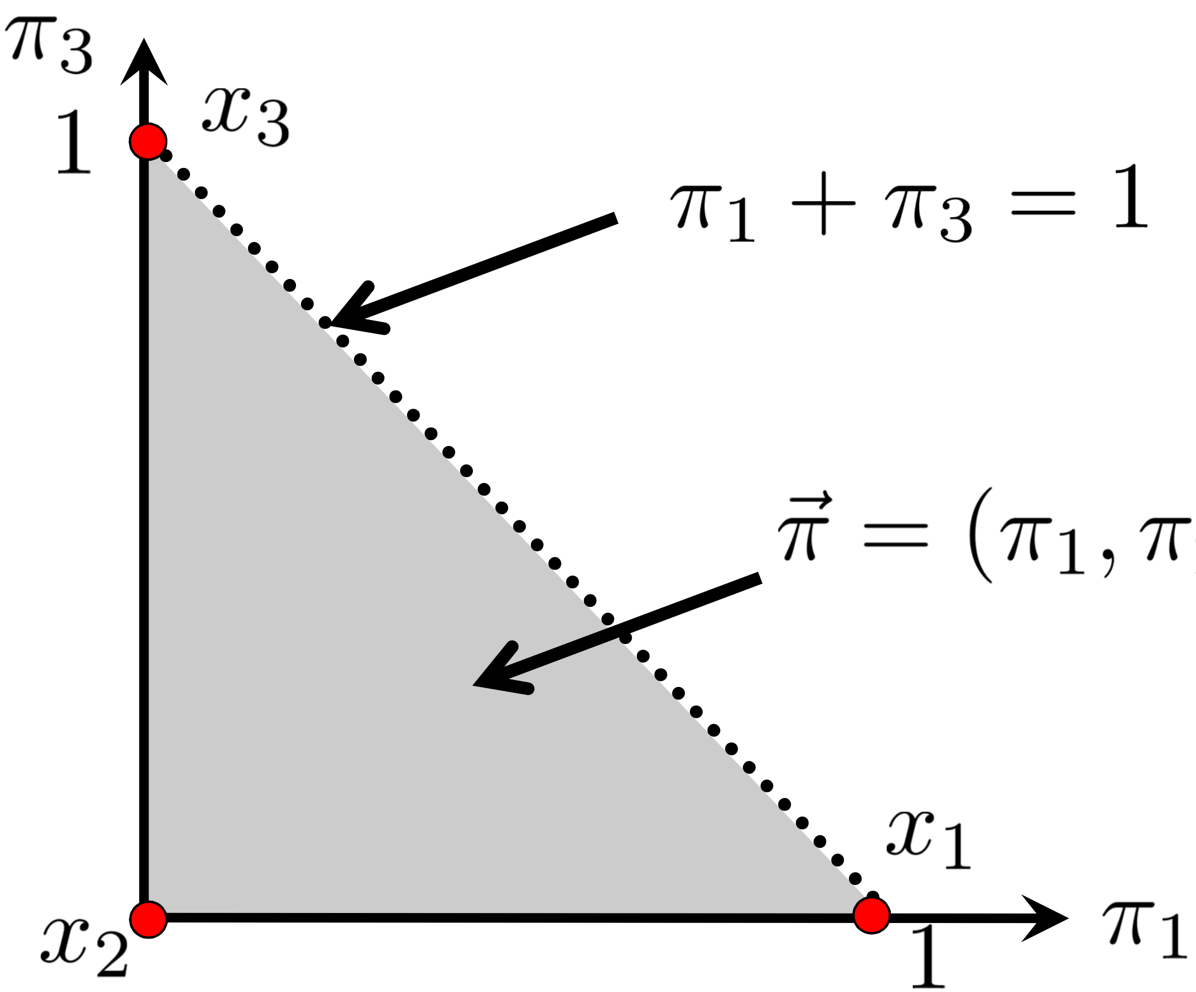
States and Probabilities

- Consequence x_s happens in state $s = 1, \cdots, S$
- Assign (subjective) probability π_s to state *s*
- A prospect $(\vec{\pi}; \vec{x}) = \big((\pi_1, \cdots, \pi_S); (x_1, \cdots, x_S)\big)$
  - People have preferences for these prospects
- Under the Axioms of Consumer Choice, exists continuous $U(\vec{\pi}; \vec{x})$ representing these pref.
- If we fix consequences; focus on probabilities

$$U(\vec{\pi}; \vec{x}) = U(\vec{\pi}) = U(\pi_1, \pi_2, \pi_3)$$

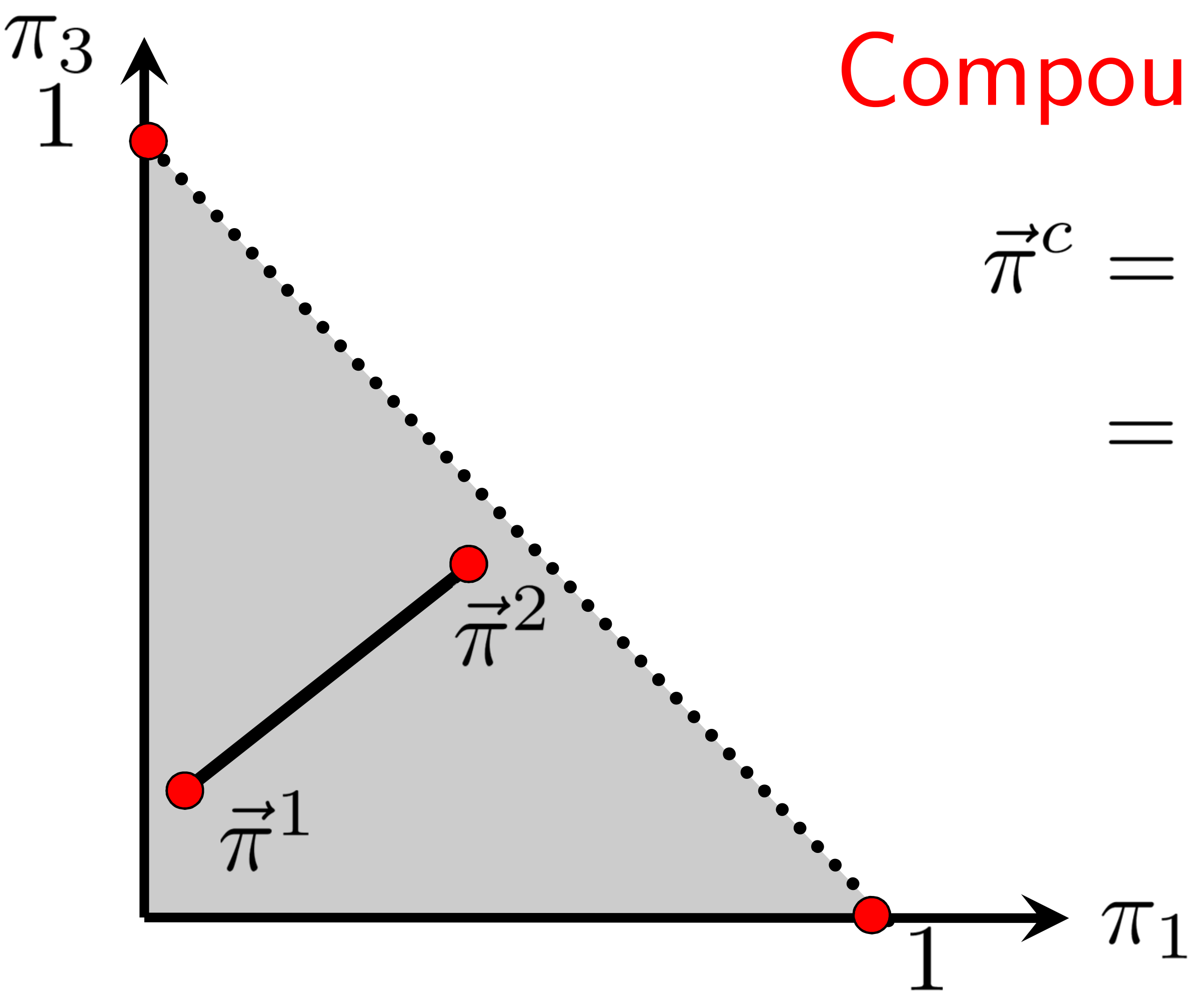
States and Probabilities

- Assume $x_3 \succ x_2 \succ x_1$, can show all possible probabilities on 2D: $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$



Compound Prospect (Compound Lottery)

- If I offer you $\vec{\pi}^1 = (\pi_1^1, \pi_2^1, \pi_3^1)$ with prob. p_1, and $\vec{\pi}^2 = (\pi_1^2, \pi_2^2, \pi_3^2)$ with probability $p_2 = 1 - p_1$

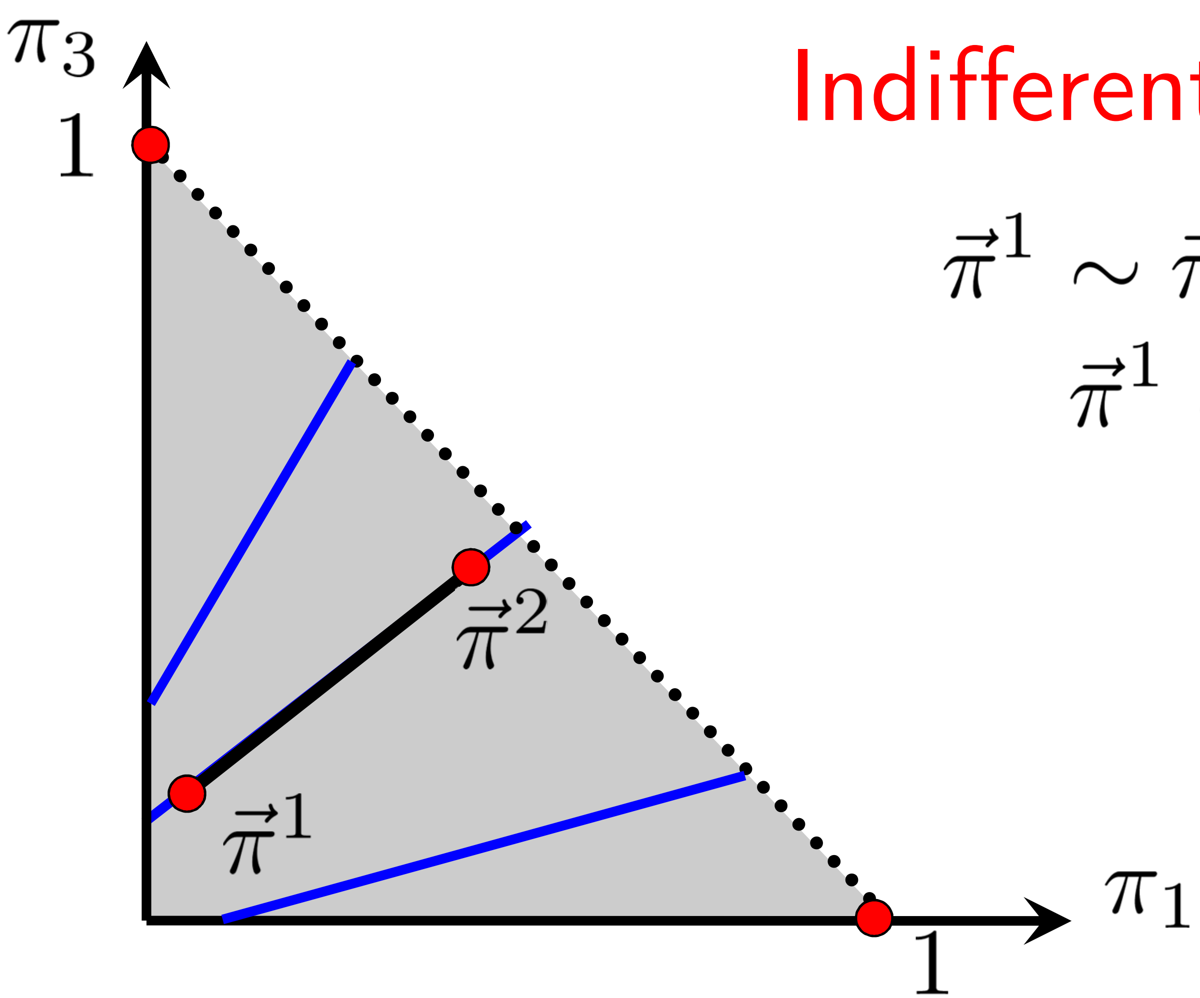


Compound Prospect:

$$\vec{\pi}^c = \left(p_1, p_2 : \vec{\pi}^1, \vec{\pi}^2\right)$$
$$= p_1\vec{\pi}^1 + (1 - p_1)\vec{\pi}^2$$

Are Indifference Curves Linear?

- If you are indifferent between $\vec{\pi}^1$ and $\vec{\pi}^2$
- How would you feel about randomizing them?



Indifferent !!

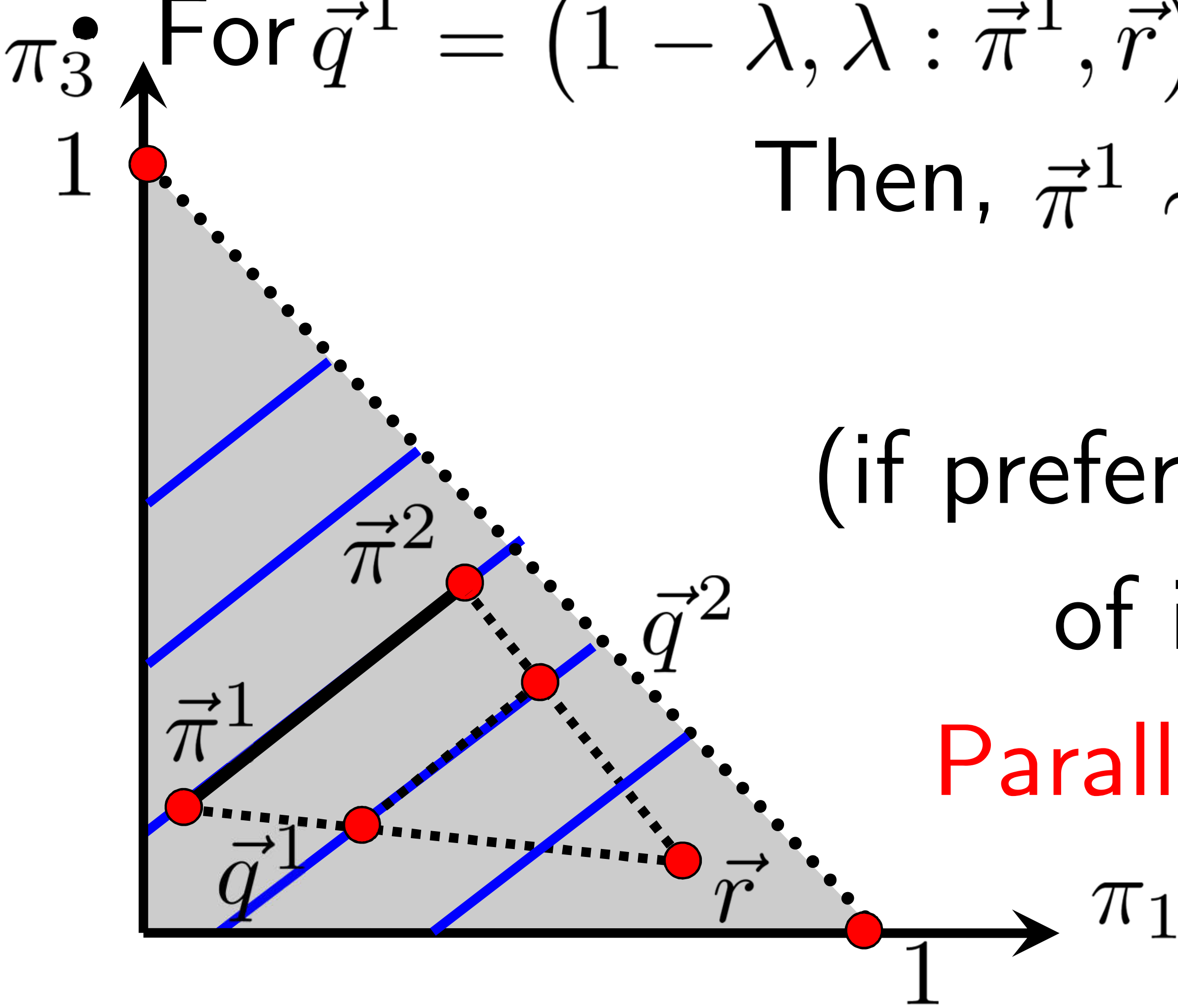
$$\vec{\pi}^1 \sim \vec{\pi}^2 \Rightarrow$$
$$\vec{\pi}^1 \sim \left(p_1, 1-p_1 : \vec{\pi}^1, \vec{\pi}^2\right)$$

Indifference Curves

Are Linear!

When Are Indifference Curves Parallel?

- Consider a third prospect $\vec{r}$
- For $\vec{q}^1 = (1-\lambda, \lambda : \vec{\pi}^1, \vec{r})$, $\vec{q}^2 = (1-\lambda, \lambda : \vec{\pi}^2, \vec{r})$



Then, $\vec{\pi}^1 \sim \vec{\pi}^2 \Rightarrow \vec{q}^1 \sim \vec{q}^2$

$\vec{\pi}^1 \succsim \vec{\pi}^2 \Rightarrow \vec{q}^1 \succsim \vec{q}^2$

(if preferences are independent of irrelevant alternatives)

Parallel Indifference Curves!

Independence Axiom(s)

- (IA) If $\vec{\pi}^1 \succsim \vec{\pi}^2$, then for any prospect $\vec{r}$ and probabilities $p_1, p_2 > 0, p_1 + p_2 = 1$

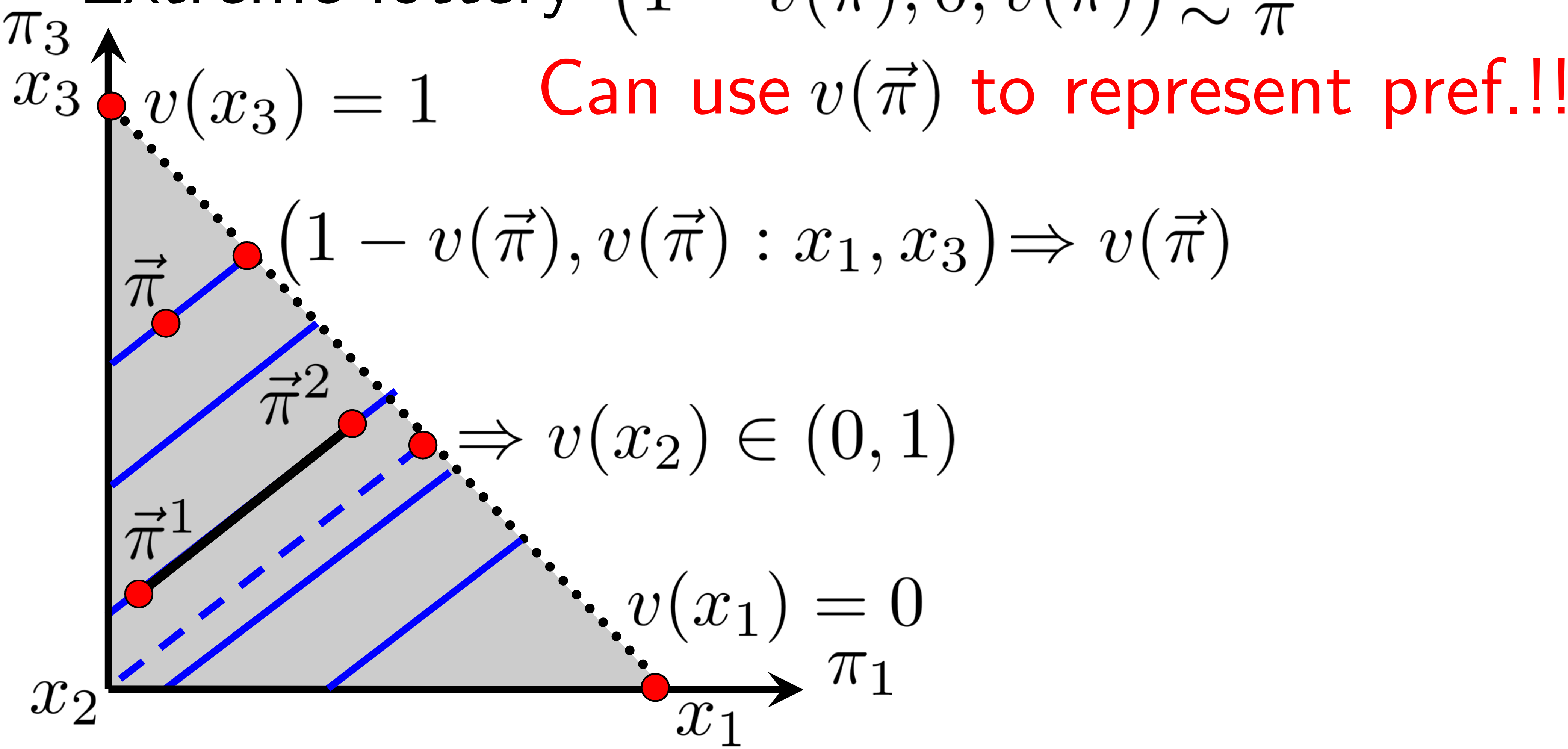
$$\vec{q}^1 = (p_1, p_2 : \vec{\pi}^1, \vec{r}) \succsim (p_1, p_2 : \vec{\pi}^2, \vec{r}) = \vec{q}^2$$

- (IA') If $\vec{\pi}^m \succsim \vec{\mu}^m, m = 1, \cdots, M$, then for any probability vector $p = (p_1, \cdots, p_M)$

$$(p_1, \cdots, p_M : \vec{\pi}^1, \cdots, \vec{\pi}^M) \succsim (p_1, \cdots, p_M : \vec{\mu}^1, \cdots, \vec{\mu}^M)$$

Expected Utility

- For any prospect $\vec{\pi}$, consider (on $\pi_1 + \pi_3 = 1$):
- Extreme lottery $\left(1 - v(\vec{\pi}), 0, v(\vec{\pi})\right) \sim \vec{\pi}$



Expected Utility

- In general, for any prospect $\vec{p} = (p_1, \cdots, p_S)$
- The consumer is indifferent between $\vec{p}$ and playing the extreme lottery

$$\left(\sum_{s=1}^{S} p_s v(x_s), 0, \cdots, 0, 1 - \sum_{s=1}^{S} p_s v(x_s) \right)$$

- Hence, we can represent her preferences with the above expected win probabilities
  - Expected Utility!!

Expected Utility Rule

- Assume (IA'), then
- Preferences over prospects

$$(\vec{p};\vec{x}) = (p_1,\cdots,p_S;x_1,\cdots,x_S)$$

- Can be represented by the Von Neumann-Morgenstern utility function

$$u(\vec{p},\vec{x}) = \sum_{s=1}^{S} p_s v(x_s)$$

- Proof:

Expected Utility Rule

- Proof: S consequences, best is x^*, worse is x_*
- Can assign probability for extreme lotteries:

$$\vec{e}^s \equiv \left(v(x_s), 1 - v(x_s) : x^*, x_*\right) \sim x_s$$

- (IA') implies $(\vec{p}; \vec{x}) \sim (p_1, \cdots, p_S : \vec{e}^1, \cdots, \vec{e}^S)$

$$\sim \left(u(\vec{p}, \vec{x}), 1 - u(\vec{p}, \vec{x}) : x^*, x_*\right)$$

where $u(\vec{p}, \vec{x}) = \sum_{s=1}^{S} p_s v(x_s)$

– (by reducing compound prospects)

Experimental Anomalies

- Allais Paradox
- Ellsberg Paradox
- Bayes' Rule Paradoxes
  - Soft vs. Hard Probabilities
  - Game Show Paradox
- Rabin Paradox

Allais Paradox

- Consider four prospects:

A. $1 million for sure

B. 90% chance $5 million (& 10% chance zero)

  - Among A and B, you choose...

C. 10% chance $1 million (& 90% chance zero)

D. 9% chance $5 million (& 91% chance zero)

  - Among C and D, you choose...

- Is this consistent with Expected Utility???

Allais Paradox * 1,000

A. $1 billion for sure

B. 90% chance $5 billion (& 10% chance zero)

– Among A and B, you choose…

C. 10% chance $1 billion (& 90% chance zero)

D. 9% chance $5 billion (& 91% chance zero)

– Among C and D, you choose…

- Are your answers (still) consistent with Expected Utility? Why or why not?

Ellsberg Paradox

- One urn: 30 Black balls, and 60 "other balls"
  - Other balls could be either Red or Green

1. One ball is drawn. You win $100 if the ball is (a) Black or (b) Green. You pick...?
2. Now you win $50 if the ball is "either Red or another color you choose." Would you choose (a) Black or (b) Green?

- What did you choose? Did it violate EU?

Ellsberg Paradox

1. One ball is drawn. You win $100 if the ball is (a) Black or (b) Green.

- Picking Black = Believe <30 Green balls

2. Now you win if "either Red or another color." You choose (a) Black or (b) Green?

- Picking Green = Believe >30 Green balls
- Since it is the same urn, this is inconsistent!
  - Can this be due to hedging (risk aversion)?
  - Maybe, but can fix this by paying only 1 round...

Bayes' Rule Paradoxes: Soft vs. Hard Prob.

- Two urns, each contain 100 balls.

1. Urn 1 has 60 Yellow balls.
2. Urn 2 has 75 or 25 Yellow balls with equal chance.

- You win a prize if you draw a Yellow ball.
- A ball is drawn from Urn 2 and it is Yellow.
- Which Urn should you choose?
- Did you do Bayesian updating correctly?

Bayes' Rule Paradoxes: Soft vs. Hard Prob.

- Prior to draw, Pr(draw a Y) $= 0.5$. After:
- Pr(Y | 75 - Y) $= 0.75$, Pr(Y | 25 - Y) $= 0.25$
- Pr(75 - Y |Y) $= 0.5$ x 0.75 / $0.5 = 0.75$
- Pr(25 - Y |Y) $= 1 - 0.75 = 0.25$
- Pr(draw another Y |Y) =
  Pr(75 - Y |Y) x Pr(Y | 75 - Y) +
  Pr(25 - Y |Y) x Pr(Y | 25 - Y)
  $= 0.75$ x $0.75 + 0.25$ x $0.25 = 0.625 > 0.6$
- So you should pick Urn 2!! (Did you do that?)

Bayes' Rule Paradoxes: Game Show Paradox

One door hides the prize (a car).
Remaining two doors hides a goat (non-prize).

Suppose you choose door number 1...

Game Show Paradox (Monty Hall Problem)

Door 3 is opened for you...

Obviously the car is not behind door 3...
Would you want to switch to door 2?

Depends on how door is opened...

- Rule to open one door:

  The Host must open one "other" door without the prize. If he has a choice between more than one door, he will randomly open one of the possible (goat) doors.

- The Game Show Paradox is also known as the Monty Hall Problem, named after the name of the TV show host "Monty Hall"

If You Picked the Right Door (33.3%)

Host randomizes between door 2 and 3

If host opens door 3... (Prob=33.3%*50%)

You should not switch (but you don't know)

If You Picked the Wrong Door (66.7%)

Host cannot open door 2 (contains car)

See host opening door 3... (Prob.=66.70%*100%)

You should switch (but you don't know)

Bayesian Updating in the Monty Hall Problem

- Pr(host opens Door 3 & car in Door 1)

  $= 33.3\%*50\% = 16.7\%$

- Pr(host opens Door 3 & car in Door 2)

  $= 33.3\%*100\% = 33.3\%$

- Pr(host opens Door 3 & car in Door 3) $= 0$
  - Host never opens Door 3!

- Pr (car in Door 1 | Host opens Door 3)

  $= [16.7\%] / [16.7\% + 33.3\% + 0\%] = 1/3$

Game Show Paradox Plus: Modified Monty Hall

One door hides the prize (a car).
Remaining two doors hides a goat (non-prize).
Door 3 is transparent (and you see the goat)

Suppose you choose door number 1...

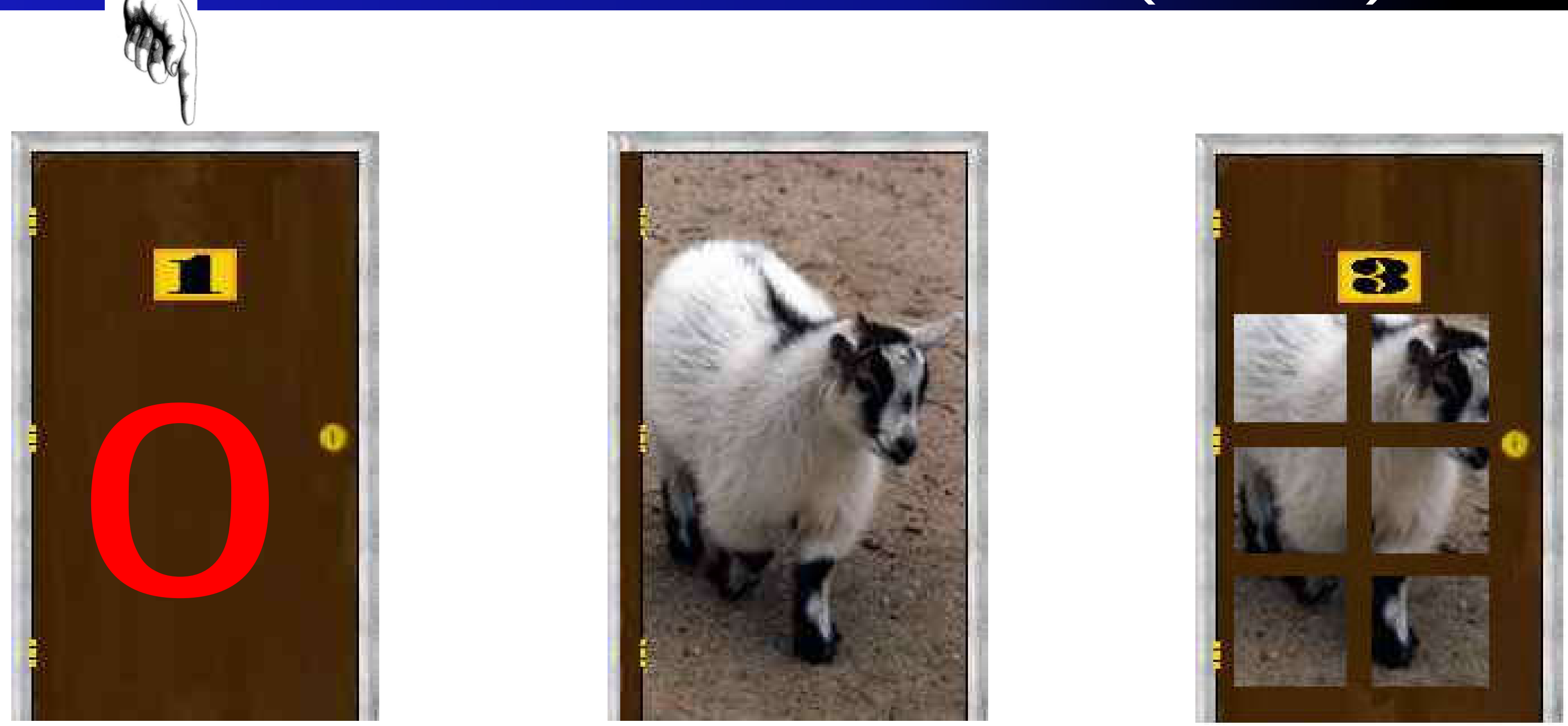
Game Show Paradox Plus: Modified Monty Hall

Door 3 is opened for you...

Obviously the car is not behind door 3 (and you knew that already)...

Would you want to switch to door 2?

If You Picked the Right Door (50%)



Host randomizes between door 2 and 3 (50-50)

If host opens door 2... (Prob.=50%*50%)

You should definitely not switch!

If You Picked the Right Door (50%)

Host randomizes between door 2 and 3

If host opens door 3... (Prob=50%*50%)

You should still not switch (but you don't know)

If You Picked the Wrong Door (50%)

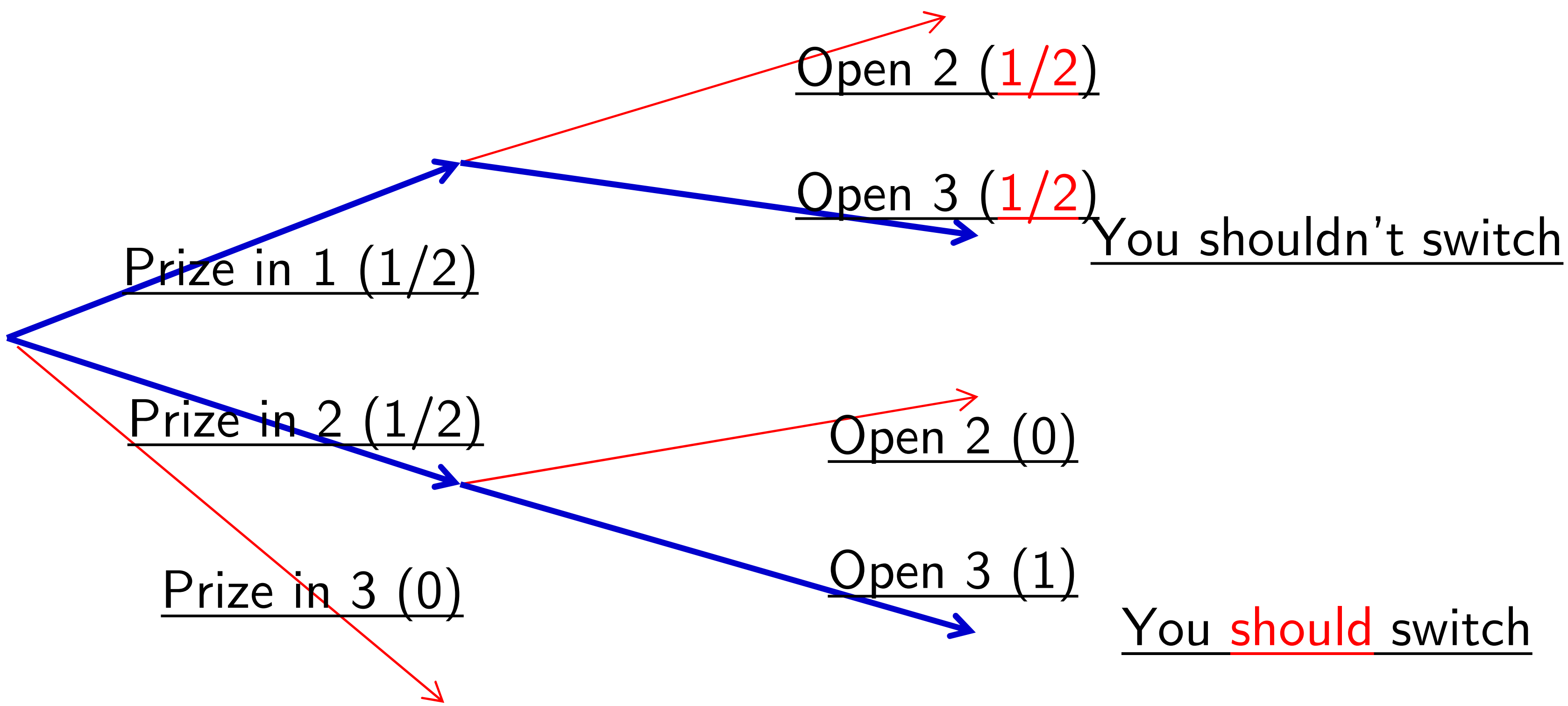
Host cannot open door 2 (contains car)

See host opening door 3... (Prob.=50%*100%)

You should switch (but you don't know)

Bayesian Solution (Monty Hall Plus)

Door #3 is transparent…



$$\text{P(Winning if you choose to switch)} = \frac{\frac{1}{2}\times 1}{\frac{1}{2}\times 1 + \frac{1}{2}\times\frac{1}{2}}$$

Rabin Paradox: Which Cells Will You Accept?

Payoff if Green Ball	Number of Green Balls (out of 100)					Payoff if Red Ball
100	52	55	60	66	70	-100
1000	13	20	33	46	57	-100
5000	7	18	33	46	57	-100
25000	7	18	33	46	57	-100

Rabin Paradox

- Suppose your risk preference follows EU.
- For initial Wealth is ω
- Consider the prospect $(p, 1-p : \omega + g, \omega - g)$
- If you reject this lottery, this implies:

$$v(\omega) \geq (1-p) \cdot v(\omega - g) + p \cdot v(\omega + g)$$

- Or,

$$\left[v(\omega + g) - v(\omega)\right] \leq \frac{1-p}{p} \cdot \left[v(\omega) - v(\omega - g)\right]$$

........(1)

Rabin Paradox

- Now consider initial wealth $\omega' = \omega + g$
- If you reject the prospect $(p, 1-p : \omega' + g, \omega' - g)$
- Then: $v(\omega') \geq (1-p) \cdot v(\omega' - g) + p \cdot v(\omega' + g)$
- Or,

$$\left[v(\omega + 2g) - v(\omega + g)\right] = \left[v(\omega' + g) - v(\omega')\right]$$
$$\leq \frac{1-p}{p} \cdot \left[v(\omega') - v(\omega' - g)\right]$$
$$= \frac{1-p}{p} \cdot \left[v(\omega + g) - v(\omega)\right]$$

Rabin Paradox

- Combining the two inequalities:

$$
\begin{aligned}
&[v(\omega + 2g) - v(\omega + g)] \\
&\qquad \leq \frac{1-p}{p} \cdot [v(\omega + g) - v(\omega)] \\
&\qquad\qquad \leq \left(\frac{1-p}{p}\right)^2 \cdot [v(\omega) - v(\omega - g)] \ldots .(2)
\end{aligned}
$$

- Only required one to reject the fair gamble at both wealth levels ω and $\omega' = \omega + g$

Rabin Paradox

- Suppose you reject the fair gamble at all wealth levels between ω and $\omega^{(n)} = \omega + ng$
- Then,

$$[v(\omega + ng) - v(\omega + (n-1)g)]$$

$$\leq \frac{1-p}{p} \cdot [v(\omega + (n-1)g) - v(\omega + (n-2)g)]$$

$$\leq \cdots \leq \left(\frac{1-p}{p}\right)^n \cdot [v(\omega) - v(\omega - g)] \text{.......(n)}$$

Rabin Paradox

- Summing (1) through (n):

$$[v(\omega+g)-v(\omega)]+[v(\omega+2g)-v(\omega+g)]$$
$$+\cdots+[v(\omega+ng)-v(\omega+(n-1)g)]$$
$$=[v(\omega+ng)-v(\omega)]$$
$$\leq\left[\tfrac{1-p}{p}+\cdots+\left(\tfrac{1-p}{p}\right)^n\right]\cdot[v(\omega)-v(\omega-g)]=$$
$$\tfrac{1-p}{p}\cdot[v(\omega)-v(\omega-g)]+\left(\tfrac{1-p}{p}\right)^2\cdot[v(\omega)-v(\omega-g)]$$
$$+\cdots+\left(\tfrac{1-p}{p}\right)^n\cdot[v(\omega)-v(\omega-g)]$$

Rabin Paradox

Let $s(n,p) = 1 + \frac{1-p}{p} + \cdots + \left(\frac{1-p}{p}\right)^n$

$$[v(\omega + ng) - v(\omega)]$$
$$\leq [v(\omega) - v(\omega - g)] \cdot \left[\frac{1-p}{p} + \cdots + \left(\frac{1-p}{p}\right)^n\right]$$
$$\Rightarrow [v(\omega + ng) + (s(n,p) - 1)v(\omega - g)] \leq s(n,p) \cdot v(\omega)$$

- Or, $v(\omega) \geq \frac{1}{s(n,p)} v(\omega + ng) + \left(1 - \frac{1}{s(n,p)}\right) v(\omega - g)$
- This means rejecting

$$\left(\frac{1}{s(n,p)}, 1 - \frac{1}{s(n,p)} : \omega + ng, \omega - g\right)$$

Rabin Paradox

- We have shown that:
- If you reject prospect $(p, 1-p : \omega + g, \omega - g)$
- For all wealth levels $[\omega, \omega + ng]$

→You would also reject the more favorable prospect $(\frac{1}{s(n,p)}, 1 - \frac{1}{s(n,p)} : \omega + ng, \omega - g)$

$$s(n,p) = 1 + \frac{1-p}{p} + \cdots + \left(\frac{1-p}{p}\right)^n \to \frac{1}{1 - \frac{1-p}{p}}$$

- This is true for any large n!

$$\frac{1}{s(n,p)} \to \frac{2p-1}{p}$$

Rabin Paradox: Which Cells Will You Accept?

Payoff if Green Ball	Number of Green Balls (out of 100)					Payoff if Red Ball
100	~~52~~	~~55~~	~~60~~	66	70	-100
1000	~~13~~	~~20~~	~~33~~	46	57	-100
5000	~~7~~	~~18~~	~~33~~	46	57	-100
25000	~~7~~	~~18~~	~~33~~	46	57	-100

Continuous Probability Distribution

- Let state $s \in \mathcal{S} = [\alpha, \beta]$
- CDF is $F(t) = \Pr\{s \leq t\}$ $\quad F(\alpha) = 0, F(\beta) = 1$
- Probability of being in $C = [s, s']$ is:
- Probability Measure $\pi(C) = F(s') - F(s)$

- Can generalize and assign probability measures over closed convex hypercube $C \in \mathbb{R}^n$

Support of the Continuous Distribution

- x is in the support of the distribution if for every neighborhood $N(x, \delta)$ of x, $\pi(N(x, \delta)) > 0$
- Example: $\mathcal{S} = [0, 3]$

$$F(\theta) = \begin{cases} \frac{1}{2}\theta, & 0 \le \theta \le 1 \\ \frac{1}{2}, & 1 < \theta < 2 \\ \frac{1}{2}(\theta - 1), & 2 \le \theta \le 3 \end{cases}$$

- What is the support?

$$[0, 1] \cup [2, 3]$$

Summary of 7.1

- Preferences over prospects
- Indifference Curves
  - Linear: "Reduction of Compound Lotteries"
  - Parallel: "Independent of Irrelevant Alternatives"
- Expected Utility
- Anomalies: Allais paradox, Ellsberg paradox, Bayes' Rule paradoxes (Soft vs. Hard prob. and Monty Hall Problem) and Rabin paradox
- Continuous State Space
- Homework: Exercise 7.1-4 (Optional: 7.1-3)

In-Class Homework: Exercise 7.1-1 IA ⇔ IA'

a) For M = 2, show that IA implies IA'

- (IA) If $\pi^1 \succsim \pi^2$, then for any prospect r and probabilities $p_1, p_2 > 0, p_1 + p_2 = 1$

$$q^1 = (p_1, p_2 : \pi^1, r) \succsim (p_1, p_2 : \pi^2, r) = q^2$$

- (IA') If $\pi^m \succsim \hat{\pi}^m, m = 1, \cdots, M$, then for any probability vector $p = (p_1, \cdots, p_M)$

$$(p : \pi^1, \cdots, \pi^M) \succsim (p : \hat{\pi}^1, \cdots, \hat{\pi}^M)$$

b) Show that if the proposition holds for M = k-1, then it must also hold for M = k.

In-Class Homework: Exercise 7.1-2 Allais

A. $1 million for sure	– (	0,	1,	0)
B. 90% chance $5 million	– (	0.90,	0,	0.10)
C. 10% chance $1 million	– (	0,	0.10,	0.90)
D. 9% chance $5 million	– (	0.09,	0,	0.91)

1. Draw tree diagrams showing that C and D can be represented as compound gambles between A and B, respectively, and (0,0,1), where the probability of (0,0,1) is the same.
2. Show that the ranking of A and B should be the same as the ranking of C and D.