Consumer Choice with N Commodities

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(Lecture 6, Micro Theory I)

From 2 Goods to N Goods...

- More applications of tools learned before...
- Questions we ask: What is needed to...

1. Obtain the compensated law of demand?
2. Have a concave minimized expenditure function?
3. Recover consumer's demand?
4. "Use" a representative agent (in macro)?

Key Problems to Consider

- Revealed Preference: Only assumption needed:
  - Compensated Law of Demand
  - Concave Minimized Expenditure Function
- Indirect Utility Function: (The Maximized Utility)
  - Roy's Identity: Can recover demand function from it
- Homothetic Preferences: (Revealed Preference)
  - Demand is proportional to income
  - Utility function is homogeneous of degree 1
  - Group demand as if one representative agent

Why do we care about this?

- Three separate questions:

1. How general can revealed preference be?
2. How do we back out demand from utility maximization?
3. When can we aggregate group demand as a representative agent (in macroeconomics)?

- Are these convincing?

Proposition 2.3-1 Compensated Price Change

Consider the dual consumer problem

$$M(\vec{p}, U^*) = \min_{\vec{x}} \{\vec{p} \cdot \vec{x} | U(\vec{x}) \geq U^*\}$$

For $\vec{x}^0$ be expenditure minimizing for prices $\vec{p}^0$

$\vec{x}^1$ be expenditure minimizing at prices $\vec{p}^1$

$\vec{x}^0, \vec{x}^1$ satify $U(\vec{x}) \geq U^*$

$\Rightarrow$ compensated price change is $\Delta\vec{p} \cdot \Delta\vec{x} \leq 0$

Proposition 2.3-1 Compensated Price Change

Proof:

$$\vec{p}^0 \cdot \vec{x}^0 \le \vec{p}^0 \cdot \vec{x}^1, \quad \vec{p}^1 \cdot \vec{x}^1 \le \vec{p}^1 \cdot \vec{x}^0$$

Since $\vec{x}^0$ is expenditure minimizing for prices $\vec{p}^0$

$\vec{x}^1$ is expenditure minimizing at prices $\vec{p}^1$

$$\Rightarrow -\vec{p}^0 \cdot (\vec{x}^1 - \vec{x}^0) \le 0, \quad \vec{p}^1 \cdot (\vec{x}^1 - \vec{x}^0) \le 0$$

$$\Rightarrow \Delta\vec{p} \cdot \Delta\vec{x} = (\vec{p}^1 - \vec{p}^0) \cdot (\vec{x}^1 - \vec{x}^0) \le 0$$

Proposition 2.3-1 Compensated Price Change

- This is true for any pair of price vectors
- For $\vec{p}^0 = (\overline{p}_1, \cdots, \overline{p}_{j-1}, p_j^0, \overline{p}_{j+1}, \cdots, \overline{p}_n)$
  $\vec{p}^1 = (\overline{p}_1, \cdots, \overline{p}_{j-1}, p_j^1, \overline{p}_{j+1}, \cdots, \overline{p}_n)$
- We have the (compensated) law of demand:
  $$\Delta p_j \cdot \Delta x_j \leq 0$$
- Note that we did not need differentiability to get this, just revealed preferences!!
- But if differentiable, we have $\dfrac{\partial x_j^c}{\partial p_j} \leq 0$

$1^{\text{st}}/2^{\text{nd}}$ Derivatives of Expenditure Function

But what is $\frac{\partial x_j^c}{\partial p_j}$?

Consider the dual problem as a maximization:

$$-M(\vec{p}, U^*) = \max_{\vec{x}} \{-\vec{p} \cdot \vec{x} | U(\vec{x}) \geq U^*\}$$

Lagrangian is $\mathfrak{L} = -\vec{p} \cdot \vec{x} + \lambda(U(\vec{x}) - U^*)$

Envelope Theorem yields $-\frac{\partial M}{\partial p_j} = \frac{\partial \mathfrak{L}}{\partial p_j} = -x_j^c$

$$\Rightarrow \frac{\partial}{\partial p_i}\left(\frac{\partial M}{\partial p_j}\right) = \frac{\partial x_j^c}{\partial p_i}$$

1st/2nd Derivatives of Expenditure Function

Hence, compensated law of demand yields

$$\frac{\partial x_j^c}{\partial p_j} = \frac{\partial^2 M}{\partial p_j^2} \leq 0$$

$\Rightarrow$ Expenditure function concave for each p_j.

Is the entire Expenditure function concave?

Requires the matrix of second derivatives

$$\left[\frac{\partial^2 M}{\partial p_i \partial p_j}\right] = \left[\frac{\partial x_j^c}{\partial p_i}\right]$$ to be negative semi-definite

Prop. 2.3-2 Concave Expenditure Function

$M(\vec{p}, U^*)$ is a concave function over $\vec{p}$.

i.e. For any $\vec{p}^0, \vec{p}^1$,

$$M(\vec{p}^\lambda, U^*) \geq (1-\lambda)M(\vec{p}^0, U^*) + \lambda M(\vec{p}^1, U^*)$$

We can show this with only revealed preferences...
(even without assuming differentiability!)

Prop. 2.3-2 Concave Expenditure Function

Proof: For $\vec{x}^{\lambda}$ that solves $M(\vec{p}^{\lambda}, U^*)$, (feasible!)

$$M(\vec{p}^0, U^*) = \vec{p}^0 \cdot \vec{x}^0 \leq \vec{p}^0 \cdot \vec{x}^{\lambda},$$

$$M(\vec{p}^1, U^*) = \vec{p}^1 \cdot \vec{x}^1 \leq \vec{p}^1 \cdot \vec{x}^{\lambda}$$

Since $M(\vec{p}, U^*)$ minimizes expenditure.

Hence,

$$(1-\lambda)M(\vec{p}^0, U^*) + \lambda M(\vec{p}^1, U^*)$$

$$\leq \left[(1-\lambda)\vec{p}^0 \cdot \vec{x}^{\lambda}\right] + \left[\lambda \vec{p}^1 \cdot \vec{x}^{\lambda}\right]$$

$$= \vec{p}^{\lambda} \cdot \vec{x}^{\lambda} = M(\vec{p}^{\lambda}, U^*)$$

What Have We Learned?

- Method of Revealed Preferences
- Used it to obtain:

1. Compensated Price Change
2. Compensated Law of Demand
3. Concave Expenditure Function
   - Special Case assuming differentiability

- Next: How can we get demand from utility?

Indirect Utility Function

Let $\vec{x}^* = \vec{x}(\vec{p}, I)$ be the demand for consumer $U(\cdot)$ with income I, facing price vector $\vec{p}$.

$$V(\vec{p}, I) = \max_{\vec{x}} \left\{ U(\vec{x}) | \vec{p} \cdot \vec{x} \leq I, \vec{x} \geq \vec{0} \right\}$$
$$= U(\vec{x}^*(\vec{p}, I))$$

is maximized $U(\vec{x})$, aka **indirect utility function**.

Why should we care about this function?

Proposition 2.3-3 Roy's Identity

$$x_j^*(\vec{p}, I) = -\frac{\frac{\partial V}{\partial p_j}}{\frac{\partial V}{\partial I}}$$

Get this directly from indirect utility function...

Proposition 2.3-3 Roy's Identity

Proof:

$$V(\vec{p}, I) = \max_{\vec{x}} \left\{ U(\vec{x}) \middle| \vec{p} \cdot \vec{x} \leq I, \vec{x} \geq \vec{0} \right\}$$

Lagrangian is $\mathfrak{L}(\vec{x}, \lambda) = U(\vec{x}) + \lambda(I - \vec{p} \cdot \vec{x})$

Envelope Theorem yields $\dfrac{\partial V}{\partial I} = \dfrac{\partial \mathfrak{L}}{\partial I}(\vec{x}^*, \lambda^*) = \lambda^*$

And $\dfrac{\partial V}{\partial p_j} = \dfrac{\partial \mathfrak{L}}{\partial p_j}(\vec{x}^*, \lambda^*) = -\lambda^* x_j^*(\vec{p}, I)$

$$\Rightarrow x_j^*(\vec{p}, I) = -\frac{\frac{\partial V}{\partial p_j}}{\frac{\partial V}{\partial I}}$$

Example: Unknown Utility…

Consider indirect utility function

$$V(\vec{p}, I) = \prod_{i=1}^{n} \left(\frac{\alpha_i I}{p_i} \right)^{\alpha_i} \quad \text{where} \quad \sum_{i=1}^{n} \alpha_i = 1$$

What's the demand (and original utility) function?

$$\ln V = \ln I - \sum_{i=1}^{n} \alpha_i \ln p_i + \sum_{i=1}^{n} \alpha_i \ln \alpha_i$$

$$\Rightarrow \frac{\partial}{\partial I} \ln V = \frac{1}{V} \frac{\partial V}{\partial I} = \frac{1}{I},$$

$$\frac{\partial}{\partial p_i} \ln V = \frac{1}{V} \frac{\partial V}{\partial p_i} = -\frac{\alpha_i}{p_i}$$

By Roy's Identity,

$$x_i^* = -\frac{\frac{\partial V}{\partial p_j}}{\frac{\partial V}{\partial I}} = \frac{\alpha_i I}{p_i}$$

Example: Unknown Utility???

- Plugging back in

$$U(\vec{x}) = V = \prod_{i=1}^{n} \left(\frac{\alpha_i I}{p_i} \right)^{\alpha_i} = \prod_{i=1}^{n} (x_i)^{\alpha_i}$$

- What is this utility function?
- Cobb-Douglas!

- Note: This is an example where demand is proportion to income. In fact, we have…

Definition: Homothetic Preferences

Strictly monotonic preference $\succsim$ is **homothetic** if, for any $\theta > 0$ and $\vec{x}^{\,0}, \vec{x}^{\,1}$ such that $\vec{x}^{\,0} \succsim \vec{x}^{\,1}$,

$$\theta\vec{x}^{\,0} \succsim \theta\vec{x}^{\,1}$$

In fact, if $\vec{x}^{\,0} \sim \vec{x}^{\,1}$,

$$\text{Then, } \theta\vec{x}^{\,0} \sim \theta\vec{x}^{\,1}$$

Why Do We Care About This?

- Proposition 2.3-4:
  - Demand proportional to income
- Proposition 2.3-5:
  - Homogeneous functions represent homothetic preferences
- Proposition 2.3-6:
  - Homothetic preferences are represented by functions that are homogeneous of degree 1
- Proposition 2.3-7: Representative Agent

Prop. 2.3-4: Demand Proportional to Income

If preferences are homothetic,

and $\vec{x}^*$ is optimal given income I,

Then $\theta\vec{x}^*$ is optimal given income θI.

Proof:

Let $\vec{x}^{**}$ be optimal given income θI,

Then $\vec{x}^{**} \succsim \theta\vec{x}^*$ since $\theta\vec{x}^*$ is feasible with θI.

By revealed preferences, $\vec{x}^* \succsim \dfrac{1}{\theta}\vec{x}^{**}$ ($\because \frac{1}{\theta}\vec{x}^{**}$ feasible)

By homotheticity, $\theta\vec{x}^* \succsim \vec{x}^{**}$

Thus, $\theta\vec{x}^* \sim \vec{x}^{**}$ (optimal for income θI)

Prop. 2.3-5: Homogeneous Func/Homothetic Pref

If preferences are represented by $U(\lambda\vec{x}) = \lambda^k U(\vec{x})$,

Then preferences are homothetic.

Proof:

Suppose $\vec{x} \succsim \vec{y}$,

Then $U(\vec{x}) \geq U(\vec{y})$.

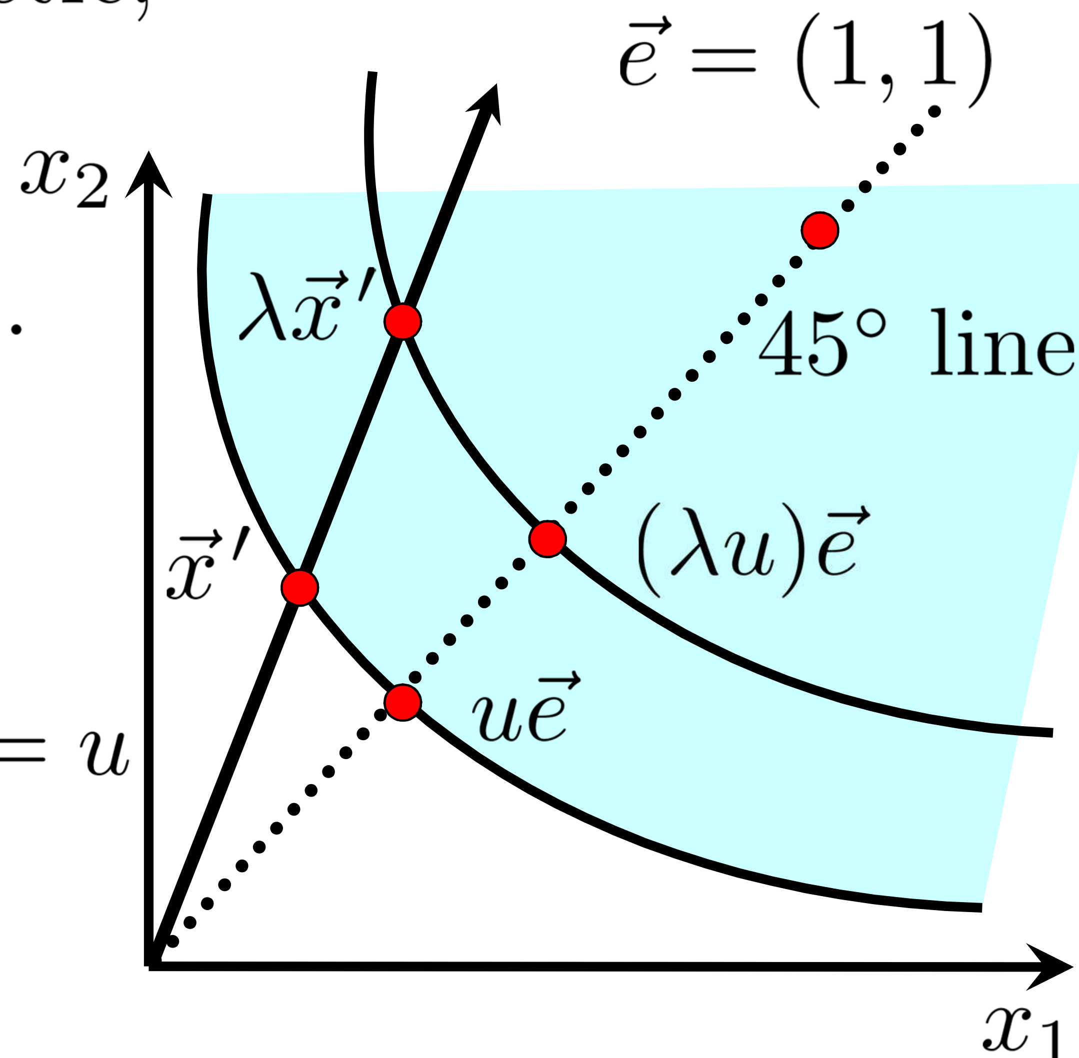
Since $U(\vec{x})$ is homogeneous,

$$U(\lambda\vec{x}) = \lambda^k U(\vec{x}) \geq \lambda^k U(\vec{y}) = U(\lambda\vec{y})$$

Thus, $\lambda\vec{x} \succsim \lambda\vec{y}$. i.e. Preferences are homothetic.

Prop. 2.3-6: Homothetic Pref. Representation

If preferences are homothetic,
They can be represented
by a function that is
homogeneous of degree 1.



Proof: $\vec{e} = (1, \cdots, 1)$
For $\vec{x}'$, exists $u\vec{e} \sim \vec{x}'$
For utility function $U(\vec{x}') = u$
By homotheticity,

$$\lambda\vec{x}' \sim (\lambda u)\vec{e}$$

Hence, $U(\lambda\vec{x}') = \lambda u = \lambda U(\vec{x}')$

Prop. 2.3-7: Representative Preferences

$$\vec{x}(\vec{p}, I) = \arg\max_{\vec{x}} \left\{ U(\vec{x}) \middle| \vec{p} \cdot \vec{x} \leq I \right\},\ U \text{ homothetic}$$

$$\Rightarrow \sum_{h=1}^{H} \vec{x}(\vec{p}, I^h) = \vec{x}(\vec{p}, I^R),\ \ I^R = \sum_{h=1}^{H} I^h$$

If a group of consumers have the same homothetic preferences represented by U,

Then group demand is equal to demand of a representative member holding all the income.

Prop. 2.3-7: Representative Preferences

$$\vec{x}(\vec{p}, I) = \arg\max_{\vec{x}} \left\{ U(\vec{x}) \middle| \vec{p} \cdot \vec{x} \le I \right\}, \; U \text{ homothetic}$$

$$\Rightarrow \sum_{h=1}^{H} \vec{x}(\vec{p}, I^h) = \vec{x}(\vec{p}, I^R), \;\; I^R = \sum_{h=1}^{H} I^h$$

Proof: $\vec{x}(p, 1) = \arg\max_{\vec{x}} \left\{ U(\vec{x}) \middle| \vec{p} \cdot \vec{x} \le 1 \right\}$

Preferences (U) homothetic $\Rightarrow \vec{x}(\vec{p}, I^h) = I^h \vec{x}(\vec{p}, 1)$

$$\Rightarrow \sum_{h=1}^{H} \vec{x}(\vec{p}, I^h) = \sum_{h=1}^{H} I^h \vec{x}(\vec{p}, 1) = I^R \vec{x}(\vec{p}, 1)$$

$$= \vec{x}(p, I^R) \text{ by homotheticity}$$

Summary of 2.3

- Revealed Preference:
  - Compensated Law of Demand
  - Concave Minimized Expenditure Function
- Indirect Utility Function:
  - Roy’s Identity: Recovering demand function
- Homothetic Preferences:
  - Demand is proportional to income
  - Utility function is homogeneous of degree 1
  - Group demand as if one representative agent
- Homework: (Optional: Exercise 2.3-3)

Homework: 2008 Midterm Q2 Roy's Identity

1. Draw their income expansion path for two consumers, A and B, with utility functions:
$$u_A(x_1^A, x_2^A) = -\frac{A_1}{x_1^A} - \frac{A_2}{x_2^A} \text{ if } x_1^A \cdot x_2^A > 0,$$
$$u_B(x_1^B, x_2^B) = \min\{2x_1^B, 3x_2^B\}.$$
2. Derive the indirect utility function $V_i(\vec{p}, I)$
   – Can you use Roy's Identity to derive each consumer's demand? Why or why not?
3. Derive $x_i^{h^*}(\vec{p}, I)$, consumer h's demand functions for commodity i

In-Class Homework: RPP and Exercise 2.3-1

Consider firm problem $\Pi(\vec{p}) = \max_{\vec{y}} \{\vec{p} \cdot \vec{y} | \vec{y} \in \mathcal{Y}^f\}$

For $\vec{y}^0$ be profit maximizing for prices $\vec{p}^0$

$\vec{y}^1$ be profit maximizing at prices $\vec{p}^1$

$$\vec{y}^0, \vec{y}^1 \in \mathcal{Y}^f \quad \Rightarrow \Delta\vec{p} \cdot \Delta\vec{y} \geq 0$$

- $$U(\vec{x}) = \times_{j=1}^{n} x_j^{\alpha_j}, \alpha_1 + \cdots + \alpha_n = 1$$

a) Solve for the indirect utility function $V_i(\vec{p}, I)$

b) Explain why you can "invert" your results to obtain the expenditure function

c) Hence solve for the Expenditure Function

In-Class Homework: Exercise 2.3-2

- Bev has a utility function $U(\vec{x}) = \sqrt{x_1 x_2} + x_3$

a) Suppose she allocates y towards the purchase of commodity 1 and 2 and purchases x_3 units of commodity 3. Show that her resulting utility is

$$U^*(x_3, y) = \frac{y}{2\sqrt{p_1 p_2}} + x_3$$

b) Given this preliminary optimization problem has been solved, her budget constraint is $p_3 x_3 + y \leq I$ Solve for her optimizing values of x_3 and y.

   – Under what conditions, if any, is she strictly worse off if she is told that she can consume at most 2 of the 3 available commodities?