Shadow Prices

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(Lecture 2, Micro Theory I)

A Peak-Load Pricing Problem

- Consider the problem faced by Chunghwa Telecom (CHT):
- By building base stations, CHT can provide cell phone service to a certain region
  - An establish network can provide service both in the day and during the night
  - Marginal cost is low (zero?!); setup cost is huge
- Marketing research reveal unbalanced demand
  - Day – peak; Night – off-peak (or vice versa?)

A Peak-Load Pricing Problem

- If you are the CEO of CHT, how would you price usage of your service?
  - Price day and night the same (or different)?
- Economic intuition should tell you to set off-peak prices lower than peak prices
  - But how low?
- All new 4G services (LTE) are facing a similar problem now...

More on Peak-Load Pricing

- Other similar problems include:
  - How should Taipower price electricity in the summer and winter?
  - How should a theme park set its ticket prices for weekday and weekends?
- Even if demand estimations are available, you will still need to do some math to find optimal prices...
  - Either to maximize profit or social welfare

A Peak-Load Pricing Problem

- Back to CHT:
- Capacity constraints:

$$q_j \leq q_0, j = 1, ..., n$$

- CHT's Cost function:

$$C(q_0, \vec{q}) = F + c_0 q_0 + \vec{c} \cdot \vec{q}$$

- Demand for cell phone service: $p_j(\vec{q})$
- Total Revenue: $R(\vec{q}) = \vec{p} \cdot \vec{q}$

A Peak-Load Pricing Problem

- The monopolist profit maximization problem:

$$\max_{q_0,\vec{q}} \left\{ R(\vec{q}) - F - c_0 q_0 - \vec{c}\cdot\vec{q} \middle| q_0 - q_j \geq 0, j = 1, ..., n \right\}$$

- How do you solve this problem?
- When does FOC guarantee a solution?
- What does the Lagrange multiplier mean?
- What should you do when FOC "fails"?

Need: Lagrange Multiplier Method

1. Write Constraints as $h_i(\vec{x}) \geq 0, i = 1, \ldots, m$

$$\vec{h}(\vec{x}) = (h_1(\vec{x}), \ldots, h_m(\vec{x}))$$

2. Shadow prices $\vec{\lambda} = (\lambda_1, \ldots, \lambda_m)$

- Lagrangian $\mathcal{L}(\vec{x}, \vec{\lambda}) = f(\vec{x}) + \vec{\lambda} \cdot \vec{h}(\vec{x})$
- FOC:

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \vec{\lambda} \cdot \frac{\partial \vec{h}}{\partial x_j} \leq 0, \text{ with equality if } \overline{x}_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = h_i(\vec{\overline{x}}) \geq 0, \text{ with equality if } \lambda_i > 0.$$

Solving Peak-Load Pricing

- The monopolist profit maximization problem:

$$\max_{q_0,\vec{q}} \left\{ R(\vec{q}) - F - c_0 q_0 - \vec{c}\cdot\vec{q} \,\middle|\, q_0 - q_j \geq 0, j = 1, ..., n \right\}$$

- The Lagrangian is

$$\mathcal{L}(q_0,\vec{q}) = R(\vec{q}) - F - c_0 q_0 - \sum_{j=1}^{n} c_j q_j + \sum_{j=1}^{n} \lambda_j (q_0 - q_j)$$

$$= R(\vec{q}) - \sum_{j=1}^{n} (c_j + \lambda_j) q_j + \left(\sum_{j=1}^{n} \lambda_j - c_0 \right) q_0 - F$$

Solving Peak-Load Pricing

- FOC:

$$\frac{\partial \mathcal{L}}{\partial q_j} = MR_j - c_j - \lambda_j \leq 0, \text{ with equality if } q_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 \leq 0, \text{ with equality if } q_0 > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- For positive production, FOC becomes:

$$\frac{\partial \mathcal{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0, \text{ since } q_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0, \text{ since } q_0 > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- Meaning of FOC:

$$\frac{\partial \mathcal{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0, \text{ since } q_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0, \text{ since } q_0 > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Since $c_0 > 0$, at least 1 period has shadow price > 0!

Solving Peak-Load Pricing

- Meaning of FOC:

$$\frac{\partial \mathcal{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0,$$

Hit capacity at positive shadow price!

$$\frac{\partial \mathcal{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0,$$

Off-peak shadow price = 0

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- Meaning of FOC:

$$\frac{\partial \mathcal{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0, \quad MR_i(\overline{q}) = c_i + \lambda_i$$

Peak MR = MC + capacity cost

$$\frac{\partial \mathcal{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0,$$

Peak periods share capacity cost via shadow price

$$MR_j(\overline{q}) = c_j \text{ equality if } \lambda_j > 0.$$

Off-peak: MR=MC!

Solving Peak-Load Pricing

- Economic Insight of FOC:
- Marginal decision of the manager: MR = MC
- Off-peak: MR = operating MC
  - Since didn't hit capacity
- Peak: Need to increase capacity
  - MR of all peak periods =
    cost of additional capacity
    + operating MC of all peak periods
- What's the theory behind this?

Constrained Optimization: Economic Intuition

- Single Constraint Problem:

$$\max_{\vec{x}} \left\{ f(\vec{x}) \middle| \vec{x} \geq 0, b - g(\vec{x}) \geq 0 \right\}$$

- Interpretation: a profit maximizing firm
  - Produce non-negative output $\vec{x} \geq 0$
  - Subject to resource constraint $g(\vec{x}) \leq b$
- Example: linear constraint $\vec{a} \cdot \vec{x} = \sum_{j=1}^{n} a_j x_j \leq b$
- Each unit of x_j requires a_j units of resource b .

Constrained Optimization: Economic Intuition

- Single Constraint Problem:

$$\max_{\vec{x}} \left\{ f(\vec{x}) \middle| \vec{x} \geq 0, b - g(\vec{x}) \geq 0 \right\}$$

- Interpretation: a utility maximizing consumer
  - Consume non-negative input $\vec{x} \geq 0$
  - Subject to budget constraint $g(\vec{x}) \leq b$
- Example: linear constraint $\vec{a} \cdot \vec{x} = \sum_{j=1}^{n} a_j x_j \leq b$
- Each unit of x_j requires a_j units of currency b .

Constrained Optimization: Economic Intuition

- Suppose $\vec{x}^*$ solves the problem
- If one increases x_j, profit changes by $\frac{\partial f}{\partial x_j}$
- Additional resources needed: $\frac{\partial g}{\partial x_j}$
- Cost of additional resources: $\lambda \frac{\partial g}{\partial x_j}$
  - (Market/shadow price is λ)
- Net gain of increasing x_j is $\frac{\partial f}{\partial x_j}(\vec{\bar{x}}) - \lambda \frac{\partial g}{\partial x_j}(\vec{\bar{x}})$

Necessary Conditions for x_j^*

- Firm will increase x_j^* if marginal net gain > 0
  - i.e. If x_j^* is optimal $\Rightarrow \dfrac{\partial f}{\partial x_j}(\vec{x}^*) - \lambda \dfrac{\partial g}{\partial x_j}(\vec{x}^*) \leq 0$
- Firm will decrease x_j^* if marginal net gain < 0 (unless x_j^* is already zero)
  - i.e. $x_j^* > 0 \Rightarrow \dfrac{\partial f}{\partial x_j}(\vec{x}^*) - \lambda \dfrac{\partial g}{\partial x_j}(\vec{x}^*) \geq 0$

$$\frac{\partial f}{\partial x_j}(\vec{x}^*) - \lambda \frac{\partial g}{\partial x_j}(\vec{x}^*) \leq 0, \text{ with equality if } x_j^* > 0.$$

Necessary Conditions for x_j^*

- If x_j^* is strictly positive, marginal net gain $= 0$
  - i.e. $$x_j^* > 0 \Rightarrow \frac{\partial f}{\partial x_j}(\vec{x}^*) - \lambda \frac{\partial g}{\partial x_j}(\vec{x}^*) = 0$$
- If x_j^* is zero, marginal net gain ≤ 0
  - i.e. $$x_j^* = 0 \Rightarrow \frac{\partial f}{\partial x_j}(\vec{x}^*) - \lambda \frac{\partial g}{\partial x_j}(\vec{x}^*) \leq 0$$

$$\frac{\partial f}{\partial x_j}(\vec{x}^*) - \lambda \frac{\partial g}{\partial x_j}(\vec{x}^*) \leq 0, \text{ with equality if } x_j^* > 0.$$

Necessary Conditions for x_j^*

- If resource doesn't bind, opportunity cost $\lambda = 0$
  - i.e. $b - g(\vec{x}^*) > 0 \Rightarrow \lambda = 0$

- Or, in other words,

$$b - g(\vec{x}^*) \geq 0 \text{ with equality if } \lambda > 0.$$

  - This is logically equivalent to the first statement.

Lagrange Multiplier Method

1. Write constraint as $h(\vec{x}) \geq 0$ $\tilde{h}(\vec{x}) \leq 0$
2. Lagrange multiplier = shadow price λ

- Lagrangian $\mathcal{L}(\vec{x}, \lambda) = f(\vec{x}) + \lambda \cdot h(\vec{x})$
- FOC: $\mathcal{L}(\vec{x}, \lambda) = f(\vec{x}) - \lambda \cdot \tilde{h}(\vec{x})$

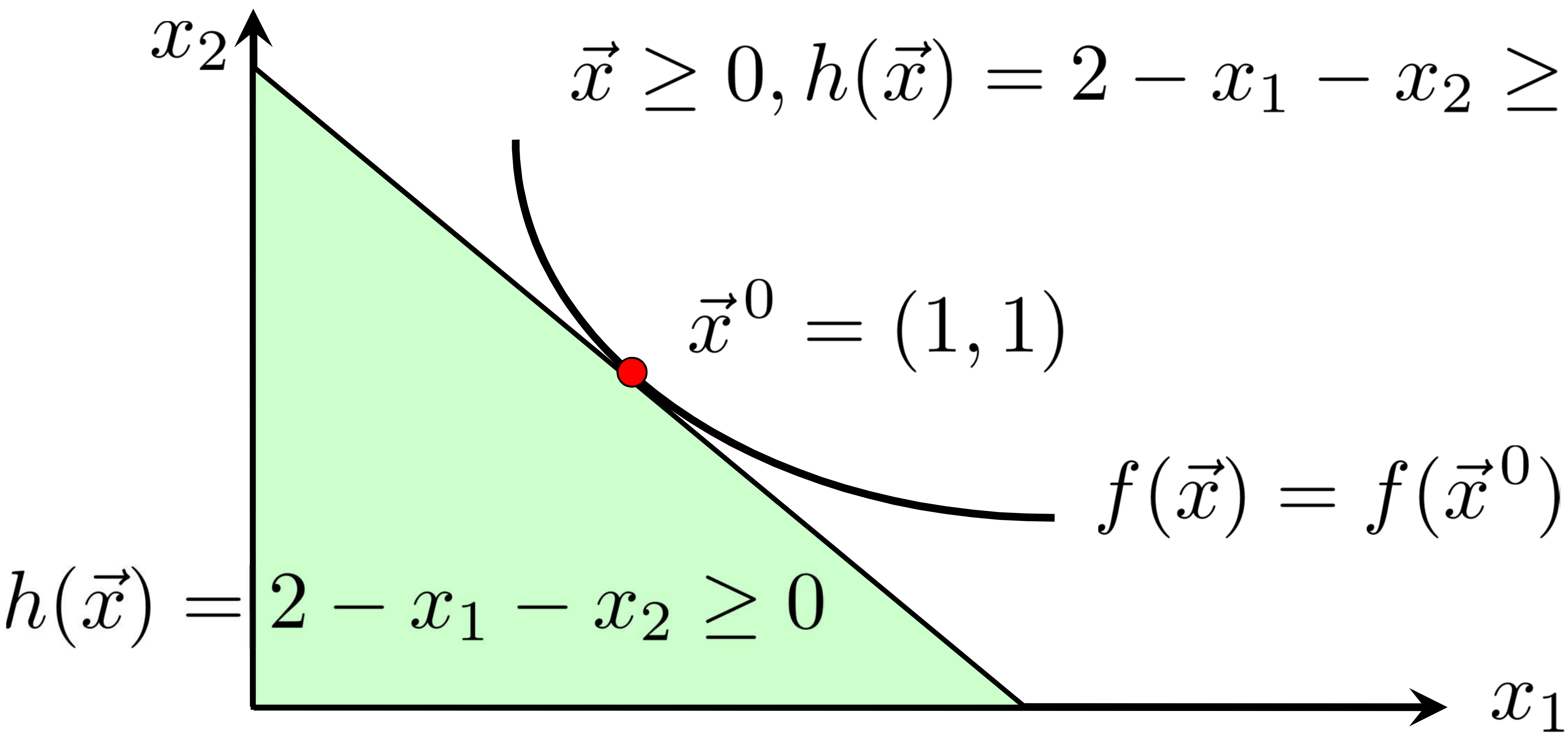
$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \lambda \cdot \frac{\partial h}{\partial x_j} \leq 0, \text{ with equality if } x_j^* > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = h(\vec{x}^*) \geq 0, \text{ with equality if } \lambda > 0.$$

Example 1

- A consumer problem:

$$\max_{\vec{x}} \left\{ f(\vec{x}) = \ln(1+x_1)(1+x_2) \,\middle|\, \vec{x} \geq 0, h(\vec{x}) = 2 - x_1 - x_2 \geq 0 \right\}$$



Example 1

- Maximum at $\vec{x}^* = (1, 1)$
- Lagrangian:

$$\mathcal{L}(\vec{x}, \lambda) = \ln(1 + x_1) + \ln(1 + x_2) + \lambda(2 - x_1 - x_2)$$

- FOC:

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{1}{1 + x_j} - \lambda \leq 0, \text{ with equality if } x_j^* > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 2 - x_1 - x_2 \geq 0, \text{ with equality if } \lambda > 0.$$

Lagrange Multiplier w/ Multiple Constraints

1. Write Constraints as $h_i(\vec{x}) \geq 0, i = 1, \cdots, m$
$$\vec{h}(\vec{x}) = \big(h_1(\vec{x}), \cdots, h_m(\vec{x})\big)$$
2. Shadow prices $\vec{\lambda} = (\lambda_1, \cdots, \lambda_m)$

- Lagrangian $\mathcal{L}(\vec{x}, \vec{\lambda}) = f(\vec{x}) + \vec{\lambda} \cdot \vec{h}(\vec{x})$
- FOC:

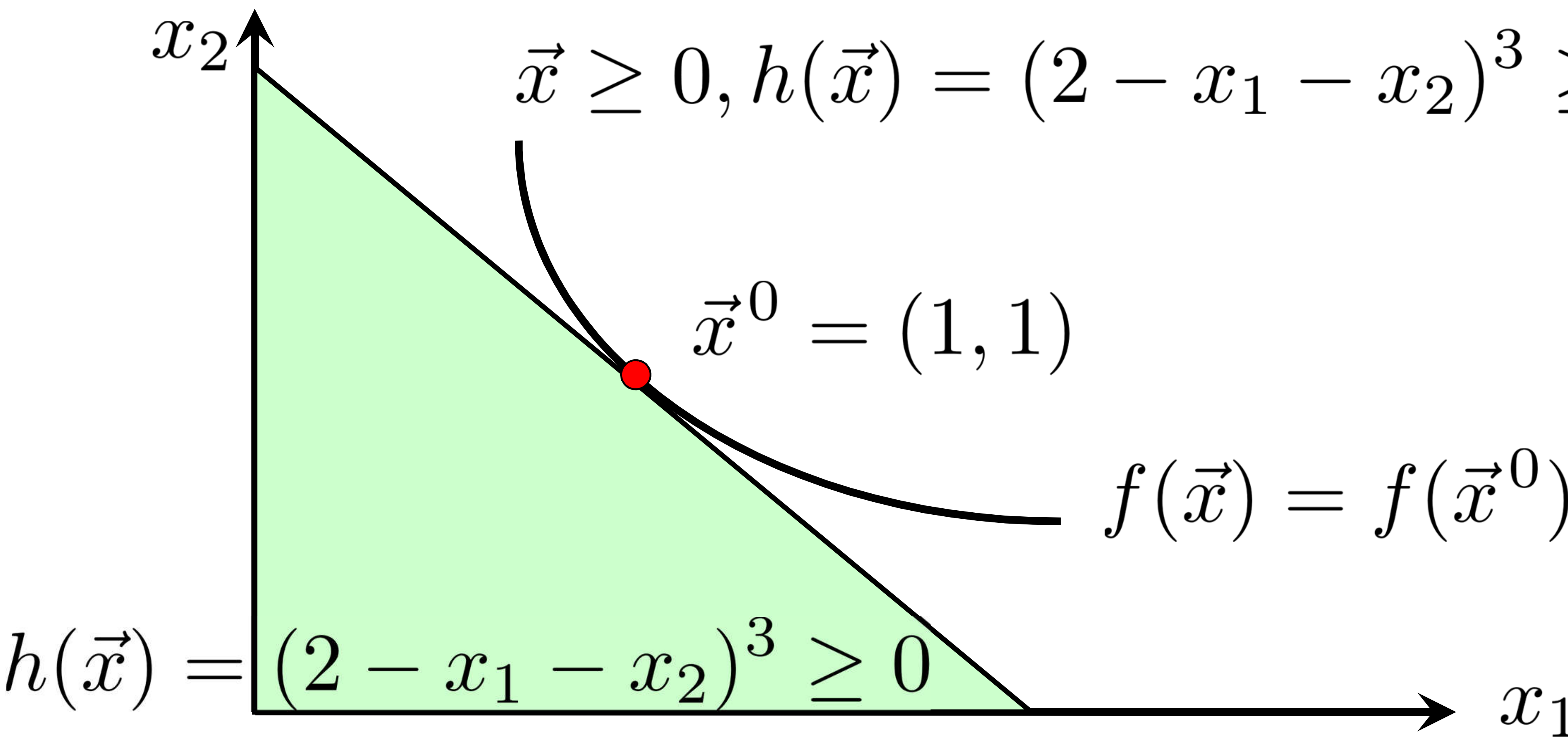
$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \vec{\lambda} \cdot \frac{\partial \vec{h}}{\partial x_j} \leq 0, \text{ with equality if } x_j^* > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = h_i(\vec{x}^*) \geq 0, \text{ with equality if } \lambda_i > 0.$$

When Intuition Breaks Down? Example 2

- A "new" problem:

$$\max_{\vec{x}} \left\{ f(\vec{x}) = \ln(1+x_1)(1+x_2) \,\middle|\, \vec{x} \geq 0, h(\vec{x}) = (2 - x_1 - x_2)^3 \geq 0 \right\}$$



When Intuition Breaks Down? Example 2

$$\mathcal{L}(\vec{x}, \lambda) = \ln(1 + x_1) + \ln(1 + x_2) + \lambda(2 - x_1 - x_2)^3$$

- FOC is violated at $\vec{x}^* = (1, 1)!$

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{1}{1 + x_j} - 3\lambda(2 - x_1 - x_2)^2 = \frac{1}{2}$$

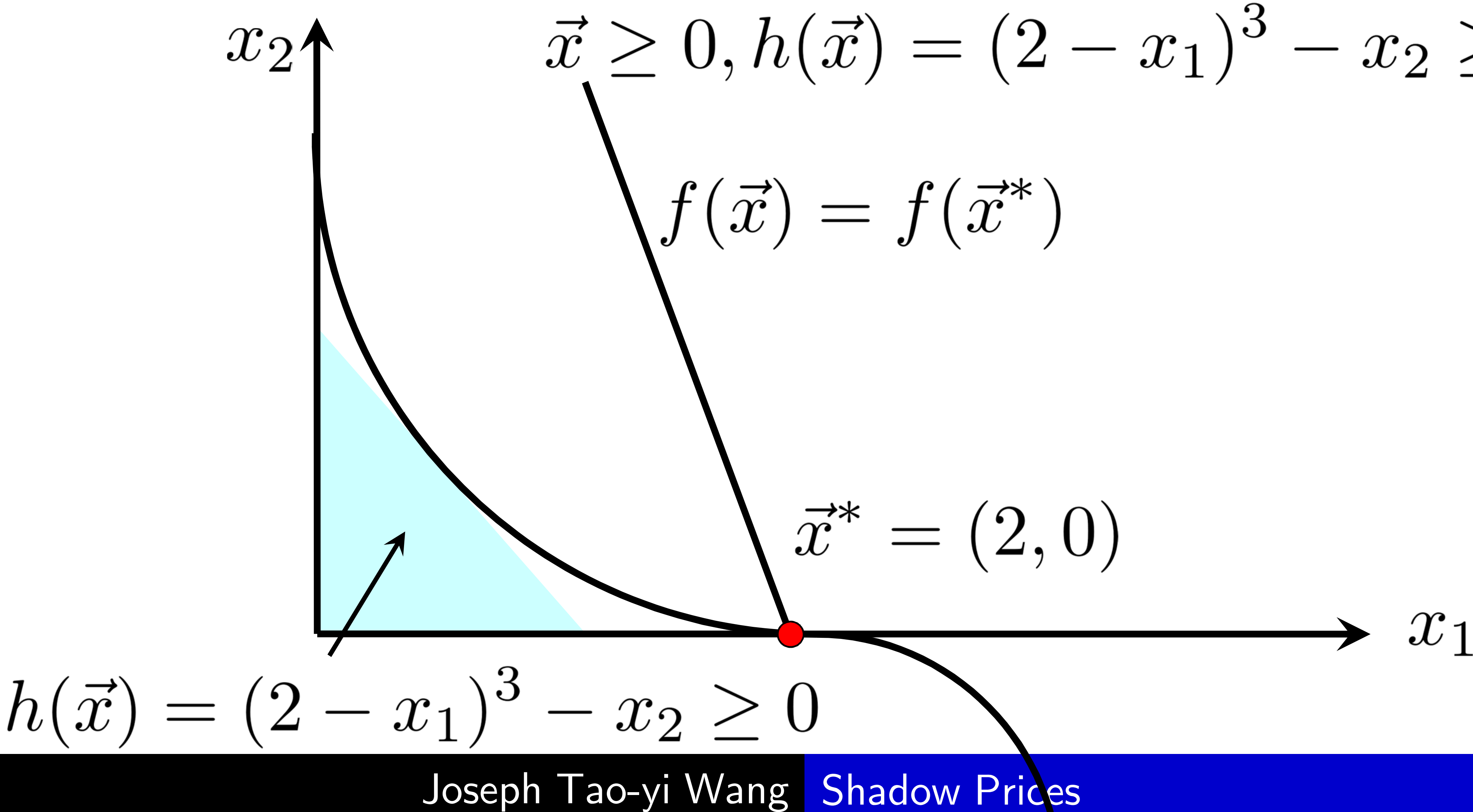
- How could this be?
- Because "linearization" fails if gradient = 0...

$$\frac{\partial h}{\partial \vec{x}} = \vec{0} \text{ at } \vec{x} = (1, 1)$$

$$\overline{h}(\vec{x}) = h(\vec{x}^*) + \frac{\partial h}{\partial \vec{x}}(\vec{x}^*) \cdot (\vec{x} - \vec{x}^*) = h(1, 1) = 0$$

Other Break Downs? See Example 3

$$\max_{\vec{x}} \left\{ f(\vec{x}) = 12x_1 + x_2 \,\middle|\, \vec{x} \geq 0, h(\vec{x}) = (2 - x_1)^3 - x_2 \geq 0 \right\}$$



Other Break Downs? See Example 3

- Lagrangian $\mathcal{L}(\vec{x},\lambda) = 12x_1 + x_2 + \lambda\left[(2-x_1)^3 - x_2\right]$
- FOC is violated!

$$\frac{\partial \mathcal{L}}{\partial x_1} = 12 - 3\lambda(2-\overline{x}_1)^2 = 12 \text{ at } \vec{x}^* = (2,0)$$

- What's the problem this time?
- Not the gradient... $\dfrac{\partial h}{\partial \vec{x}}(\vec{x}^*) = (0,-1)$

- But "Linearized feasible set" has no interior!

Other Break Downs? See Example 3

– What's the problem this time?

- Gradient is $\dfrac{\partial h}{\partial \vec{x}}(\vec{x}^*) = (0, -1)$
- Linear approximation of the constraint is:

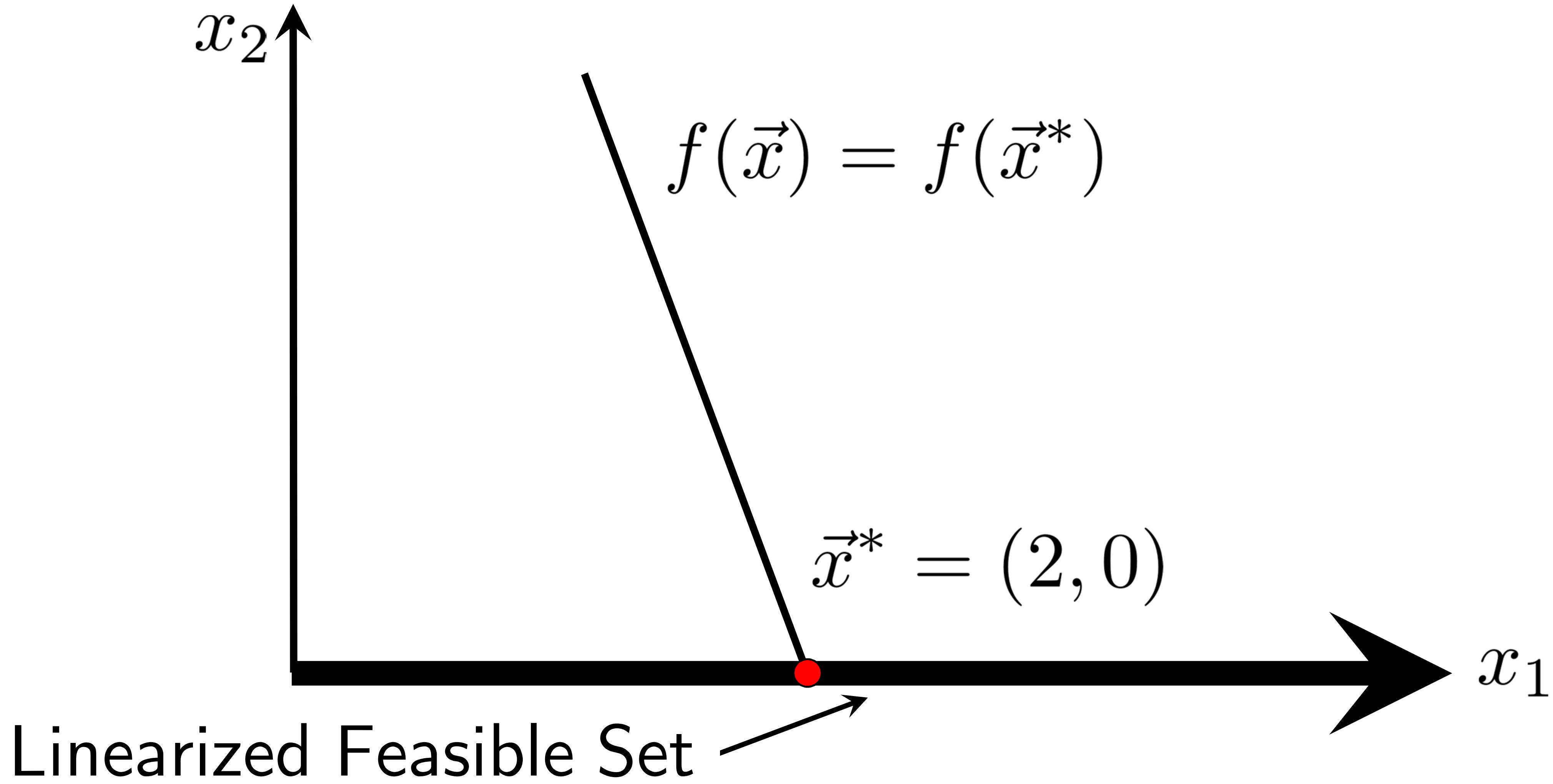
$$\frac{\partial h}{\partial \vec{x}}(\vec{x}^*) \cdot (\vec{x} - \vec{x}^*)$$

$$= \frac{\partial h}{\partial x_1}(\vec{x}^*) \cdot (x_1 - 2) + \frac{\partial h}{\partial x_2}(\vec{x}^*) \cdot x_2$$

$$= -x_2 \geq 0 \quad \Rightarrow x_2 = 0$$

Other Break Downs? See Example 3

$$\max_{\vec{x}} \left\{ f(\vec{x}) = 12x_1 + x_2 \middle| \vec{x} \geq 0, h(\vec{x}) = (2 - x_1)^3 - x_2 \geq 0 \right\}$$



Linearized Feasible Set $\overline{X}$

- Set of constraints binding at $\vec{x}^*$: $h_i(\vec{x}^*) = 0$
  - For $i \in B = \{i | i = 1, ..., m, h_i(\vec{x}^*) = 0\}$
- Replace binding constraints by linear approx.

$$\overline{h}_i(\vec{x}) = \underline{\underline{h_i(\vec{x}^*)}} + \frac{\partial h_i}{\partial \vec{x}}(\vec{x}^*) \cdot (\vec{x} - \vec{x}^*) \geq 0$$

- These constraints also bind, and

$$\frac{\partial h_i}{\partial \vec{x}}(\vec{x}^*) \cdot (\vec{x} - \vec{x}^*) \geq 0, i \in B$$

  - since $\underline{\underline{h_i(\vec{x}^*)}} = 0$

Linearized Feasible Set $\overline{X}$

- Note: These are "true" constraints if gradient

$$\frac{\partial h_i}{\partial \vec{x}}(\vec{x}^*) \neq \vec{0}$$

- $\overline{X} =$ Linearized Feasible Set

  $=$ Set of non-negative vectors satisfying

$$\frac{\partial h_i}{\partial \vec{x}}(\vec{x}^*) \cdot (\vec{x} - \vec{x}^*) \geq 0, i \in B$$

Constraint Qualifications

- Set of feasible vectors:

$$X = \{\vec{x} \big| \vec{x} \geq 0, h_i(\vec{x}) \geq 0\}$$

- Constraint Qualifications hold at $\vec{x}^* \in \overline{X}$ if

(i) Binding constraints have non-zero gradients

$$\frac{\partial h_i}{\partial \vec{x}}(\vec{x}^*) \neq \vec{0}$$

(ii) The linearized feasible set $\overline{X}$ at $\vec{x}^*$ has a non-empty interior.

  – CQ guarantees FOC to be necessary conditions

Proposition 1.2-1 Kuhn-Tucker Conditions

- Suppose $\vec{x}^*$ solves
  $$\max_{\vec{x}} \left\{ f(\vec{x}) \middle| \vec{x} \in X \right\}, X = \text{feasible set}$$
- If the constraint qualifications hold at $\vec{x}^*$
- Then there exists shadow price vector $\vec{\lambda} \geq 0$
- Such that (for *j=1,…,n, i=1,…,m*)
  $$\frac{\partial \mathcal{L}}{\partial x_j}(\vec{x}^*, \vec{\lambda}) \leq 0, \text{ with equality if } x_j^* > 0.$$
  $$\frac{\partial \mathcal{L}}{\partial \lambda_i} = (\vec{x}^*, \vec{\lambda}) \geq 0, \text{ with equality if } \lambda_i > 0.$$

Lemma 1.2-2 [Special Case] Quasi-Concave

- If for each binding constraint at $\vec{x}^*$, h_i is quasi-concave and $\frac{\partial h_i}{\partial \vec{x}}(\vec{x}^*) \neq \vec{0}$
- Then, $\boxed{X \subset \overline{X}}$
  - Tangent Hyperplanes
    - $=$ Supporting Hyperplanes!
- Hence, if X has a non-empty interior, then so does the linearized set $\overline{X}$
  - Thus we have…

Prop 1.2-3 [Quasi-Concave] Constraint Qualif.

- Suppose feasible set has non-empty interior

$$X = \{\vec{x} | \vec{x} \geq 0, h_i(\vec{x}) \geq 0\}$$

- Constraint Qualifications hold at $\vec{x}^* \in \overline{X}$ if

- Binding constraints h_i are quasi-concave,
- And the gradient $\dfrac{\partial h_i}{\partial \vec{x}}(\overline{\vec{x}}) \neq \vec{0}$

Proposition 1.2-4 Sufficient Conditions

- $\vec{x}^*$ solves
  $$\max_{\vec{x}} \left\{ f(\vec{x}) \middle| \vec{x} \geq 0, h_i(\vec{x}) \geq 0, i = 1, ..., m \right\}$$
- If $f, h_i, i = 1, ..., m$ are quasi-concave,
- The Kuhn-Tucker conditions hold at $\vec{x}^*$,
- Binding constraints have $\frac{\partial h_i}{\partial \vec{x}}(\vec{x}^*) \neq \vec{0}$
- And $\frac{\partial f}{\partial \vec{x}}(\vec{x}^*) \neq \vec{0}$.

Summary of 1.2

- Consumer = Producer
- Lagrange multiplier = Shadow prices
- FOC = "MR – MC = 0": Kuhn-Tucker
- When does this intuition fail?
  - Gradient = 0
  - Linearized feasible set has no interior

→Constraint Qualification: when it flies...
  - CQ for quasi-concave constraints

- Sufficient Conditions (Proof in Section 1.4)

Summary of 1.2

- Peak-Load Pricing requires Kuhn-Tucker
- MR= "effective" MC
- Off-peak shadow price (for capacity) = 0
- Peak periods share additional capacity cost
- Can you think of real world situations that requires something like peak-load pricing?
  - After you start your new job making $$$$...
- Homework: Exercise 1.2-2 (Optional 1.2-3)