Supporting Prices and Convexity

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(Lecture 1, Micro Theory I)

Overview of Chapter 1

- Theory of Constrained Maximization
  - Why should we care about this?
- What is Economics?
- Economics is the study of how society manages its scarce resources (Mankiw, Ch.1)
  - "Economics is the science which studies human behavior as a relationship between given ends and scarce means which have alternative uses." (Lionel Robbins, 1932)

Overview of Chapter 1

- Other Historical Accounts:
  - Economics is the "study of how societies use scarce resources to produce valuable commodities and distribute them among different people." (Paul A. Samuelson, 1948)
- I think Economics is the study of institutions & human behavior (reaction to institutions)
- Either way, constrained maximization is key!

Tools Introduced in Chapter 1

1. Supporting Hyperplanes (and Convexity)
2. First Order Conditions (Kuhn-Tucker)
3. Envelope Theorem

- But why do I need to know the math?
- When does Coase conjecture work?
  - It depends—Math makes these predictions precise
- What happens if you ignore the conditions required for theory to work? (Recall 2008/09!)

Publication Reward Problem

- Example: How should NTU reward its professors to publish journal articles?
  - Should NTU pay, say, NT$300,000 per article published in Science or Nature?
- Well, it depends...
- Peek the answer ahead:
  - Yes, if the production set is convex.
  - No, if, for example, there is initial increasing returns to scale.

Supporting Prices

- More generally,
- can prices and profit maximization provide appropriate incentives to induce all possible efficient production plans?
  - Is there a price vector that supports each efficient production plan?
- (Yes, but when?)
- Need some definitions first...

Production Plan and Production Set

- A plant can:
- produce n outputs $\vec{q} = (q_1, \cdots, q_n)$
- using up to m inputs $\vec{z} = (z_1, \cdots, z_m)$
- Production Plan $(\vec{z}, \vec{q})$
- Production Set $\mathcal{Y} \subset \mathbb{R}_+^{m+n}$

  $=$ Set of all Feasible Production Plan
- Production Vector (treat inputs as negative)

  $\vec{y} = (-\vec{z}, \vec{q}) = (-z_1, \cdots, -z_m, q_1, \cdots, q_n)$

Production Set and Profits

- Production vector

$$\vec{y} = (y_1, \cdots, y_{m+n}) = (-z_1, \cdots, -z_m, q_1, \cdots, q_n)$$

- Production Set $\mathcal{Y} \subset \mathbb{R}^{m+n}$

  = Set of Feasible Production Plan

- Price vector $\vec{p} = (p_1, \cdots, p_{m+n})$

- Profit $\Pi = \underbrace{\sum_{i=m+1}^{m+n} p_i y_i}_{\text{total revenue}} - \underbrace{\sum_{i=1}^{m} p_i z_i}_{\text{total cost}} = \vec{p} \cdot \vec{y}$

EX: Production Function & Production Set

- A professor has 25 units of "brain-power"
- Allocates z_1 units to produce TSSCI papers
- Produce $q_1 = 4\sqrt{z_1}$ (Production Function)
- Production Set

$$\mathcal{Y}_1 = \left\{(z_1, q_1) \middle| z_1 \geq 0, q_1 \leq 4\sqrt{z_1}\right\}$$

- Treating inputs as negatives, $\vec{y} = (-\vec{z}, \vec{q})$
- Production Set is

$$\mathcal{Y}_1 = \left\{(y_1, y_2) \middle| -16y_1 - y_2^2 \geq 0\right\}$$

Production Efficiency

- A production plan $\vec{y}$ is wasteful if another plan in $\mathcal{Y}$ achieves larger output with smaller input
- $\vec{\bar{y}}$ is production efficient (=non-wasteful) if

  There is no $\vec{y} \in \mathcal{Y}$ such that $\vec{y} > \vec{\bar{y}}$

  – Note: $\vec{y} \geq \vec{\bar{y}}$ if $y_j \geq \bar{y}_j$ for all j

  – $\vec{y} > \vec{\bar{y}}$ if inequality is strict for some j

  – $\vec{y} \gg \vec{\bar{y}}$ if inequality is strict for all j

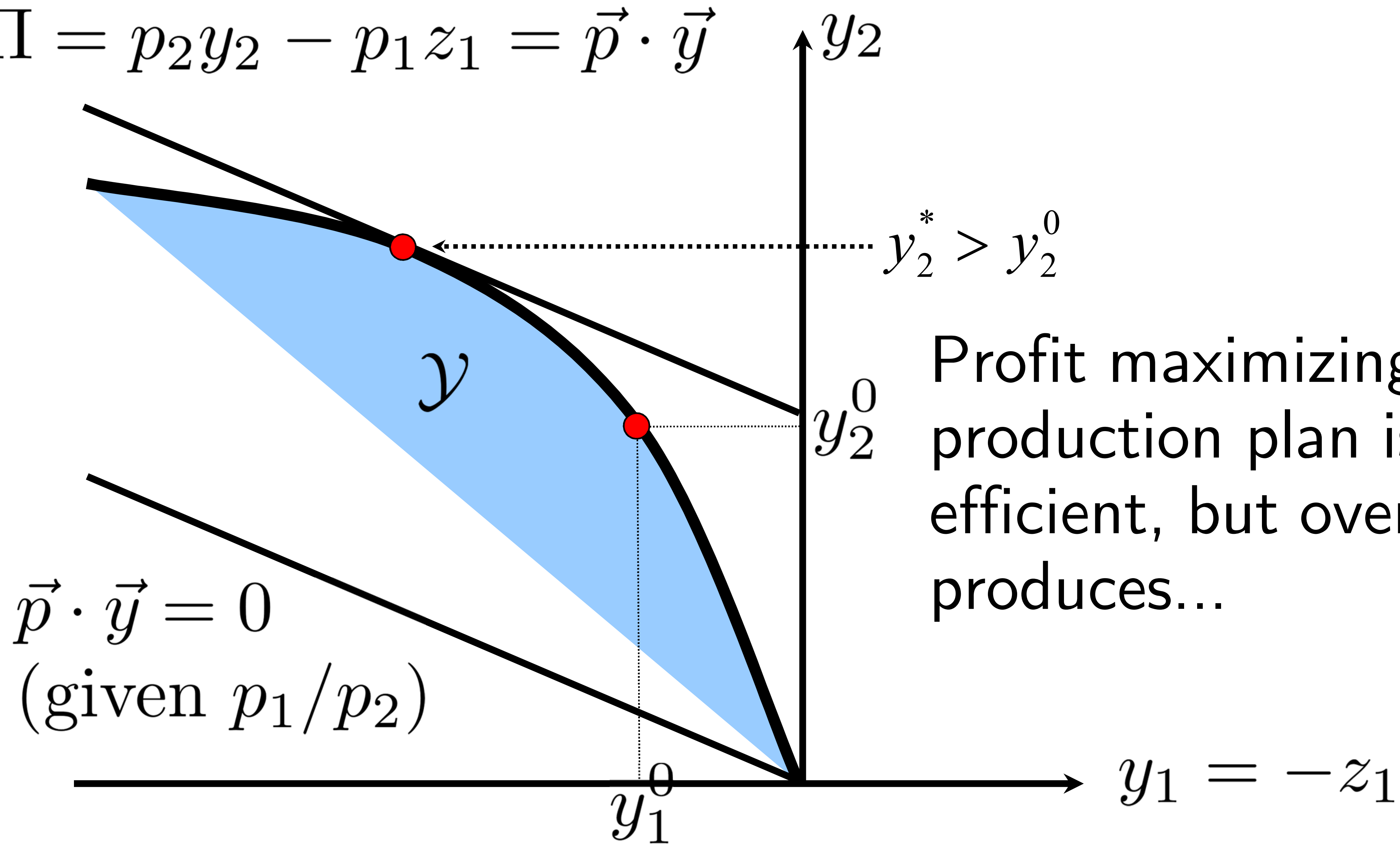
Can Prices Support Efficient Production?

- A professor has 25 units of "brain-power"
- Allocates y_1 units to produce TSSCI papers
- Price of brain-power is p_1
- Production Set $\mathcal{Y}_1$

- Can we induce production target y_2^0?
- With piece-rate prize p_2?

Can Prices Support Efficient Production?

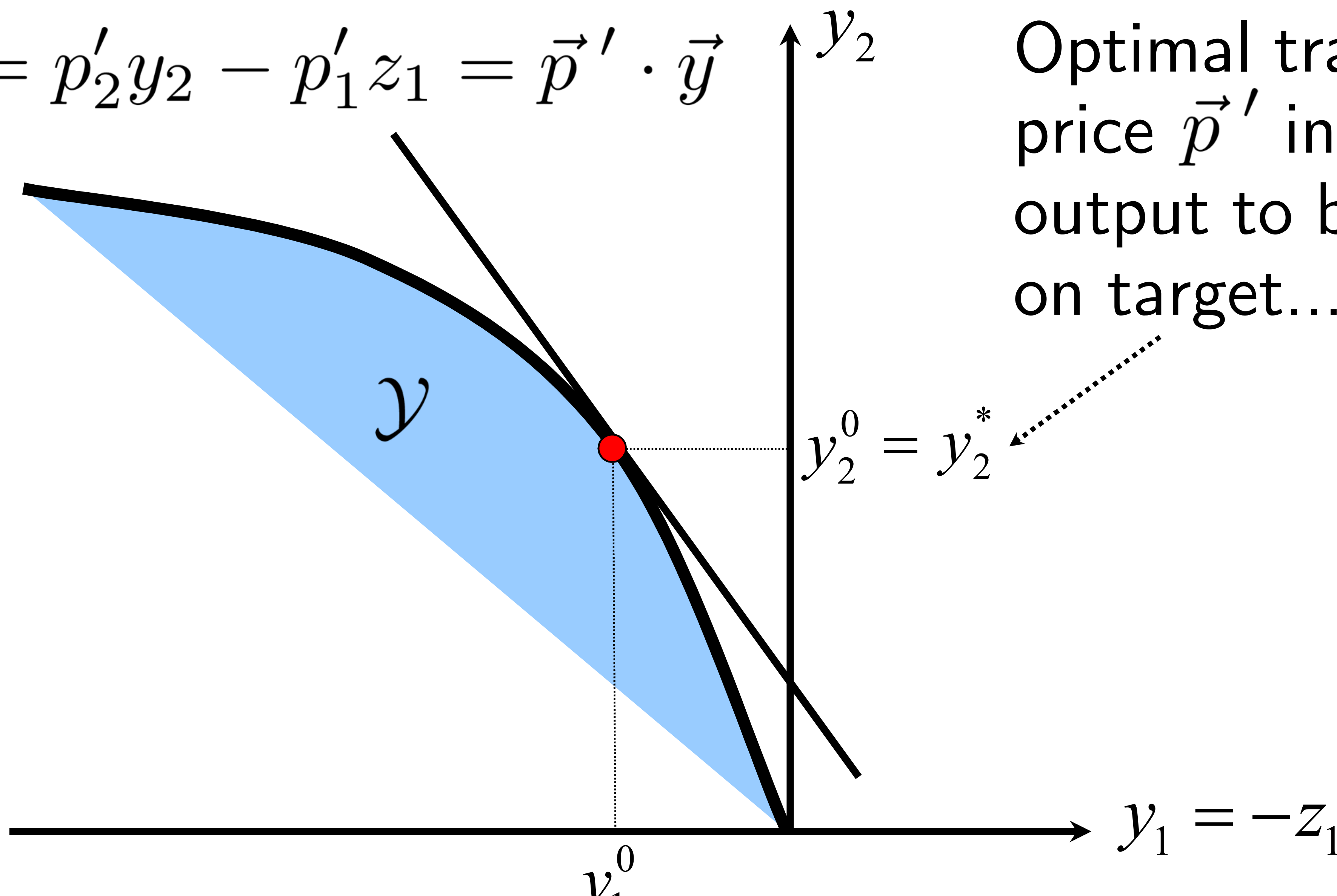
$$\Pi = p_2 y_2 - p_1 z_1 = \vec{p} \cdot \vec{y}$$



Profit maximizing production plan is efficient, but over-produces...

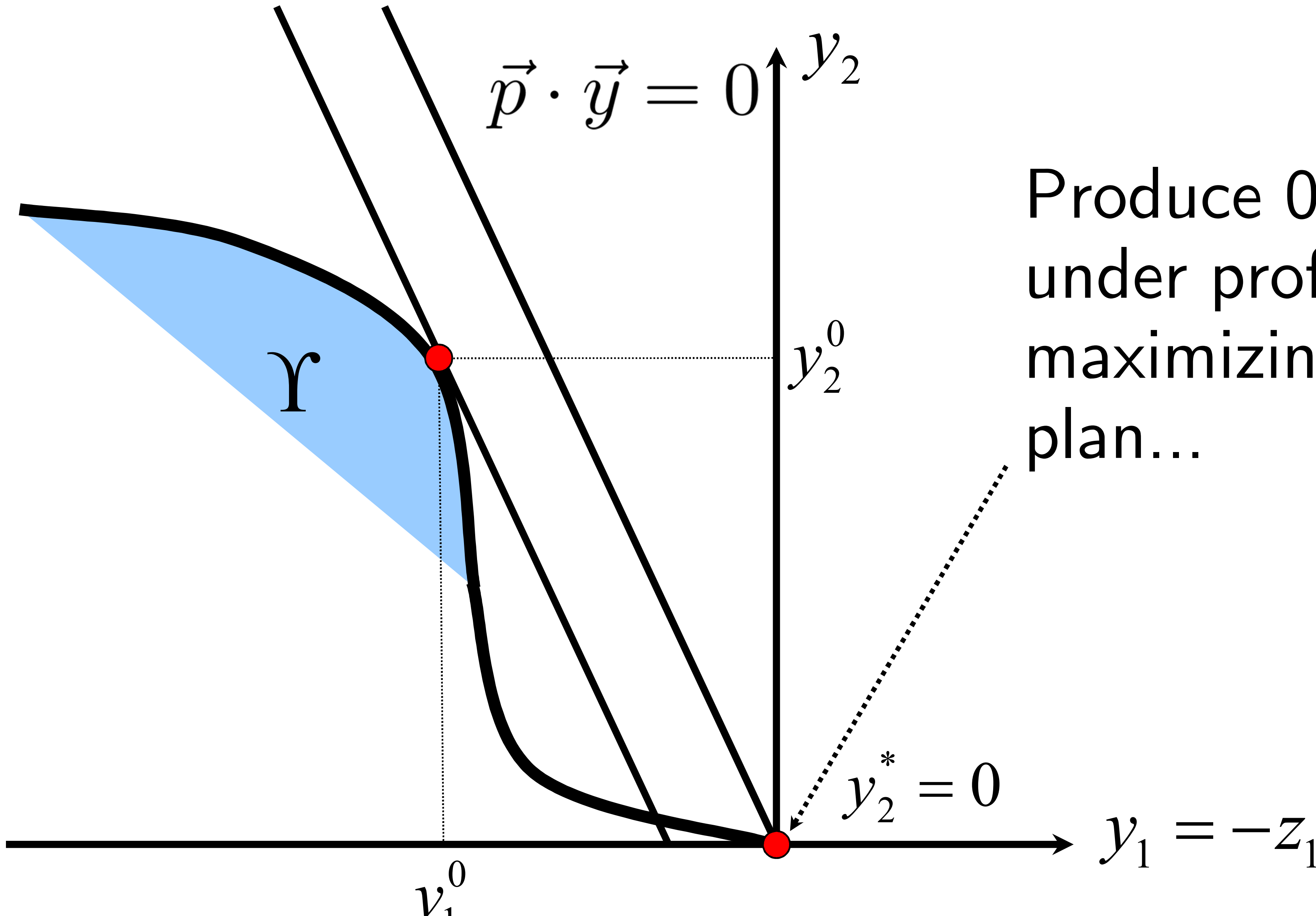
Too High? Let's Lower the Transfer Price...

$$\Pi = p_2' y_2 - p_1' z_1 = \vec{p}\,' \cdot \vec{y}$$



Optimal transfer price $\vec{p}\,'$ induces output to be right on target...

Will this Always Work?



Produce 0 under profit maximizing plan...

What Made It Fail?

- The last production set was NOT convex.
- Production Set $\mathcal{Y}_1$ is convex if for any $\vec{y}^0, \vec{y}^1$
- Its convex combination (for $0 < \lambda < 1$)

$$\vec{y}^\lambda = (1-\lambda)\vec{y}^0 + \lambda\vec{y}^1 \in \mathcal{Y}_1$$

  – (is also in the production set)
- Is it true that we can use prices to guide production decisions as long as production sets are convex?

Supporting Hyperplane Theorem

Proposition 1.1-1 (Supporting Hyperplane)

- Suppose $\mathcal{Y} \subset \mathbb{R}^n$ is non-empty and convex,
- And $\vec{y}^0$ lies on the boundary of $\mathcal{Y}$
- Then, there exists $\vec{p} \neq 0$ such that

i. For all $\vec{y} \in \mathcal{Y}$, $\vec{p} \cdot \vec{y} \leq \vec{p} \cdot \vec{y}^0$

ii. For all $\vec{y} \in \text{int}\mathcal{Y}$, $\vec{p} \cdot \vec{y} < \vec{p} \cdot \vec{y}^0$

  – Can we obtain part (ii)???

- Proof: For the general case, see Appendix C.

Supporting Hyperplane Thm (Special Case)

- Consider special case where
- Production set $\mathcal{Y}$ is the upper contour set

$$\mathcal{Y} = \left\{ \vec{y} | g(\vec{y}) \geq g(\vec{y}^0) \right\}, \ g \text{ is differentiable}$$

- Suppose the gradient vector is non-zero at $\vec{y}^0$
- The linear approximation of g at $\vec{y}^0$ is:

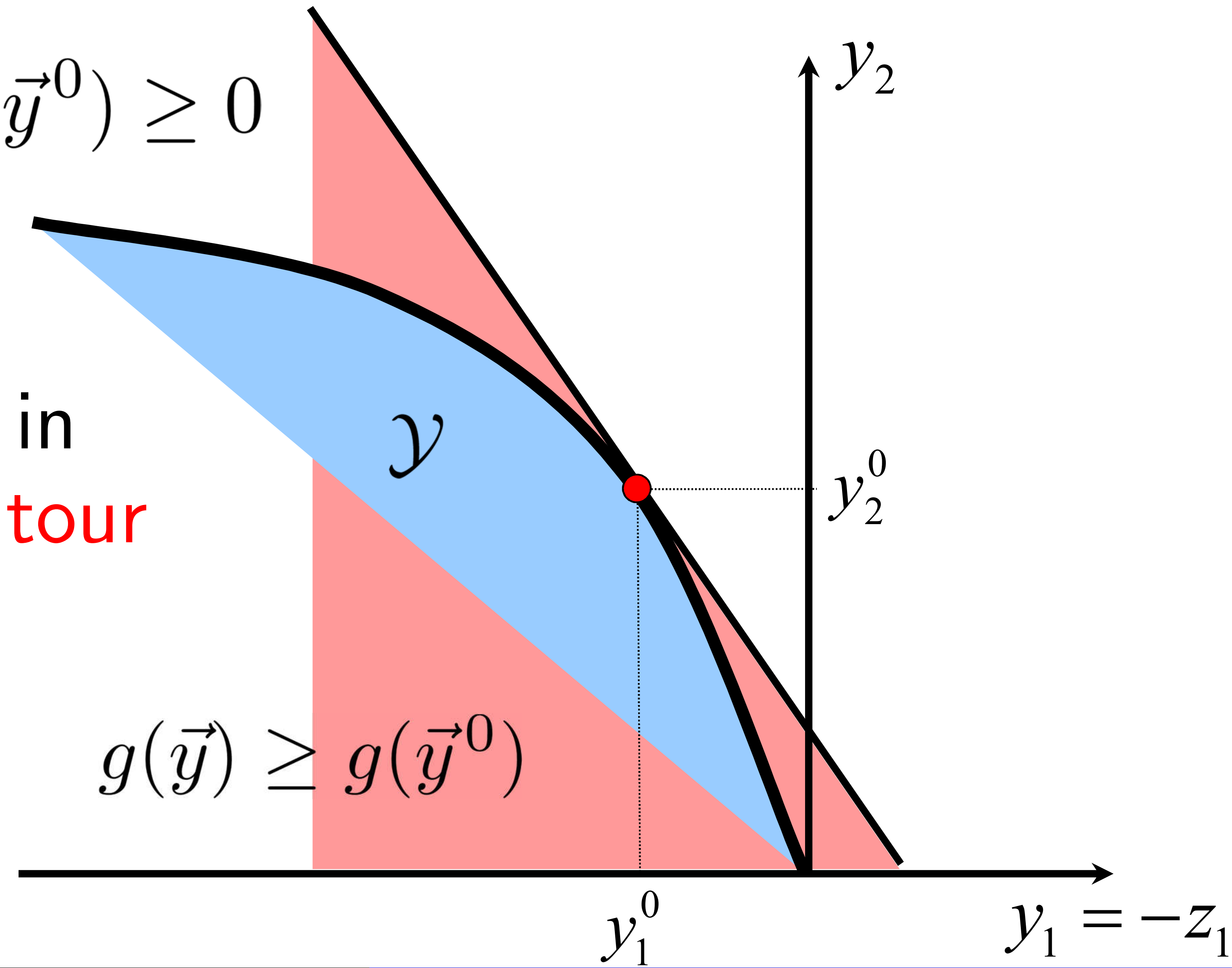
$$\overline{g}(\vec{y}) = g(\vec{y}^0) + \frac{\partial g}{\partial \vec{y}}(\vec{y}^0) \cdot (\vec{y} - \vec{y}^0)$$

- If $\mathcal{Y}$ is convex, it lies in upper contour set of $\overline{g}$

Supporting Hyperplane Thm (Special Case)

$$\frac{\partial g}{\partial \vec{y}}(\vec{y}^0) \cdot (\vec{y} - \vec{y}^0) \geq 0$$

Convex $\mathcal{Y}$ lies in the upper contour set of $\underline{\underline{\overline{g}}}$



Special Case of Supporting Hyperplane Thm

- Lemma 1.1-2
- If g is differentiable and $\mathcal{Y} = \{\vec{y} | g(\vec{y}) \geq g(\vec{y}^0)\}$ is convex, then

$$\vec{y} \in \mathcal{Y} \;\; \Rightarrow \;\; \frac{\partial g}{\partial \vec{y}}(\vec{y}^0) \cdot (\vec{y} - \vec{y}^0) \geq 0$$

- This tells us how to calculate the supporting prices (under this special case):
- For boundary point $\vec{y}^0$, choose $\vec{p} = -\frac{\partial g}{\partial \vec{y}}(\vec{y}^0)$

From Lemma to Supporting Hyperplane Thm

- If g is differentiable and $\mathcal{Y} = \{\vec{y} | g(\vec{y}) \geq g(\vec{y}^0)\}$ is convex, then set $\vec{p} = -\frac{\partial g}{\partial \vec{y}}(\vec{y}^0)$
- By lemma: $\vec{y} \in \mathcal{Y} \;\; \Rightarrow \;\; -\vec{p} \cdot (\vec{y} - \vec{y}^0) \geq 0$
  $\Rightarrow \;\; \vec{p} \cdot \vec{y} \leq \vec{p} \cdot \vec{y}^0$
- This gives us part (i) of S. H. T.
  – What about part (ii)? See Prop. 1-1.3...

Supporting Hyperplane Theorem

Proposition 1.1-1 (Supporting Hyperplane)

- Suppose $\mathcal{Y} \subset \mathbb{R}^n$ is non-empty and convex,
- And $\vec{y}^0$ lies on the boundary of $\mathcal{Y}$
- Then, there exists $\vec{p} \neq 0$ such that

i. For all $\vec{y} \in \mathcal{Y}$, $\vec{p} \cdot \vec{y} \leq \vec{p} \cdot \vec{y}^0$

- Proof: For the general case, see Appendix C.

From Lemma to Supporting Hyperplane Thm

- If g is differentiable and $\mathcal{Y} = \{\vec{y} | g(\vec{y}) \geq g(\vec{y}^0)\}$ is convex, then $\vec{y} \in \mathcal{Y}$
$$\Rightarrow \quad -\vec{p} \cdot (\vec{y} - \vec{y}^0) \geq 0$$
$$\Rightarrow \quad \vec{p} \cdot \vec{y} \leq \vec{p} \cdot \vec{y}^0$$
- Attempt part (ii) of S. H. T.
- Note: $\vec{y} \in \text{int}\mathcal{Y} \; \Rightarrow \; \exists \; \vec{y}' = \vec{y} + \vec{\epsilon} \in \mathcal{Y}, \vec{\epsilon} \gg 0$
- And $\vec{p} \cdot \vec{y}' = \vec{p} \cdot \vec{y} + \vec{p} \cdot \vec{\epsilon} \leq \vec{p} \cdot \vec{y}^0$
- If $\vec{p} \cdot \vec{\epsilon} > 0 \Rightarrow \quad \vec{p} \cdot \vec{y} < \vec{p} \cdot \vec{y}^0$
  - Why is this the case? See Prop. 1-1.3...

Supporting Hyperplane Theorem

Proposition 1.1-1 (Supporting Hyperplane)

- Suppose $\mathcal{Y} \subset \mathbb{R}^n$ is non-empty and convex,
- And $\vec{y}^0$ lies on the boundary of $\mathcal{Y}$
- Then, there exists $\vec{p} \neq 0$ such that

i. For all $\vec{y} \in \mathcal{Y}$, $\vec{p} \cdot \vec{y} \leq \vec{p} \cdot \vec{y}^0$

ii. For all $\vec{y} \in \text{int}\mathcal{Y}$, $\vec{p} \cdot \vec{y} < \vec{p} \cdot \vec{y}^0$

- Proof: For the general case, see Appendix C.

Proof of Lemma 1.1-2

- If g is differentiable and $\mathcal{Y} = \{\vec{y} | g(\vec{y}) \geq g(\vec{y}^0)\}$ is convex, then

$$\vec{y} \in \mathcal{Y} \quad \Rightarrow \quad \frac{\partial g}{\partial \vec{y}}(\vec{y}^0) \cdot (\vec{y} - \vec{y}^0) \geq 0$$

  – Proof:

- For $\vec{y} \in \mathcal{Y}$, consider $\vec{y}^\lambda = (1-\lambda)\vec{y}^0 + \lambda \vec{y} \in \mathcal{Y}$
- So, $g(\vec{y}^\lambda) - g(\vec{y}^0) \geq 0$
- Define $h(\lambda) \equiv g(\vec{y}^\lambda) = g(\vec{y}^0 + \lambda(\vec{y} - \vec{y}^0))$

Proof of Lemma 1.1-2

$$\vec{y} \in \mathcal{Y} \;\Rightarrow\; \vec{y}^{\lambda} = (1-\lambda)\vec{y}^0 + \lambda\vec{y} \in \mathcal{Y}$$

$$h(\lambda) \equiv g(\vec{y}^{\lambda}) = g(\vec{y}^0 + \lambda(\vec{y} - \vec{y}^0))$$

$$\frac{h(\lambda) - h(0)}{\lambda} = \frac{g((\vec{y}^0 + \lambda(\vec{y} - \vec{y}^0)) - g(\vec{y}^0)}{\lambda} \geq 0 \to \frac{dh}{d\lambda}$$

- By Lemma. Therefore, by chain rule:

$$\left.\frac{dh}{d\lambda}(\lambda)\right|_{\lambda=0} = \left.\frac{\partial g}{\partial \vec{y}}(\vec{y}^0 + \lambda(\vec{y} - \vec{y}^0)) \cdot (\vec{y} - \vec{y}^0)\right|_{\lambda=0}$$

$$= \frac{\partial g}{\partial \vec{y}}(\vec{y}^0) \cdot (\vec{y} - \vec{y}^0) \geq 0. \quad \square$$

Example

- A professor has $z = 25$ units of "brain-power"
- Allocates z_2 units to produce TSSCI papers
- Produce $y_2 = 2\sqrt{z_2}$ number of TSSCI papers
- Allocates z_3 units to produce SSCI papers
- Produce $y_3 = \sqrt{z_3}$ number of SSCI papers
- Set of feasible plans is ($y_1 = -z$)

$$\mathcal{Y} = \left\{ \vec{y} \middle| g(\vec{y}) = -y_1 - \frac{1}{4} y_2^2 - y_3^2 \geq 0 \right\}$$

Example

- Professor W is working at full capacity

- Professor W's output is $\vec{y}^0 = (-25, 8, 3)$
  - 8 TSSCI papers and 3 SSCI papers!
- What reward scheme can support this?

$$\vec{p} = -\frac{\partial g}{\partial \vec{y}}(\vec{y}^0) = (1, \frac{1}{2}y_2^0, 2y_3^0) = (1, \underline{\underline{4, 6}})$$

- To instead induce $(y_2^1, y_3^1) = (2, 2\sqrt{6}) \approx (2, 5)$
  - Approx. 2 TSSCI papers and 5 SSCI papers

$$\vec{p} = (1, \frac{1}{2}y_2^1, 2y_3^1) = (1, 1, 4\sqrt{6}) \approx (1, \underline{\underline{1, 10}})$$

Positive Prices (Free Disposal)

- Supporting Hyperplane theorem has economic meaning if prices are positive
  - Need another assumption
- Free Disposal
- For any feasible production plan $\vec{y} \in \mathcal{Y}$ and any
- $\vec{\delta} > 0$, the production plan $\vec{y} - \vec{\delta}$ is also feasible

Supporting Prices

- With free disposal, we can prove:

Proposition 1.1-3 (Supporting Prices)

- If $\vec{y}^0$ is a boundary point of a convex set $\mathcal{Y}$
- And the free disposal assumption holds,
- Then, there exists a price vector $\vec{p} > \vec{0}$ such
- that $\vec{p} \cdot \vec{y} \leq \vec{p} \cdot \vec{y}^0$ for all $\vec{y} \in \mathcal{Y}$
- Moreover, if $\vec{0} \in \mathcal{Y}$, then $\vec{p} \cdot \vec{y}^0 \geq 0$
- Finally, for all $\vec{y} \in \text{int}\mathcal{Y}$, $\vec{p} \cdot \vec{y} < \vec{p} \cdot \vec{y}^0$ - part (ii)

Supporting Prices

- Proof: Supporting Hyperplane Theorem says:
- There is some $\vec{p} \neq \vec{0}$ such that, for all $\vec{y} \in \mathcal{Y}$,
- $\vec{p} \cdot (\vec{y}^0 - \vec{y}) \geq 0$. Now need to show $p_i \geq 0$
- By free disposal, $\vec{y}' = \vec{y}^0 - \vec{\delta} \in \mathcal{Y}, \forall \vec{\delta} > \vec{0}$
- Set $\vec{\delta} = (1, 0, \cdots, 0)$, $\vec{p} \cdot (\vec{y}^0 - \vec{y}') = p_1 \geq 0$
- Set $\vec{\delta} = (0, 1, 0, \cdots)$, $\vec{p} \cdot (\vec{y}^0 - \vec{y}') = p_2 \geq 0$
- ...
- Set $\vec{\delta} = (0, \cdots, 0, 1)$, $\vec{p} \cdot (\vec{y}^0 - \vec{y}') = p_n \geq 0$

Supporting Prices

- Since $\vec{p} \cdot \vec{y} \leq \vec{p} \cdot \vec{y}^0$ for all $\vec{y} \in \mathcal{Y}$, if $\vec{0} \in \mathcal{Y}$
- Set $\vec{y} = \vec{0}$ and we have $\vec{p} \cdot \vec{y}^0 \geq 0$

- Claim: For all $\vec{y} \in \mathrm{int}\mathcal{Y}$, $\vec{p} \cdot \vec{y} < \vec{p} \cdot \vec{y}^0$- part (ii)
- For $\vec{y} \in \mathrm{int}\mathcal{Y} \quad \Rightarrow \quad \exists \ \vec{y}' = \vec{y} + \vec{\epsilon} \in \mathcal{Y}, \vec{\epsilon} \gg 0$
- And $\quad \vec{p} \cdot \vec{y}' = \vec{p} \cdot \vec{y} + \vec{p} \cdot \vec{\epsilon} \leq \vec{p} \cdot \vec{y}^0$
- Since $\vec{p} > 0$, we have

$$\vec{p} \cdot \vec{\epsilon} > 0 \Rightarrow \quad \vec{p} \cdot \vec{y} < \vec{p} \cdot \vec{y}^0 \ \square$$

Back to Publication Rewards

- Should NTU really pay NT$300,000 per article published in Science or Nature?
  - Is the production set for Science/Nature convex?
- What would be a better incentive scheme to encourage publications in Science/Nature?
  - Efficient Wages (High Fixed Wages)?
  - Tenure?
  - Endowed Chair Professorships?

Back to Publication Rewards

- What are some tasks do you expect piece-rate incentives to work?
  - Sales
  - Real estate agents
- What about a fixed payment?
  - Secretaries and Office Staff
  - Store Clerk
- What about other incentives schemes?
  - That's for you to answer (in contract theory)!

Summary of 1.1

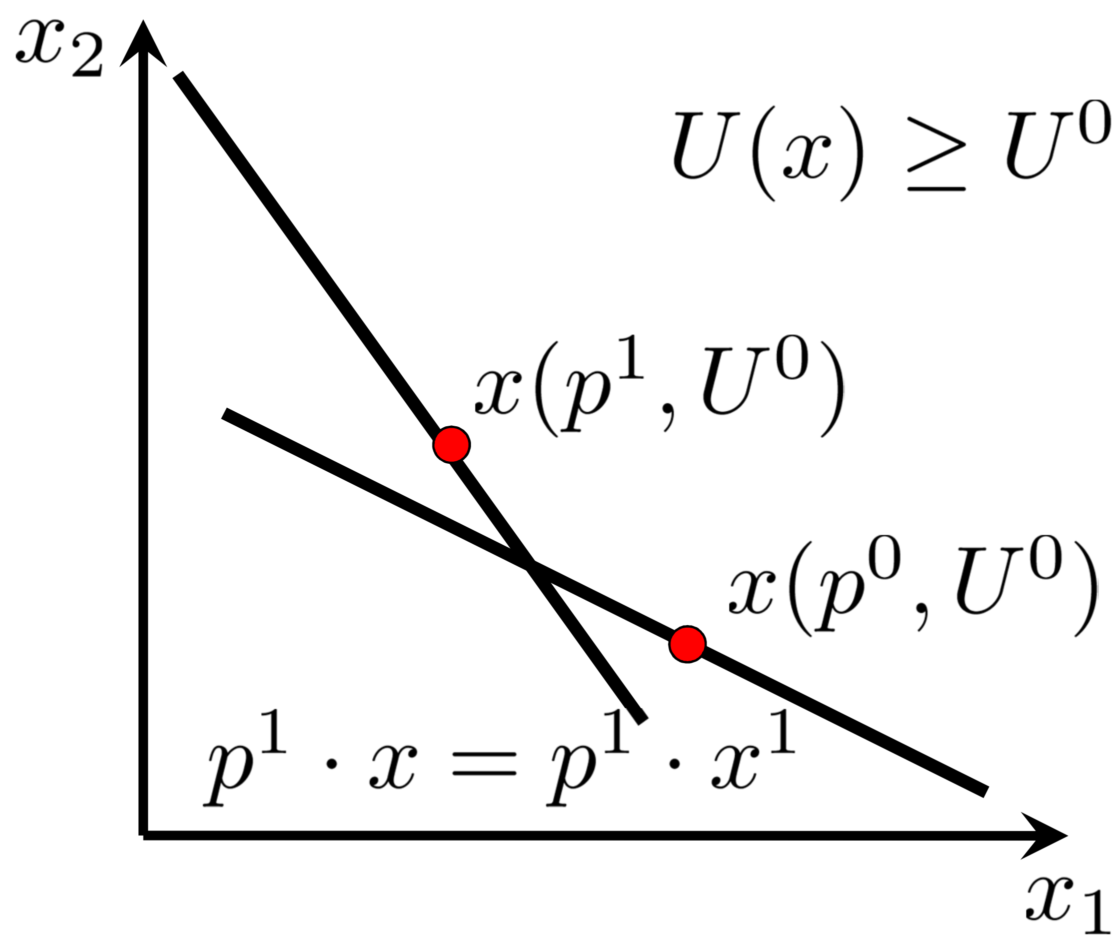
- Input = Negative Output
- Vector space of $\vec{y}$
- Convexity (quasi-concavity) is the key for supporting prices (=linearization)
- What is a good incentive scheme to induce professor to publish in Science and Nature?
- Consumer = Producer
- Homework: Exercise 1.1-4 (Optional: 1.1-6)

Another Example: Linear Model

- What if firm has n plants producing the same product q using m inputs $z = (z_1, \cdots, z_m)$?
- Need to consider activity level x_j for plant j
  - Produce output $a_{0j}x_j$ with input $a_{ij}x_j, i = 1, ..., m$
- Total output $\sum_{j=1}^{n} a_{0j}x_j$; Total input $\sum_{j=1}^{n} a_{ij}x_j, \forall i$
- Linear Production Set (convex, free disposal)

$$\mathcal{Y} = \left\{(-z, q) \middle| x \geq 0, q \leq a_0 \cdot x, Ax \leq z\right\}$$

Another Example: Linear Model



Production Set and Profits

- Production vector

$$y = (y_1, \cdots, y_{m+n}) = (-z_1, \cdots, -z_m, q_1, \cdots, q_n)$$

- Production Set $\mathcal{Y} \subset \mathbf{R}^{m+n}$

  =Set of Feasible Production Plan

- Price vector $p = (p_1, \cdots, p_{m+n})$

- Profit $\Pi = \underbrace{\sum_{i=m+1}^{m+n} p_i y_i}_{\text{total revenue}} - \underbrace{\sum_{i=1}^{m} p_i z_i}_{\text{total cost}} = p \cdot y$

Quasi-Concavity

- f is quasi-concave if the upper contour set of f set are convex. Equivalently, for any y^0, y^1 and convex combination

  $$y^\lambda = \lambda \cdot y^0 + (1-\lambda) \cdot y^1,$$

  $$f(y^\lambda) \geq \min\left\{f(y^0), f(y^1)\right\}.$$

- Why is this useful?
  - Because we have...

Separating Hyperplane Theorem

- Proposition 1.1-2:
  Suppose S and T are convex sets with a common boundary point $s^0 = t^0$ and no common interior points.
  Then there is some p such that, for all $s \in S$ and $t \in T$, $p \cdot s \leq p \cdot t$.
  (Inequality strict if either s or t is an interior.)

Separating Hyperplane Theorem

- Proof of Proposition 1.1-2:

  Define $\Upsilon = S - T$, then $s^0 - t^0 = 0 \in \Upsilon$

  If Υ is convex (verify this!!!), then...

  Supporting Hyperplane Theorem says:

  there is some $p \neq 0$ such that, for all $y \in \Upsilon$,

  $p \cdot y \leq p \cdot (s^0 - t^0) = 0.$

  Since $y = s - t$ for some $s \in S, t \in T$,

  $p \cdot s \leq p \cdot t$ for all $s \in S, t \in T$.