

Theory of Risky Choice

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(Lecture 10, Micro Theory I)

Theory of Risky Choice

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- We analyzed preferences, utility and choices
- Apply them to study risk and uncertainty
 - Preference for probabilities
 - Expected Utility
- Discuss Experimental Anomalies
 1. Allais paradox and Ellsberg paradox
 2. Bayes' Rule paradoxes: Soft vs. Hard prob.,
Game show paradox (Monty Hall problem)
 3. Rabin paradox

States and Probabilities

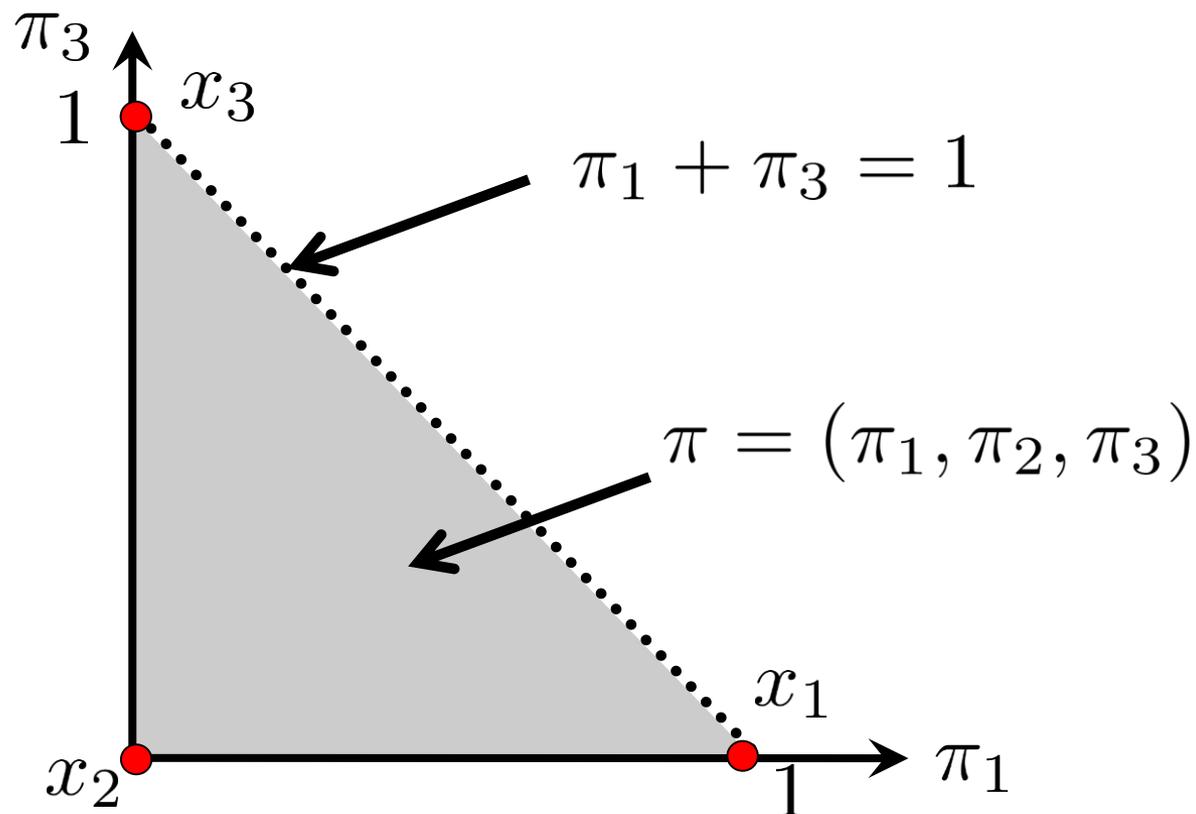
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- Consequence x_s happens in state $s = 1, \dots, S$
- Assign (subjective) **probability** π_s to state s
- A **prospect** $(\pi; x) = ((\pi_1, \dots, \pi_S); (x_1, \dots, x_S))$
 - People have preferences for these prospects
- Under the **Axioms of Consumer Choice**, exists continuous $U(\pi; x)$ representing these pref.
- If we fix consequences; focus on probabilities

$$U(\pi; x) = U(\pi) = U(\pi_1, \pi_2, \pi_3)$$

States and Probabilities

- Assume $x_1 \succ x_2 \succ x_3$, can show all possible probabilities on 2D: $\pi = (\pi_1, \pi_2, \pi_3)$

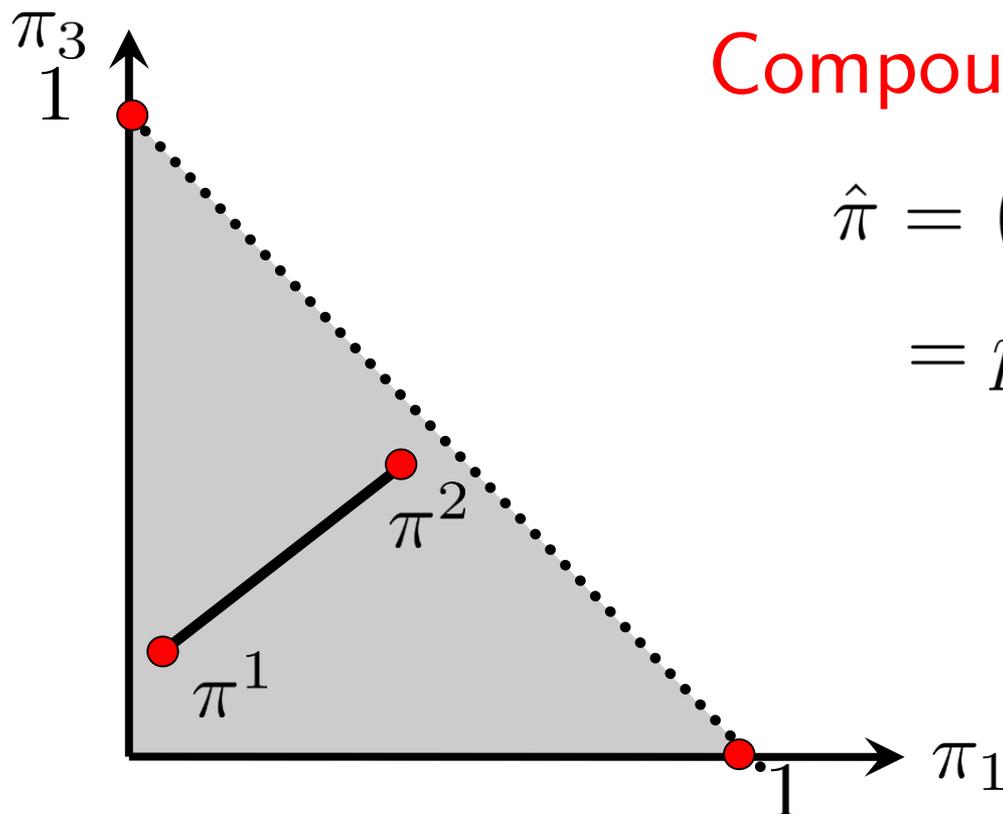


Compound Prospect (Compound Lottery)

- If I offer you $\pi^1 = (\pi_1^1, \pi_2^1, \pi_3^1)$ with prob. p_1 , and $\pi^2 = (\pi_1^2, \pi_2^2, \pi_3^2)$ with probability $p_2 = 1 - p_1$

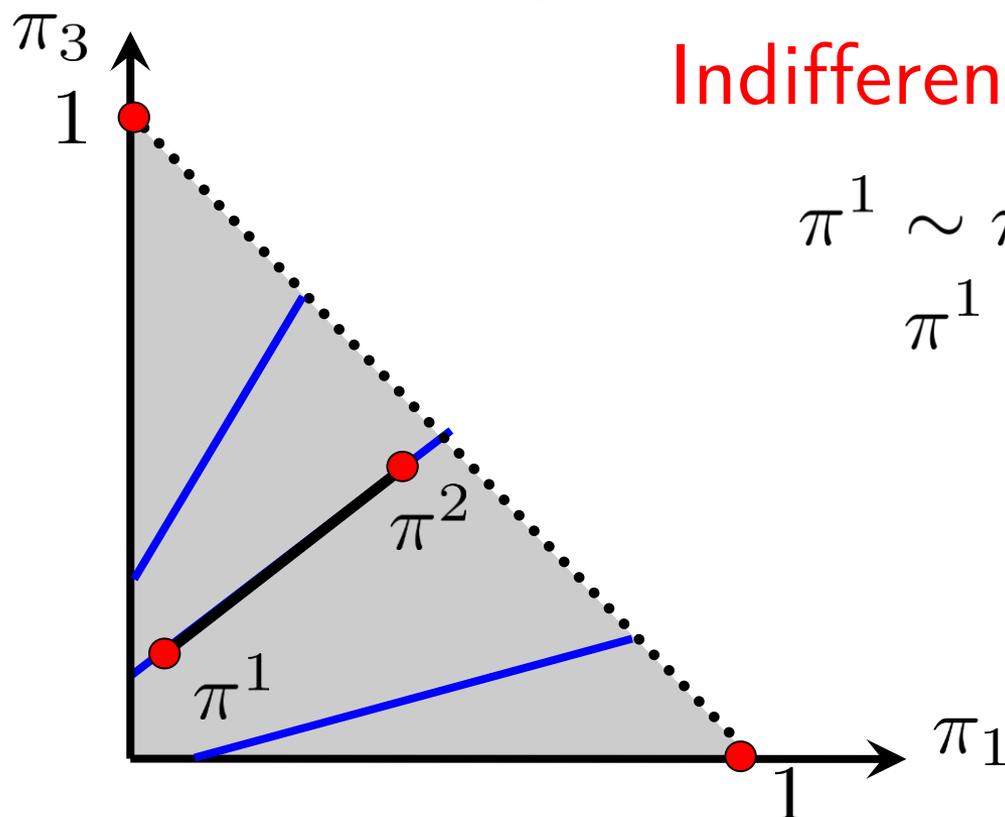
Compound Prospect:

$$\begin{aligned}\hat{\pi} &= (p_1, p_2 : \pi^1, \pi^2) \\ &= p_1 \pi^1 + (1 - p_1) \pi^2\end{aligned}$$



Are Indifference Curves Linear?

- If you are indifferent between π^1 and π^2
- How would you feel about randomizing them?



Indifferent !!

$$\pi^1 \sim \pi^2 \Rightarrow$$

$$\pi^1 \sim (p_1, 1 - p_1 : \pi^1, \pi^2)$$

Indifference Curves
Are Linear!

When Are Indifference Curves Parallel?

- Consider a third prospect r

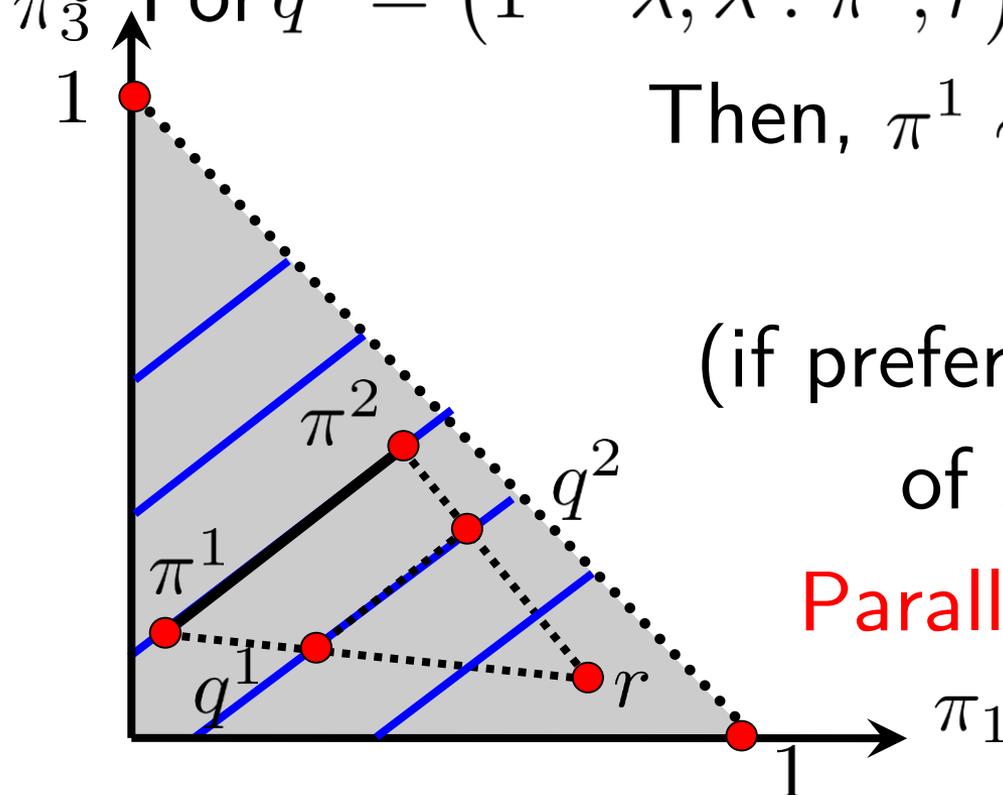
• For $q^1 = (1 - \lambda, \lambda : \pi^1, r)$, $q^2 = (1 - \lambda, \lambda : \pi^2, r)$

Then, $\pi^1 \sim \pi^2 \Rightarrow q^1 \sim q^2$

$\pi^1 \succsim \pi^2 \Rightarrow q^1 \succsim q^2$

(if preferences are independent of irrelevant alternatives)

Parallel Indifference Curves!



Independence Axiom(s)

- (IA) If $\pi^1 \succsim \pi^2$, then for any prospect r and probabilities $p_1, p_2 > 0, p_1 + p_2 = 1$

$$q^1 = (p_1, p_2 : \pi^1, r) \succsim (p_1, p_2 : \pi^2, r) = q^2$$

- (IA') If $\pi^m \succsim \hat{\pi}^m, m = 1, \dots, M$, then for any probability vector $p = (p_1, \dots, p_M)$

$$(p_1, \dots, p_M : \pi^1, \dots, \pi^M)$$

$$\succsim (p_1, \dots, p_M : \hat{\pi}^1, \dots, \hat{\pi}^M)$$

Expected Utility

- In general, for any prospect $p = (p_1, \dots, p_S)$
- The consumer is indifferent between p and playing the extreme lottery

$$\left(\sum_{s=1}^S p_s v(x_s), 0, \dots, 0, 1 - \sum_{s=1}^S p_s v(x_s) \right)$$

- Hence, we can represent her preferences with the above expected win probabilities
 - Expected Utility!!

Expected Utility Rule

- Assume (IA'), then
- Preferences over prospects

$$(p; x) = (p_1, \dots, p_S; x_1, \dots, x_S)$$

- Can be represented by the Von Neumann-Morgenstern utility function

$$u(p, x) = \sum_{s=1}^S p_s v(x_s)$$

- Proof:

Expected Utility Rule

- Proof: S consequences, best is x^* , worse is x_*
- Can assign probability for extreme lotteries:

$$e^s \equiv (v(x_s), 1 - v(x_s) : x^*, x_*) \sim x_s$$

- (IA') implies $(p; x) \sim (p_1, \dots, p_S : e^1, \dots, e^S)$
 $\sim (u(p, x), 1 - u(p, x) : x^*, x_*)$

$$\text{where } u(p, x) = \sum_{s=1}^S p_s v(x_s)$$

- (by reducing compound prospects)

Experimental Anomalies

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- Allais Paradox
- Ellsberg Paradox
- Bayes' Rule Paradoxes
 - Soft vs. Hard Probabilities
 - Game Show Paradox
- Rabin Paradox

Allais Paradox

- Consider four prospects:
 - A. \$1 million for sure
 - B. 90% chance \$5 million (& 10% chance zero)
 - C. 10% chance \$1 million (& 90% chance zero)
 - D. 9% chance \$5 million (& 91% chance zero)
- Among A and B, you choose...
- Among C and D, you choose...
- Is this consistent with Expected Utility???

Allais Paradox * 1,000

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- A. \$1 billion for sure
 - B. 90% chance \$5 billion (& 10% chance zero)
 - C. 10% chance \$1 billion (& 90% chance zero)
 - D. 9% chance \$5 billion (& 91% chance zero)
- Among A and B, you choose...
 - Among C and D, you choose...
 - Are your answers (still) consistent with Expected Utility? Why or why not?

Ellsberg Paradox

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- One urn: 30 Black balls, and 60 “other balls”
 - Other balls could be either Red or Green
- 1. One ball is drawn. You win \$100 if the ball is (a) Black or (b) Green. You pick...?
- 2. Now you win \$50 if the ball is “either Red or another color you choose.” Would you choose (a) Black or (b) Green?
- What did you choose? Did it violate EU?

Ellsberg Paradox

1. One ball is drawn. You win \$100 if the ball is (a) Black or (b) Green.
 - Picking Black = Believe < 30 Green balls
2. Now you win if “either Red or another color.” You choose (a) Black or (b) Green?
 - Picking Green = Believe > 30 Green balls
 - Since it is the same urn, this is inconsistent!
 - Can this be due to hedging (risk aversion)?
 - Maybe, but can fix this by paying only 1 round...

Bayes' Rule Paradoxes: Soft vs. Hard Prob.

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- Two urns, each contain 100 balls.
 1. Urn 1 has 60 **Yellow** balls.
 2. Urn 2 has 75 or 25 **Yellow** balls with equal chance.
- You win a prize if you draw a **Yellow** ball.
- A ball is drawn from Urn 2 and it is **Yellow**.
- Which Urn should you choose?
- Did you do Bayesian updating correctly?

Bayes' Rule Paradoxes: Soft vs. Hard Prob.

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- Prior to draw, $\Pr(\text{draw a } Y) = 0.5$. After:
- $\Pr(Y \mid 75 - Y) = 0.75$, $\Pr(Y \mid 25 - Y) = 0.25$
- $\Pr(75 - Y \mid Y) = 0.5 \times 0.75 / 0.5 = 0.75$
- $\Pr(25 - Y \mid Y) = 1 - 0.75 = 0.25$
- $\Pr(\text{draw another } Y \mid Y) =$
 $\Pr(75 - Y \mid Y) \times \Pr(Y \mid 75 - Y) +$
 $\Pr(25 - Y \mid Y) \times \Pr(Y \mid 25 - Y)$
 $= 0.75 \times 0.75 + 0.25 \times 0.25 = 0.625 > 0.6$
- So you should pick Urn 2!! (Did you do that?)

Bayes' Rule Paradoxes: Game Show Paradox

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One door hides the prize (a car).
Remaining two doors hides a goat (non-prize).

Suppose you choose door number 1...

Game Show Paradox (Monty Hall Problem)

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Door 3 is opened for you...

Obviously the car is not behind door 3...

Would you want to switch to door 2?

Depends on how door is opened...

- Rule to open one door:
The Host must open one “other” door without the prize. If he has a choice between more than one door, he will **randomly** open one of the possible (goat) doors.
- The Game Show Paradox is also known as the **Monty Hall Problem**, named after the name of the TV show host “Monty Hall”

Game Show Paradox Plus: Modified Monty Hall ²³



One door hides the prize (a car).
Remaining two doors hides a goat (non-prize).
Door 3 is transparent (and you see the goat)
Suppose you choose door number 1...

Game Show Paradox Plus: Modified Monty Hall ²⁴



Door 3 is opened for you...

Obviously the car is not behind door 3 (and you knew that already)...

Would you want to switch to door 2?

If You Picked the Right Door (50%)



Host randomizes between door 2 and 3 (50-50)

If host opens door 2... (Prob.=50%*50%)

You should definitely not switch!

If You Picked the Right Door (50%)



Host randomizes between door 2 and 3
If host opens door 3... (Prob=50%*50%)

You should still not switch (but you don't know)

If You Picked the Wrong Door (50%)

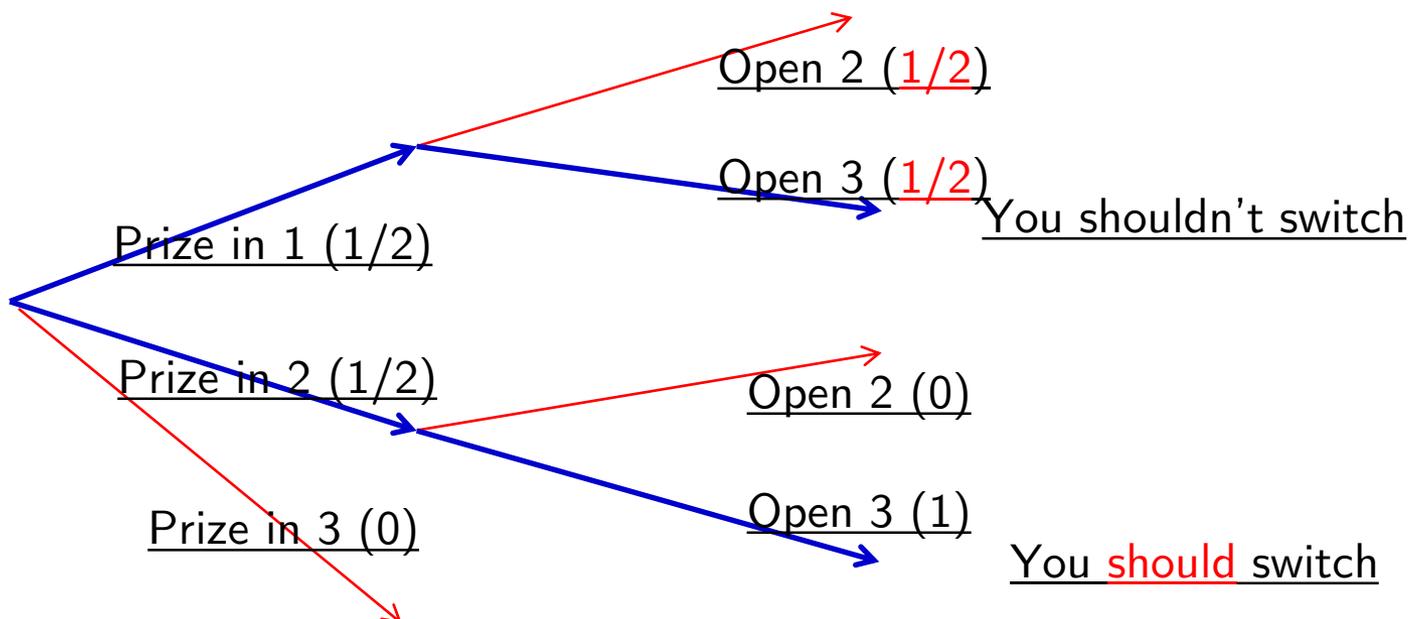
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Host cannot open door 2 (contains car)
See host opening door 3... (Prob.=50%*100%)
You should switch (but you don't know)

Bayesian Solution (Monty Hall Plus)

Door #3 is transparent...



$$P(\text{Winning if you choose to switch}) = \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2}}$$

Rabin Paradox: Which Cells Will You Accept?²⁹

Payoff if Green Ball	Number of Green Balls (out of 100)					Payoff if Red Ball
100	52	55	60	66	70	-100
1000	13	20	33	46	57	-100
5000	7	18	33	46	57	-100
25000	7	18	33	46	57	-100

Rabin Paradox

- Suppose your risk preference follows EU.
- For initial Wealth is ω
- Consider the prospect $(p, 1 - p : \omega + g, \omega - g)$
- If you reject this lottery, this implies:

$$v(\omega) \geq (1 - p) \cdot v(\omega - g) + p \cdot v(\omega + g)$$

- Or,

$$[v(\omega + g) - v(\omega)] \leq \frac{1 - p}{p} \cdot [v(\omega) - v(\omega - g)] \quad \dots\dots(1)$$

Rabin Paradox

- Now consider initial wealth $\omega' = \omega + g$
- If you reject the prospect $(p, 1 - p : \omega' + g, \omega' - g)$
- Then: $v(\omega') \geq (1 - p) \cdot v(\omega' - g) + p \cdot v(\omega' + g)$
- Or,

$$\begin{aligned}
 [v(\omega + 2g) - v(\omega + g)] &= [v(\omega' + g) - v(\omega')] \\
 &\leq \frac{1 - p}{p} \cdot [v(\omega') - v(\omega' - g)] \\
 &= \frac{1 - p}{p} \cdot [v(\omega + g) - v(\omega)]
 \end{aligned}$$

Rabin Paradox

- Combining the two inequalities:

$$\begin{aligned}
 & [v(\omega + 2g) - v(\omega + g)] \\
 & \leq \frac{1-p}{p} \cdot [v(\omega + g) - v(\omega)] \\
 & \leq \left(\frac{1-p}{p}\right)^2 \cdot [v(\omega) - v(\omega - g)] \dots (2)
 \end{aligned}$$

- Only required one to reject the fair gamble at both wealth levels ω and $\omega' = \omega + g$

Rabin Paradox

- Suppose you reject the fair gamble at all wealth levels between ω and $\omega^{(n)} = \omega + ng$
- Then,

$$\begin{aligned}
 & [v(\omega + ng) - v(\omega + (n - 1)g)] \\
 & \leq \frac{1 - p}{p} \cdot [v(\omega + (n - 1)g) - v(\omega + (n - 2)g)] \\
 & \leq \dots \leq \left(\frac{1 - p}{p}\right)^n \cdot [v(\omega) - v(\omega - g)] \dots\dots(n)
 \end{aligned}$$

Rabin Paradox

- Summing (1) through (n):

$$\begin{aligned}
 & [\cancel{v(\omega + g)} - v(\omega)] + [\cancel{v(\omega + 2g)} - \cancel{v(\omega + g)}] \\
 & \quad + \dots + [v(\omega + ng) - \cancel{v(\omega + (n-1)g)}] \\
 & = [v(\omega + ng) - v(\omega)] \\
 & \leq \left[\frac{1-p}{p} + \dots + \left(\frac{1-p}{p} \right)^n \right] \cdot [v(\omega) - v(\omega - g)] = \\
 & \frac{1-p}{p} \cdot [v(\omega) - v(\omega - g)] + \left(\frac{1-p}{p} \right)^2 \cdot [v(\omega) - v(\omega - g)] \\
 & \quad + \dots + \left(\frac{1-p}{p} \right)^n \cdot [v(\omega) - v(\omega - g)]
 \end{aligned}$$

Rabin Paradox

Let $s(n, p) = 1 + \frac{1-p}{p} + \dots + \left(\frac{1-p}{p}\right)^n$

$$[v(\omega + ng) - \underline{v(\omega)}]$$

$$\leq [\underline{v(\omega)} - v(\omega - g)] \cdot \left[\frac{1-p}{p} + \dots + \left(\frac{1-p}{p}\right)^n \right]$$

$$\Rightarrow [v(\omega + ng) + (s(n, p) - 1)v(\omega - g)] \leq s(n, p) \cdot \underline{v(\omega)}$$

- Or, $v(\omega) \geq \frac{1}{s(n, p)}v(\omega + ng) + \left(1 - \frac{1}{s(n, p)}\right)v(\omega - g)$
- This means rejecting

$$\left(\frac{1}{s(n, p)}, 1 - \frac{1}{s(n, p)} : \omega + ng, \omega - g\right)$$

Rabin Paradox

- We have shown that:
- If you reject prospect $(p, 1 - p : \omega + g, \omega - g)$
- For all wealth levels $[\omega, \omega + ng]$

→ You would also reject the more favorable prospect $(\frac{1}{s(n,p)}, 1 - \frac{1}{s(n,p)} : \omega + ng, \omega - g)$

$$s(n, p) = 1 + \frac{1-p}{p} + \dots + \left(\frac{1-p}{p}\right)^n \rightarrow \frac{1}{1 - \frac{1-p}{p}}$$

- This is true for any large $n!$ $= \frac{p}{2p - 1}$

Rabin Paradox: Which Cells Will You Accept?³⁷

Payoff if Green Ball	Number of Green Balls (out of 100)					Payoff if Red Ball
100	52	55	60	66	70	-100
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5000	7	18	33	46	57	-100
25000	7	18	33	46	57	-100

Continuous Probability Distribution

- Let state $s \in \mathcal{S} = [\alpha, \beta]$
- CDF is $F(t) = \Pr\{s \leq t\}$ $F(\alpha) = 0, F(\beta) = 1$
- Probability of being in $C = [s, s']$ is:
- **Probability Measure** $\pi(C) = F(s') - F(s)$
- Can generalize and assign probability measures over closed convex hypercube $C \in \mathbb{R}^n$

Support of the Continuous Distribution

- x is in the **support** of the distribution if for every neighborhood $N(x, \delta)$ of x , $\pi(N(x, \delta)) > 0$
- Example: $\mathcal{S} = [0, 3]$

$$F(\theta) = \begin{cases} \frac{1}{2}\theta, & 0 \leq \theta \leq 1 \\ \frac{1}{2}, & 1 < \theta < 2 \\ \frac{1}{2}(\theta - 1), & 2 \leq \theta \leq 3 \end{cases}$$

- What is the support?

$$[0, 1] \cup [2, 3]$$

Summary of 7.1

- Preferences over prospects
- Indifference Curves
 - Linear: “Reduction of Compound Lotteries”
 - Parallel: “Independent of Irrelevant Alternatives”
- Expected Utility
- Anomalies: Allais paradox, Ellsberg paradox, Bayes’ Rule paradoxes (Soft vs. Hard prob. and Monty Hall Problem) and Rabin paradox
- Continuous State Space
- Homework: Exercise 7.1-4 (Optional: 7.1-3)

In-Class Homework: Exercise 7.1-1 $IA \leftrightarrow IA'$ 41

- a) For $M = 2$, show that IA implies IA'
- (IA) If $\pi^1 \succsim \pi^2$, then for any prospect r and probabilities $p_1, p_2 > 0, p_1 + p_2 = 1$

$$q^1 = (p_1, p_2 : \pi^1, r) \succsim (p_1, p_2 : \pi^2, r) = q^2$$
 - (IA') If $\pi^m \succsim \hat{\pi}^m, m = 1, \dots, M$, then for any probability vector $p = (p_1, \dots, p_M)$

$$(p : \pi^1, \dots, \pi^M) \succsim (p : \hat{\pi}^1, \dots, \hat{\pi}^M)$$
- b) Show that if the proposition holds for $M = k-1$, then it must also hold for $M = k$.

In-Class Homework: Exercise 7.1-2 Allais

A. \$1 million for sure – (0, 1, 0)

B. 90% chance \$5 million – (0.90, 0, 0.10)

C. 10% chance \$1 million – (0, 0.10, 0.90)

D. 9% chance \$5 million – (0.09, 0, 0.91)

1. Draw tree diagrams showing that C and D can be represented as compound gambles between A and B, respectively, and $(0,0,1)$, where the probability of $(0,0,1)$ is the same.
2. Show that the ranking of A and B should be the same as the ranking of C and D.