Shadow Prices

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(Lecture 2, Micro Theory I)

A Peak-Load Pricing Problem

- Consider the problem faced by Chunghwa Telecom (CHT):
- By building base stations, CHT can provide cell phone service to a certain region
  - An establish network can provide service both in the day and during the night
  - Marginal cost is low (zero?!); setup cost is huge
- Marketing research reveal unbalanced demand
  - Day – peak; Night – off-peak (or vice versa?)

A Peak-Load Pricing Problem

- If you are the CEO of CHT, how would you price day and night usage of your service?
  - The same or different?
- Economic intuition should tell you to set off-peak prices lower than peak prices
  - But how low?
- All new 4G services (LTE) are facing a similar problem now…

More on Peak-Load Pricing

- Other similar problems include:
  - How should Taipower price electricity in the summer and winter?
  - How should a theme park set its ticket prices for weekday and weekends?
- Even if demand estimations are available, you will still need to do some math to find optimal prices...
  - Either to maximize profit or social welfare

A Peak-Load Pricing Problem

- Back to CHT:
- Capacity constraints:

$$q_j \leq q_0, j = 1, ..., n$$

- CHT's Cost function:

$$C(q_0, q) = F + c_0 q_0 + c \cdot q$$

- Demand for cell phone service: $p_j(q)$
- Total Revenue: $R(q) = p \cdot q$

A Peak-Load Pricing Problem

- The monopolist profit maximization problem:

$$\max_{q_0, q} \left\{ R(q) - F - c_0 q_0 - c \cdot q \middle| q_0 - q_j \geq 0, j = 1, ..., n \right\}$$

- How do you solve this problem?
- When does FOC guarantee a solution?
- What does the Lagrange multiplier mean?
- What should you do when FOC "fails"?

Need: Lagrange Multiplier Method

1. Write Constraints as $h_i(x) \geq 0, i = 1, ..., m$

$$h(x) = (h_1(x), ..., h_m(x))$$

2. Shadow prices $\lambda = (\lambda_1, ..., \lambda_m)$

- Lagrangian $\mathcal{L}(x, \lambda) = f(x) + \lambda \cdot h(x)$
- FOC:

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \lambda \cdot \frac{\partial h}{\partial x_j} \leq 0, \text{ with equality if } \overline{x}_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = h_i(\overline{x}) \geq 0, \text{ with equality if } \lambda_i > 0.$$

Solving Peak-Load Pricing

- The monopolist profit maximization problem:

$$\max_{q_0, q} \left\{ R(q) - F - c_0 q_0 - c \cdot q \middle| q_0 - q_j \geq 0, j = 1, ..., n \right\}$$

- The Lagrangian is

$$\mathcal{L}(q_0, q) = R(q) - F - c_0 q_0 - \sum_{j=1}^{n} c_j q_j + \sum_{j=1}^{n} \lambda_j (q_0 - q_j)$$

$$= R(q) - \sum_{j=1}^{n} (c_j + \lambda_j) q_j + \left(\sum_{j=1}^{n} \lambda_j - c_0 \right) q_0 - F$$

Solving Peak-Load Pricing

- FOC:

$$\frac{\partial \mathcal{L}}{\partial q_j} = MR_j - c_j - \lambda_j \leq 0, \text{ with equality if } q_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 \leq 0, \text{ with equality if } q_0 > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- For positive production, FOC becomes:

$$\frac{\partial \mathcal{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0, \text{ since } q_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0, \text{ since } q_0 > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- Meaning of FOC:

$$\frac{\partial \mathcal{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0, \text{ since } q_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0, \text{ since } q_0 > 0.$$

At least 1 period has shadow price > 0!

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- Meaning of FOC:

$$\frac{\partial \mathcal{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0,$$

Hit capacity at positive shadow price!

$$\frac{\partial \mathcal{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0,$$

Off-peak shadow price = 0

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- Meaning of FOC

Peak MR = MC + capacity cost

$$\frac{\partial \mathcal{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0, \quad MR_i(\overline{q}) = c_i + \lambda_i$$

$$\frac{\partial \mathcal{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0,$$

Peak periods share capacity cost via shadow price

Off-peak: MR=MC!

$MR_j(\overline{q}) = c_j$ equality if $\lambda_j > 0$.

Solving Peak-Load Pricing

- Economic Insight of FOC:
- Marginal decision of the manager: MR = MC
- Off-peak: MR = operating MC
  - Since didn't hit capacity
- Peak: Need to increase capacity
  - MR of all peak periods =
    cost of additional capacity
    + operating MC of all peak periods
- What's the theory behind this?

Constrained Optimization: Economic Intuition

- Single Constraint Problem:

$$\max_x \left\{ f(x) \middle| x \geq 0, b - g(x) \geq 0 \right\}$$

- Interpretation: a profit maximizing firm
  - Produce non-negative output $x \geq 0$
  - Subject to resource constraint $g(x) \leq b$
- Example: linear constraint $a \cdot x = \sum_{j=1}^{n} a_j x_j \leq b$
- Each unit of x_j requires a_j units of resource b .

Constrained Optimization: Economic Intuition

- Single Constraint Problem:

$$\max_x \left\{ f(x) \middle| x \geq 0, b - g(x) \geq 0 \right\}$$

- Interpretation: a utility maximizing consumer
  - Consume non-negative input $x \geq 0$
  - Subject to budget constraint $g(x) \leq b$

- Example: linear constraint $a \cdot x = \sum_{j=1}^{n} a_j x_j \leq b$

- Each unit of x_j requires a_j units of currency b .

Constrained Optimization: Economic Intuition

- Suppose $\overline{x}$ solves the problem
- If one increases x_j, profit changes by $\dfrac{\partial f}{\partial x_j}$
- Additional resources needed: $\dfrac{\partial g}{\partial x_j}$
- Cost of additional resources: $\lambda \dfrac{\partial g}{\partial x_j}$
  - (Market/shadow price is λ)
- Net gain of increasing x_j is $\dfrac{\partial f}{\partial x_j}(\overline{x}) - \lambda \dfrac{\partial g}{\partial x_j}(\overline{x})$

Necessary Conditions for $\overline{x}_j$

- Firm will increase $\overline{x}_j$ if marginal net gain > 0
  - i.e. $\overline{x}_j$ optimal $\Rightarrow \frac{\partial f}{\partial x_j}(\overline{x}) - \lambda \frac{\partial g}{\partial x_j}(\overline{x}) \leq 0$
- Firm will decrease $\overline{x}_j$ if marginal net gain < 0 (unless $\overline{x}_j$ is already zero)
  - i.e. $\overline{x}_j > 0 \Rightarrow \frac{\partial f}{\partial x_j}(\overline{x}) - \lambda \frac{\partial g}{\partial x_j}(\overline{x}) \geq 0$

$$\frac{\partial f}{\partial x_j}(\overline{x}) - \lambda \frac{\partial g}{\partial x_j}(\overline{x}) \leq 0, \text{ with equality if } \overline{x}_j > 0.$$

Necessary Conditions for $\overline{x}_j$

- If $\overline{x}_j$ is strictly positive, marginal net gain $= 0$
  - i.e. $\overline{x}_j > 0 \Rightarrow \dfrac{\partial f}{\partial x_j}(\overline{x}) - \lambda \dfrac{\partial g}{\partial x_j}(\overline{x}) = 0$
- If $\overline{x}_j$ is zero, marginal net gain ≤ 0
  - i.e. $\overline{x}_j = 0 \Rightarrow \dfrac{\partial f}{\partial x_j}(\overline{x}) - \lambda \dfrac{\partial g}{\partial x_j}(\overline{x}) \leq 0$

$$\frac{\partial f}{\partial x_j}(\overline{x}) - \lambda \frac{\partial g}{\partial x_j}(\overline{x}) \leq 0, \text{ with equality if } \overline{x}_j > 0.$$

Necessary Conditions for $\overline{x}_j$

- If resource doesn't bind, opportunity cost $\lambda = 0$
  - i.e. $b - g(\overline{x}) > 0 \Rightarrow \lambda = 0$

- Or, in other words,

$$b - g(\overline{x}) \geq 0 \text{ with equality if } \lambda > 0.$$

  - This is logically equivalent to the first statement.

Lagrange Multiplier Method

1. Write constraint as $h(x) \geq 0$
2. Lagrange multiplier = shadow price λ

- Lagrangian $\mathcal{L}(x, \lambda) = f(x) + \lambda \cdot h(x)$
- FOC:

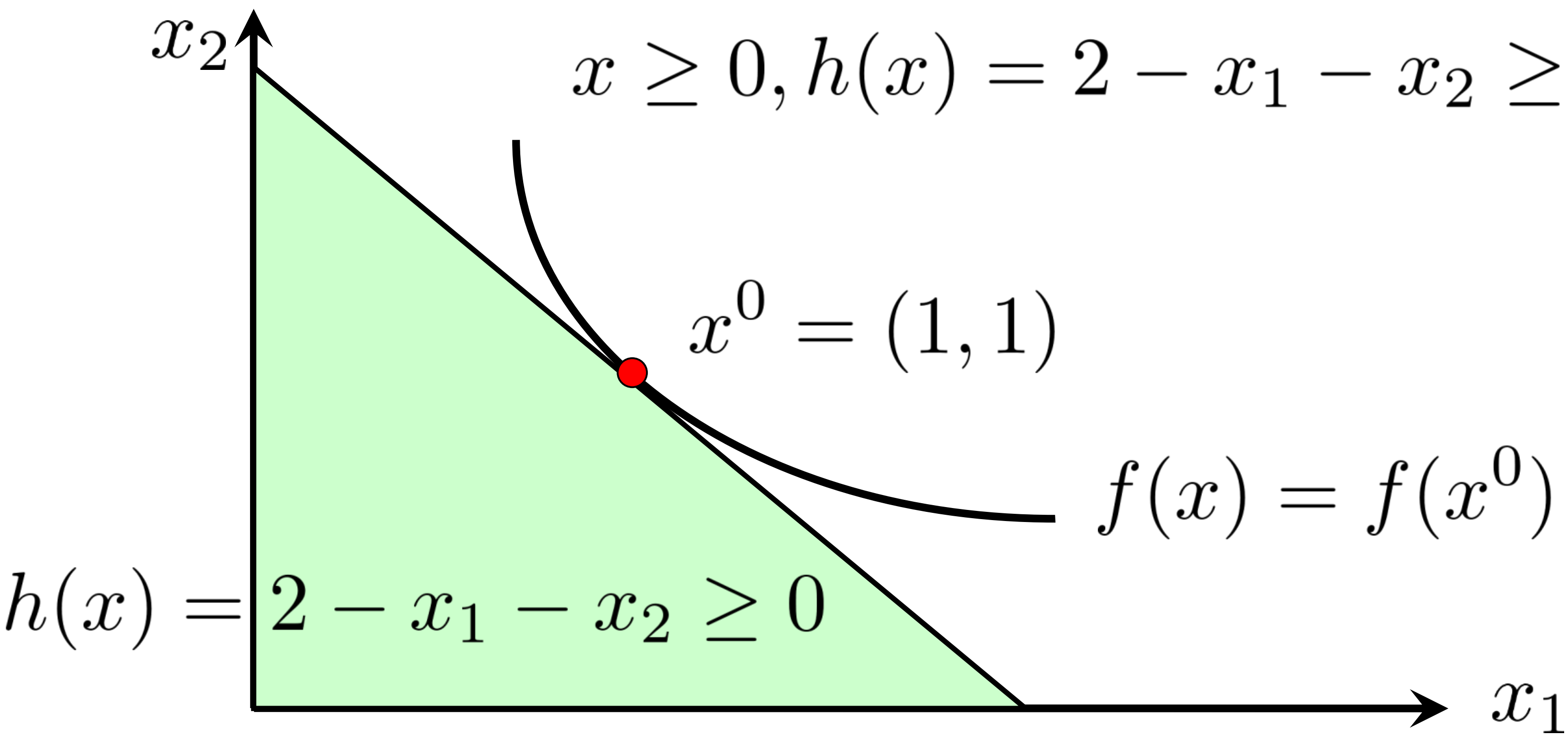
$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \lambda \cdot \frac{\partial h}{\partial x_j} \leq 0, \text{ with equality if } \overline{x}_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = h(\overline{x}) \geq 0, \text{ with equality if } \lambda > 0.$$

Example 1

- A consumer problem:

$$\max_{x} \left\{ f(x) = \ln(1+x_1)(1+x_2) \,\middle|\, x \geq 0, h(x) = 2 - x_1 - x_2 \geq 0 \right\}$$



Example 1

- Maximum at $\overline{x} = (1, 1)$
- Lagrangian:

$$\mathcal{L}(x, \lambda) = \ln(1 + x_1) + \ln(1 + x_2) + \lambda(2 - x_1 - x_2)$$

- FOC:

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{1}{x_j} + \lambda \leq 0, \text{ with equality if } \overline{x}_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 2 - x_1 - x_2 \geq 0, \text{ with equality if } \lambda > 0.$$

Lagrange Multiplier with Multiple Constraints

1. Write Constraints as $h_i(x) \geq 0, i = 1, \cdots, m$

$$h(x) = \left(h_1(x), \cdots, h_m(x)\right)$$

2. Shadow prices $\lambda = (\lambda_1, \cdots, \lambda_m)$

- Lagrangian $\mathcal{L}(x, \lambda) = f(x) + \lambda \cdot h(x)$
- FOC:

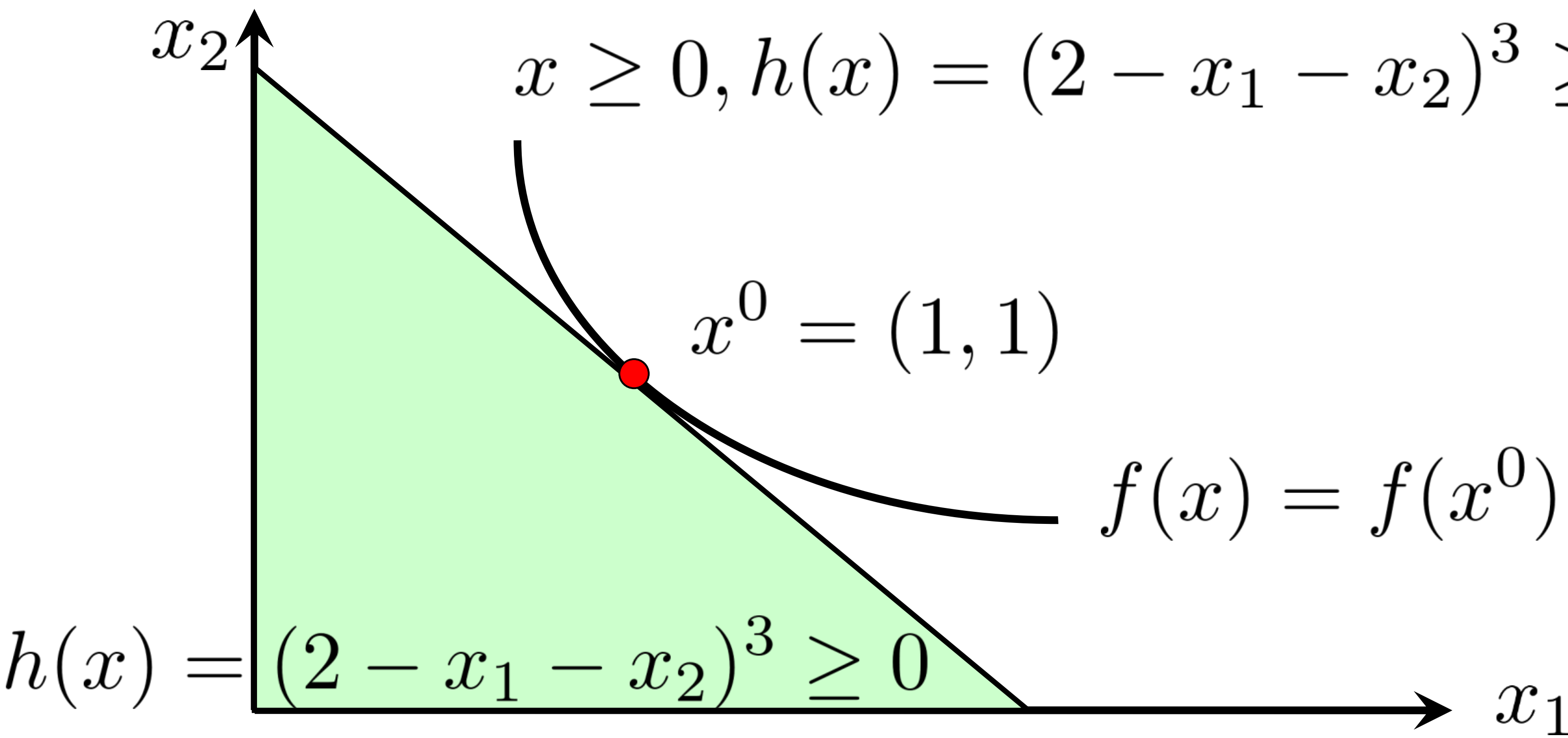
$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \lambda \cdot \frac{\partial h}{\partial x_j} \leq 0, \text{ with equality if } \overline{x}_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = h_i(\overline{x}) \geq 0, \text{ with equality if } \lambda_i > 0.$$

When Intuition Breaks Down? See Example 2

- A “new” problem:

$$\max_{x} \left\{ f(x) = \ln(1+x_1)(1+x_2) \,\middle|\, x \geq 0, h(x) = (2 - x_1 - x_2)^3 \geq 0 \right\}$$



When Intuition Breaks Down? See Example 2

- Lagrangian

$$\mathcal{L}(x,\lambda) = \ln(1+x_1) + \ln(1+x_2) + \lambda(2-x_1-x_2)^3$$

- FOC is violated!

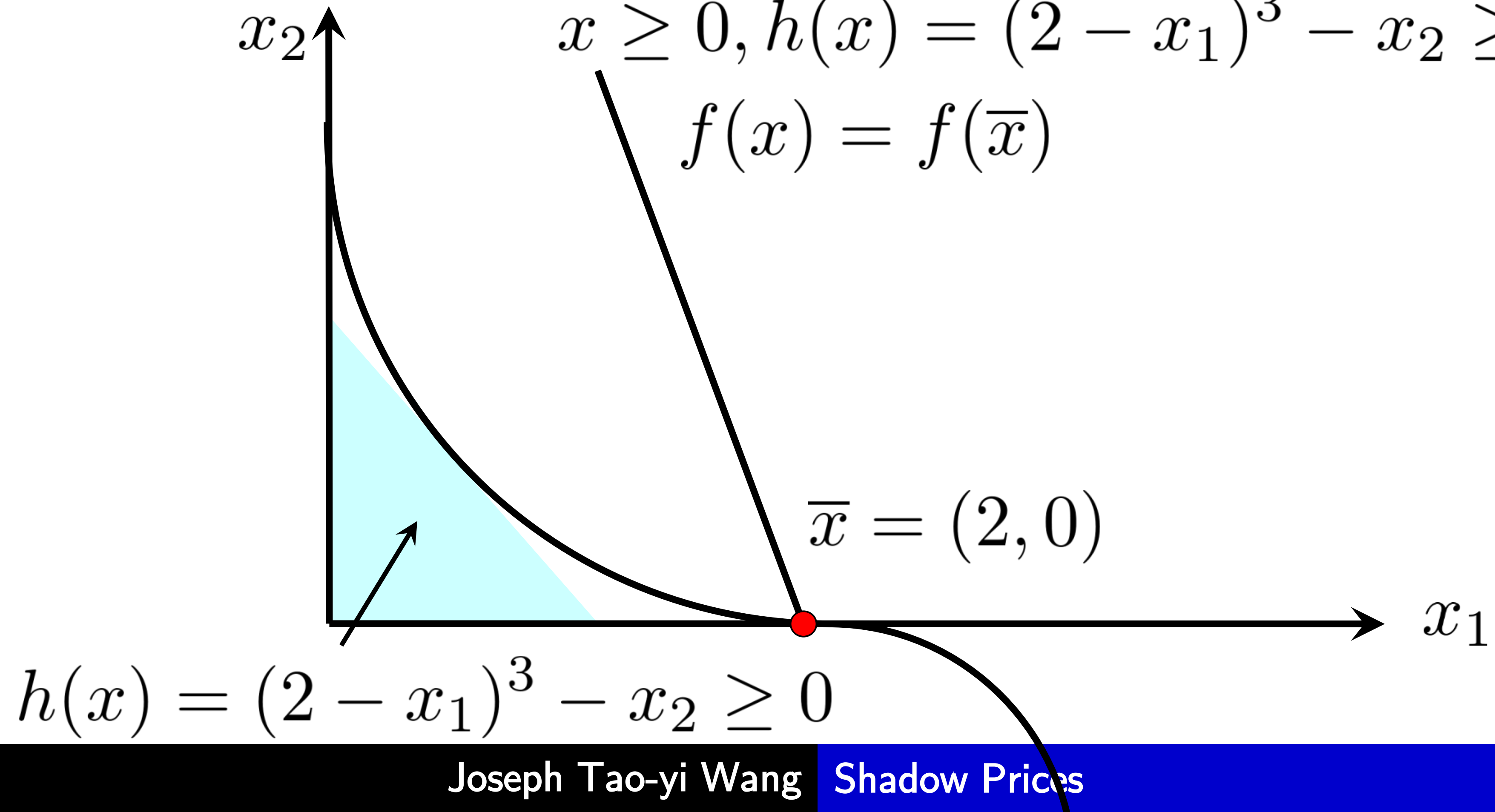
$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{1}{1+x_j} - 3\lambda(2-x_1-x_2)^2 = \frac{1}{2} \text{ at } \overline{x} = (1,1)$$

- How could this be? Because "linearization" fails if gradient = 0… $\dfrac{\partial h}{\partial x} = 0$ at $x = (1,1)$

$$\overline{h}(x) = h(\overline{x}) + \frac{\partial h}{\partial x}(\overline{x}) \cdot (x - \overline{x}) = h(1,1) = 0$$

Other Break Downs? See Example 3

$$\max_x \left\{ f(x) = 12x_1 + x_2 \,\middle|\, x \geq 0, h(x) = (2 - x_1)^3 - x_2 \geq 0 \right\}$$



Other Break Downs? See Example 3

- Lagrangian $\mathcal{L}(x, \lambda) = 12x_1 + x_2 + \lambda \left[(2 - x_1)^3 - x_2 \right]$
- FOC is violated!
$$\frac{\partial \mathcal{L}}{\partial x_1} = 12 - 3\lambda(2 - \overline{x}_1)^2 = 12 \text{ at } \overline{x} = (2, 0)$$
- What's the problem this time?
- Not the gradient... $\frac{\partial h}{\partial x}(\overline{x}) = (0, -1)$
- "Linearized feasible set" has no interior...

Other Break Downs? See Example 3

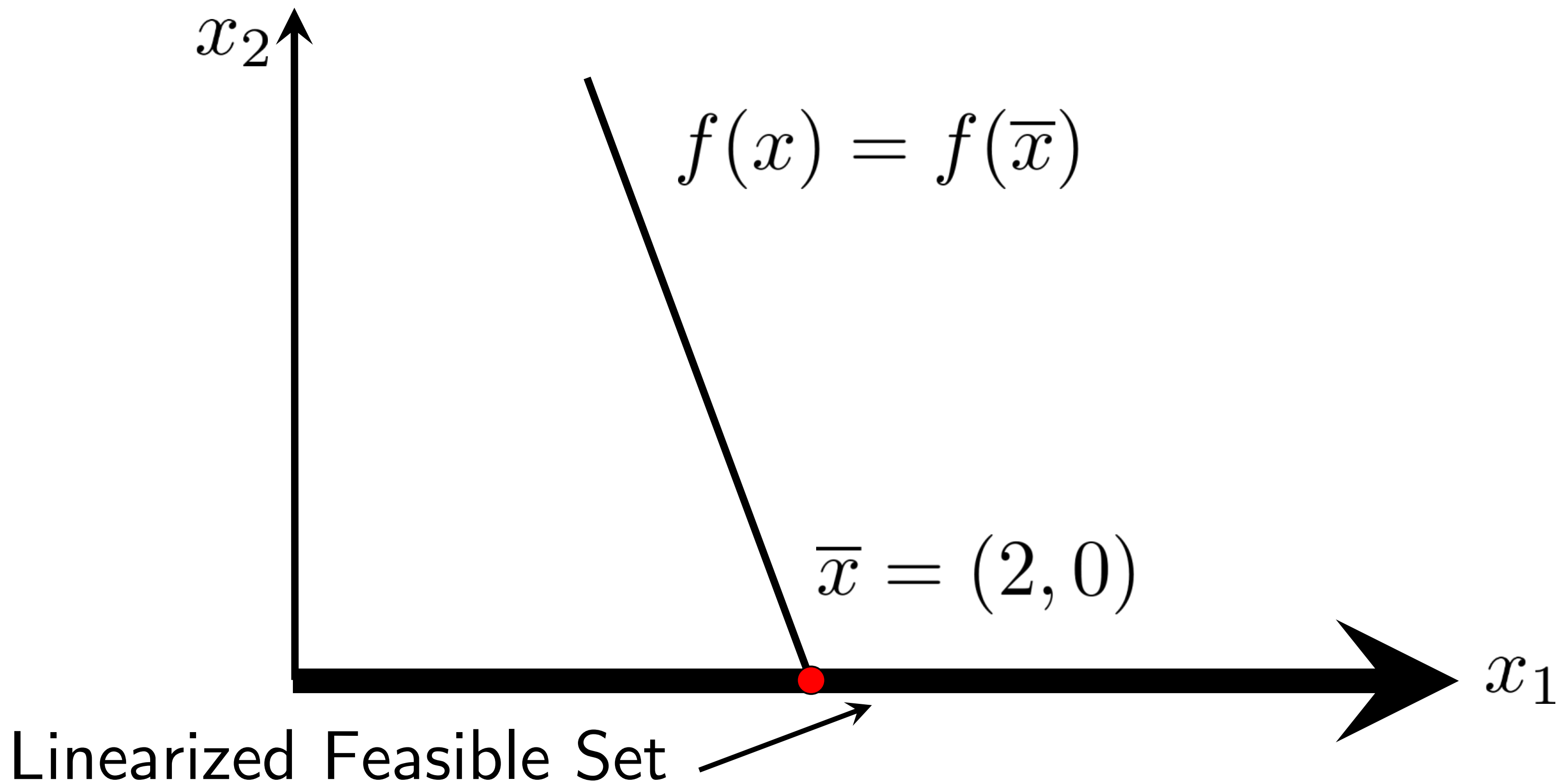
- What's the problem this time?
- Gradient is $\frac{\partial h}{\partial x}(\overline{x}) = (0, -1)$
- Hence, the linear approximation of the constraint is:

$$\frac{\partial h}{\partial x}(\overline{x}) \cdot (x - \overline{x}) = \frac{\partial h}{\partial x_1}(\overline{x}) \cdot (x_1 - 2) + \frac{\partial h}{\partial x_2}(\overline{x}) \cdot x_2$$

$$= -x_2 \geq 0 \;\; \Rightarrow x_2 = 0$$

Other Break Downs? See Example 3

$$\max_x \left\{ f(x) = 12x_1 + x_2 \middle| x \geq 0, h(x) = (2 - x_1)^3 - x_2 \geq 0 \right\}$$



Linearized Feasible Set $\overline{X}$

- Set of constraints binding at $\overline{x}$: $h_i(\overline{x}) = 0$
  - For $i \in B = \{i | i = 1, ..., m, h_i(\overline{x}) = 0\}$
- Replace binding constraints by linear approx.

$$\overline{h}_i(x) = \underline{\underline{h_i(\overline{x})}} + \frac{\partial h_i}{\partial x}(\overline{x}) \cdot (x - \overline{x}) \geq 0$$

- Since these constraints also bind, we have

$$\frac{\partial h_i}{\partial x}(\overline{x}) \cdot (x - \overline{x}) \geq 0, i \in B$$

  - Because $\underline{\underline{h_i(\overline{x})}} = 0$

Linearized Feasible Set $\overline{X}$

- Note: These are "true" constraints if gradient

$$\frac{\partial h_i}{\partial x}(\overline{x}) \neq 0$$

- $\overline{X}$ = Linearized Feasible Set

= Set of non-negative vectors satisfying

$$\frac{\partial h_i}{\partial x}(\overline{x}) \cdot (x - \overline{x}) \geq 0, i \in B$$

Constraint Qualifications

- Set of feasible vectors:

$$X = \{x \big| x \geq 0, h_i(x) \geq 0\}$$

- The Constraint Qualifications hold at $\overline{x} \in \overline{X}$ if

(i) Binding constraints have non-zero gradients

$$\frac{\partial h_i}{\partial x}(\overline{x}) \neq 0$$

(ii) The linearized feasible set $\overline{X}$ at $\overline{x}$ has a non-empty interior.

  – CQ guarantees FOC to be necessary conditions

Proposition 1.2-1 Kuhn-Tucker Conditions

- Suppose $\overline{x}$ solves

$$\max_x \left\{ f(x) \middle| x \in X \right\}, X = \text{feasible set}$$

- If the constraint qualifications hold at $\overline{x}$
- Then there exists shadow price vector $\lambda \geq 0$
- Such that (for $j=1,\ldots,n,\ i=1,\ldots,m$)

$$\frac{\partial \mathcal{L}}{\partial x_j}(\overline{x}, \lambda) \leq 0, \text{ with equality if } \overline{x}_j > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = (\overline{x}, \lambda) \geq 0, \text{ with equality if } \lambda_i > 0.$$

Lemma 1.2-2 [Special Case] Quasi-Concave

- If for each binding constraint at $\overline{x}$, h_i is quasi-concave and $\dfrac{\partial h_i}{\partial x}(\overline{x}) \neq 0$
- Then, $\boxed{X \subset \overline{X}}$
  - Tangent Hyperplanes = Supporting Hyperplanes!
- Hence, if X has a non-empty interior, then so does the linearized set $\overline{X}$
  - Thus we have...

Prop 1.2-3 [Quasi-Concave] Constraint Qualifications

- Suppose feasible set has non-empty interior
$$X = \{x \big| x \geq 0, h_i(x) \geq 0\}$$
- The Constraint Qualifications hold at $\overline{x} \in \overline{X}$ if

- Binding constraints h_i is quasi-concave, and
$$\frac{\partial h_i}{\partial x}(\overline{x}) \neq 0$$

Proposition 1.2-4 Sufficient Conditions

- $\bar{x}$ solves
  $$\max_x \left\{ f(x) \middle| x \geq 0, h_i(x) \geq 0, i = 1, ..., m \right\}$$

- If f and $h_i, i = 1, ..., m$ are quasi-concave,
- The Kuhn-Tucker conditions hold at $\overline{x}$,
- Binding constraints have $\dfrac{\partial h_i}{\partial x}(\overline{x}) \neq 0$
- And $\dfrac{\partial f}{\partial x}(\overline{x}) \neq 0$.

Summary of 1.2

- Consumer = Producer
- Lagrange multiplier = Shadow prices
- FOC = "MR – MC = 0": Kuhn-Tucker
- When does this intuition fail?
  - Gradient = 0
  - Linearized feasible set has no interior

→Constraint Qualification: when it flies…
  - CQ for quasi-concave constraints

- Sufficient Conditions (Proof in Section 1.4)

Summary of 1.2

- Peak-Load Pricing requires Kuhn-Tucker
- MR="effective" MC
- Off-peak shadow price (for capacity) = 0
- All peak periods share additional capacity cost
- Can you think of situations (after you start your new job making $$$$) that requires something similar to peak-load pricing?
- Homework: Exercise 1.2-2 (Optional 1.2-3)