The 2x2 Exchange Economy

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(Lecture 8, Micro Theory I)

Road Map for Chapter 3

- Pareto Efficiency Allocation (PEA)
  - Cannot make one better off without hurting others
- Walrasian (Price-taking) Equilibrium (WE)
  - When Supply Meets Demand
  - Focus on Exchange Economy First
- 1st Welfare Theorem:
  - Any WE is PEA (Adam Smith Theorem)
- 2nd Welfare Theorem:
  - Any PEA can be supported as a WE with transfers

2x2 Exchange Economy

- 2 Commodities: Good 1 and 2
- 2 Consumers: Alex and Bev - $h = A, B$
  - Endowment: $\omega^h = (\omega_1^h, \omega_2^h)$, $\omega_i = \omega_i^A + \omega_i^B$
  - Consumption Set: $x^h = (x_1^h, x_2^h) \in \mathbb{R}_+^2$
  - Strictly Monotonic Utility Function:
- Edgeworth Box

$$U^h(x^h) = U^h(x_1^h, x_2^h)$$

- These consumers could be representative agents, or literally TWO people (bargaining)

Why do we care about this?

- The Walrasian (Price-taking) Equilibrium (W.E.) is (a candidate of) Adam Smith's "Invisible Hand"
  - Are real market rules like Walrasian auctioneers?
  - Is Price-taking the result of competition, or competition itself?
- Illustrate W.E. in more general cases
  - Hard to graph "N goods" as 2D
- Two-party Bargaining
  - This is what Edgeworth himself really had in mind

Why do we care about this?

- Consider the following situation: You company is trying to make a deal with another company
  - You have better technology, but lack funding
  - They have plenty of funding, but low-tech
- There are "gives" and "takes" for both sides
- Where would you end up making the deal?
  - Definitely not where "something is left on the table."
- What are the possible outcomes?
  - How did you get there?

Social Choice and Pareto Efficiency

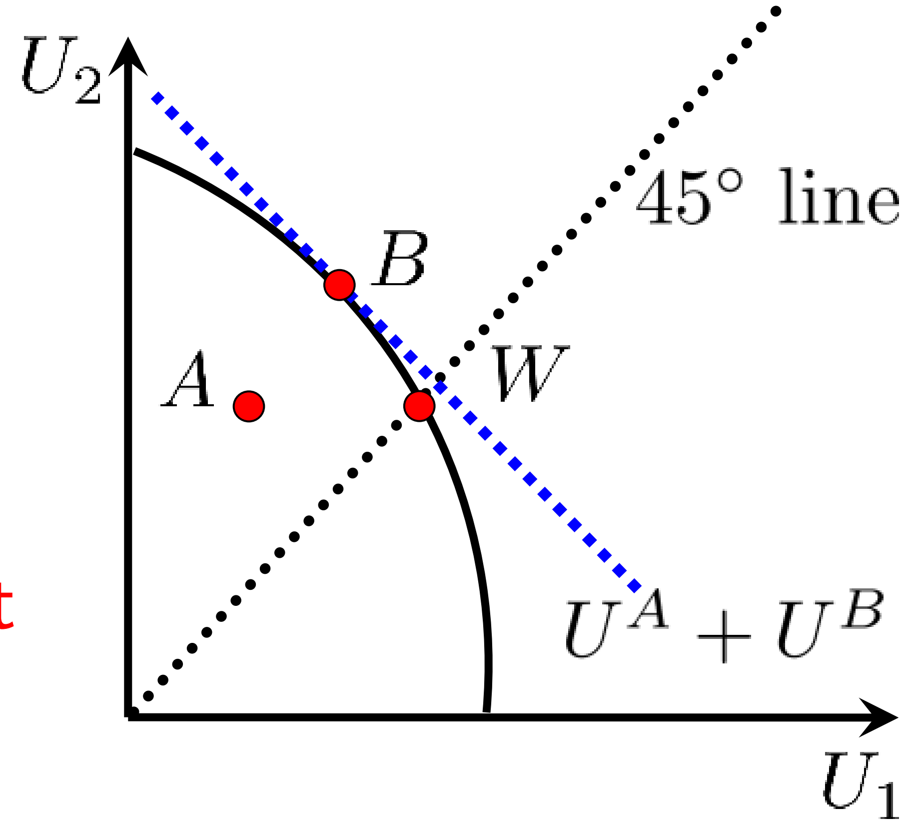
- **Benthamite**:
  - Behind Veil of Ignorance
  - Assign Prob. 50-50

$$\max \tfrac{1}{2}U^A + \tfrac{1}{2}U^B$$

- **Rawlsian**:
  - Infinitely Risk Averse

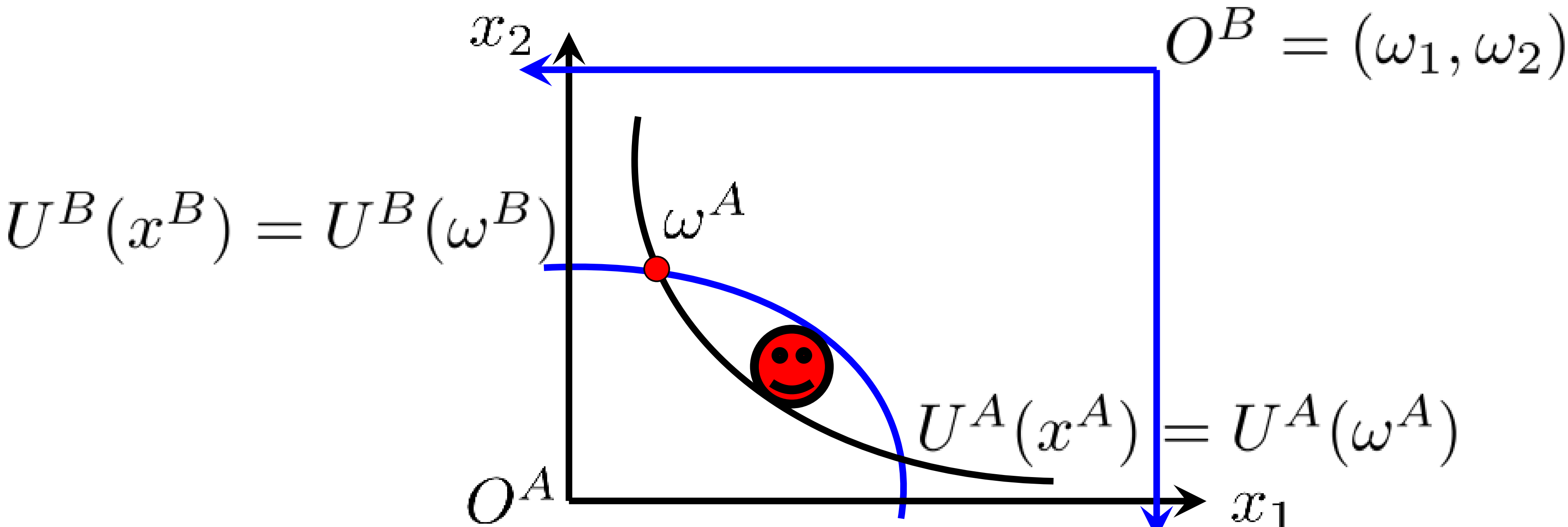
$$\max \min\{U^A, U^B\}$$

- Both are Pareto Efficient
  - But A is not



Pareto Efficiency

- A feasible allocation is Pareto efficient if
- there is no other feasible allocation that is
- strictly preferred by at least one consumer
- and is weakly preferred by all consumers.

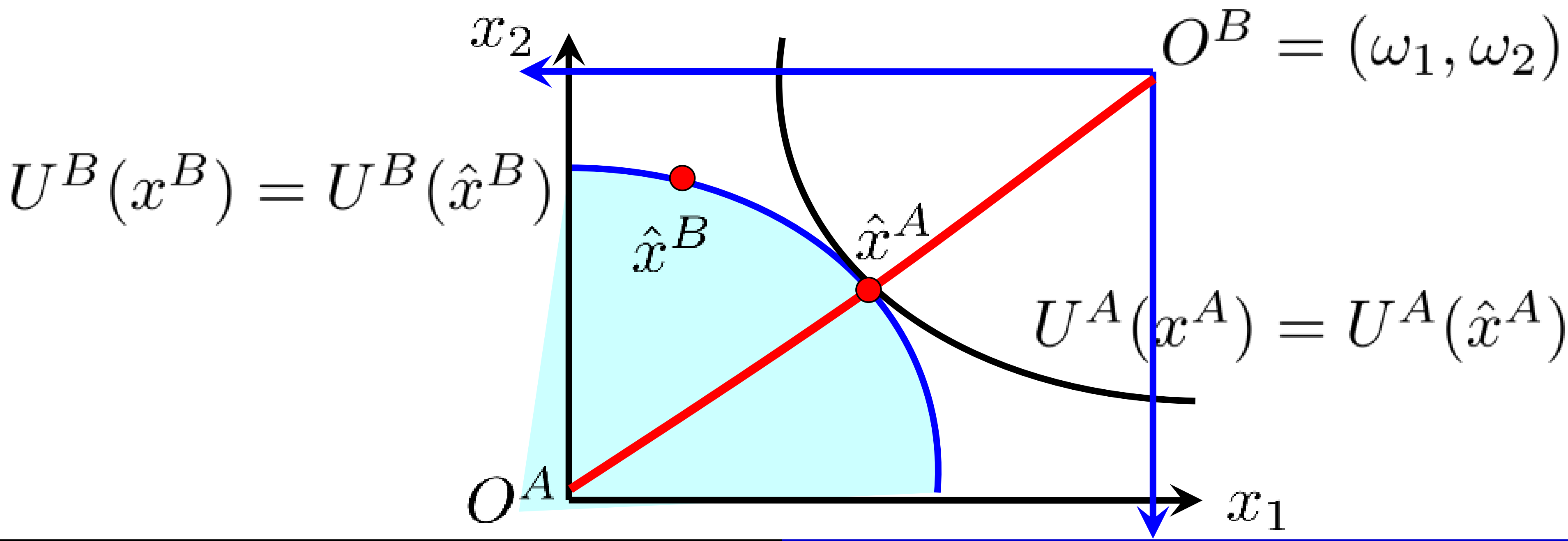


Pareto Efficient Allocations

For $\omega = (\omega_1, \omega_2)$, consider

$$\max_{x^A, x^B} \{U^A(x^A) | U^B(x^B) \geq U^B(\hat{x}^B), x^A + x^B \leq \omega\}$$

Need $MRS^A(\hat{x}^A) = MRS^B(\hat{x}^A)$ (interior solution)



Example: CES Preferences

- CES: $$U(x) = \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}}$$
- MRS: $$MRS^h(x^h) = k\left(\frac{x_2^h}{x_1^h}\right)^{1/\theta}, h = A, B$$
- Equal MRS for PEA in interior of Edgeworth box
$$\Rightarrow \frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} = \frac{x_2^A + x_2^B}{x_1^A + x_1^B} = \frac{\omega_2}{\omega_1}$$
- Thus, $$MRS^h(x^h) = k\left(\frac{\omega_2}{\omega_1}\right)^{1/\theta}, h = A, B$$

Walrasian Equilibrium - 2x2 Exchange Economy

- All Price-takers: Price vector $p \geq 0$
- 2 Consumers: Alex and Bev - $h \in \mathcal{H} = \{A, B\}$
  - Endowment: $\omega^h = (\omega_1^h, \omega_2^h)$, $\omega_i = \omega_i^A + \omega_i^B$
  - Consumption Set: $x^h = (x_1^h, x_2^h) \in \mathbb{R}_+^2$
  - Wealth: $W^h = p \cdot \omega^h$
- Market Demand: $x(p) = \sum_h x^h(p, p \cdot \omega^h)$

  (Solution to consumer problem)
- Vector of Excess Demand: $z(p) = x(p) - \omega$
  - Vector of total Endowment: $\omega = \sum_h \omega^h$

Definition: Market Clearing Prices

- Let excess demand for commodity j be $z_j(p)$
- The market for commodity j clears if

$$z_j(p) \leq 0 \text{ and } p_j \cdot z_j(p) = 0$$

  – Excess demand = 0, or it's negative (& price = 0)
    - Excess demand = shortage; negative ED means surplus
- Why is this important?

1. Walras Law
   – The last market clears if all other markets clear
2. Market clearing defines Walrasian Equilibrium

Local Non-Satiation Axiom (LNS)

For any consumption bundle $x \in C \subset \mathbb{R}^n$
and any δ-neighborhood $N(x, \delta)$ of x,
there is some bundle $y \in N(x, \delta)$ s.t. $y \succ_h x$

- LNS implies consumer must spend all income
- If not, we have $p \cdot x^h < p \cdot \omega^h$ for optimal x^h
- But then there exist δ-neighborhood $N(x^h, \delta)$
- In the budget set for sufficiently small $\delta > 0$
- LNS $\Rightarrow y \in N(x^h, \delta), y \succ_h x^h$, x^h is not optimal!

Walras Law

- For any price vector p , the market value of excess demands must be zero, because:

$$p \cdot z(p) = p \cdot (x - \omega) = p \cdot \left(\sum_h (x^h - \omega^h) \right)$$

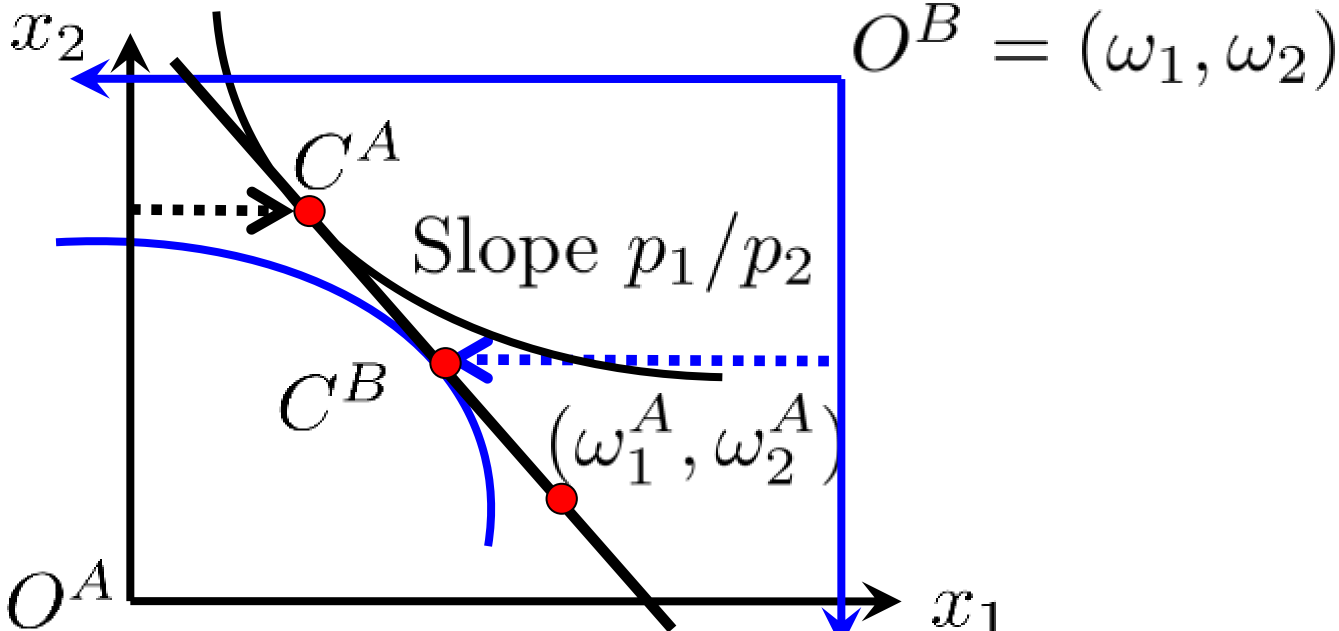
$$= \sum_h (p \cdot x^h - p \cdot \omega^h) = 0 \text{ by LNS}$$

$$= p_1 z_1(p) + p_2 z_2(p) = 0$$

- If one market clears, so must the other.

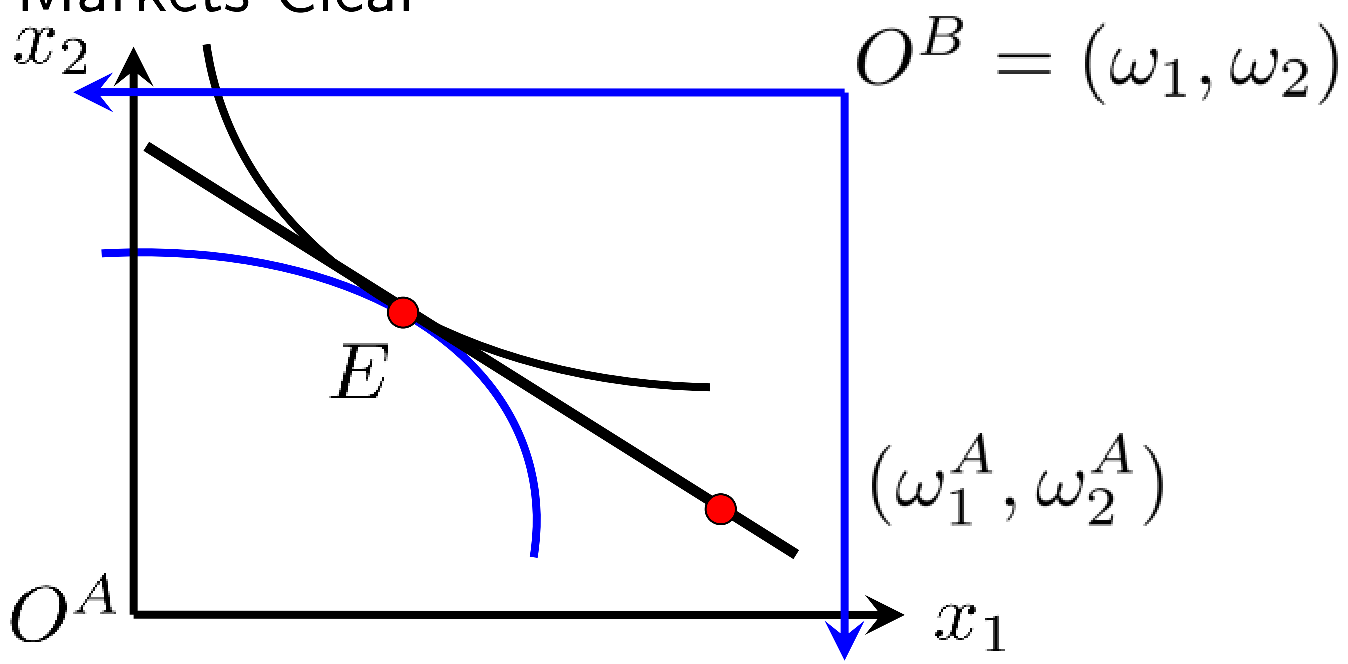
Definition: Walrasian Equilibrium

- The price vector $p \geq 0$ is a Walrasian Equilibrium price vector if all markets clear.
  - WE = price vector!!!
- EX: Excess supply (surplus) of commodity 1...



Definition: Walrasian Equilibrium

- Lower price for commodity 1 if excess supply
  - Until Markets Clear



- Cannot raise Alex's utility without hurting Bev
  - Hence, we have...

First Welfare Theorem: WE → PEA

- If preferences satisfy LNS, then a Walrasian Equilibrium allocation (in an exchange economy) is Pareto efficient.
- Sketch of Proof:

1. Any weakly (strictly) preferred bundle must cost at least as much (strictly more) as WE
2. Markets clear

→ Pareto preferred allocation not feasible

First Welfare Theorem: WE → PEA

1. Since WE allocation $\overline{x}^h$ maximizes utility, so

$$U^h(x^h) > U(\overline{x}^h) \Rightarrow p \cdot x^h > p \cdot \overline{x}^h$$

Now need to show: (Duality Lemma 2.2-3!)

$$U^h(x^h) \geq U(\overline{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \overline{x}^h$$

- Recall Proof: If not, we have $p \cdot x^h < p \cdot \overline{x}^h$
- But then LNS yields a δ-neighborhood $N(x^h, \delta)$
- In the budget set for sufficiently small $\delta > 0$
- In which a point $\tilde{x}^h$ such that

$$U^h(\tilde{x}^h) > U^h(x^h) \geq U(\overline{x}^h)$$

Contradiction!

First Welfare Theorem: WE → PEA

1. $U^h(x^h) > U(\overline{x}^h) \Rightarrow p \cdot x^h > p \cdot \overline{x}^h$

   $U^h(x^h) \geq U(\overline{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \overline{x}^h$

- Satisfied by Pareto preferred allocation (x^A, x^B)

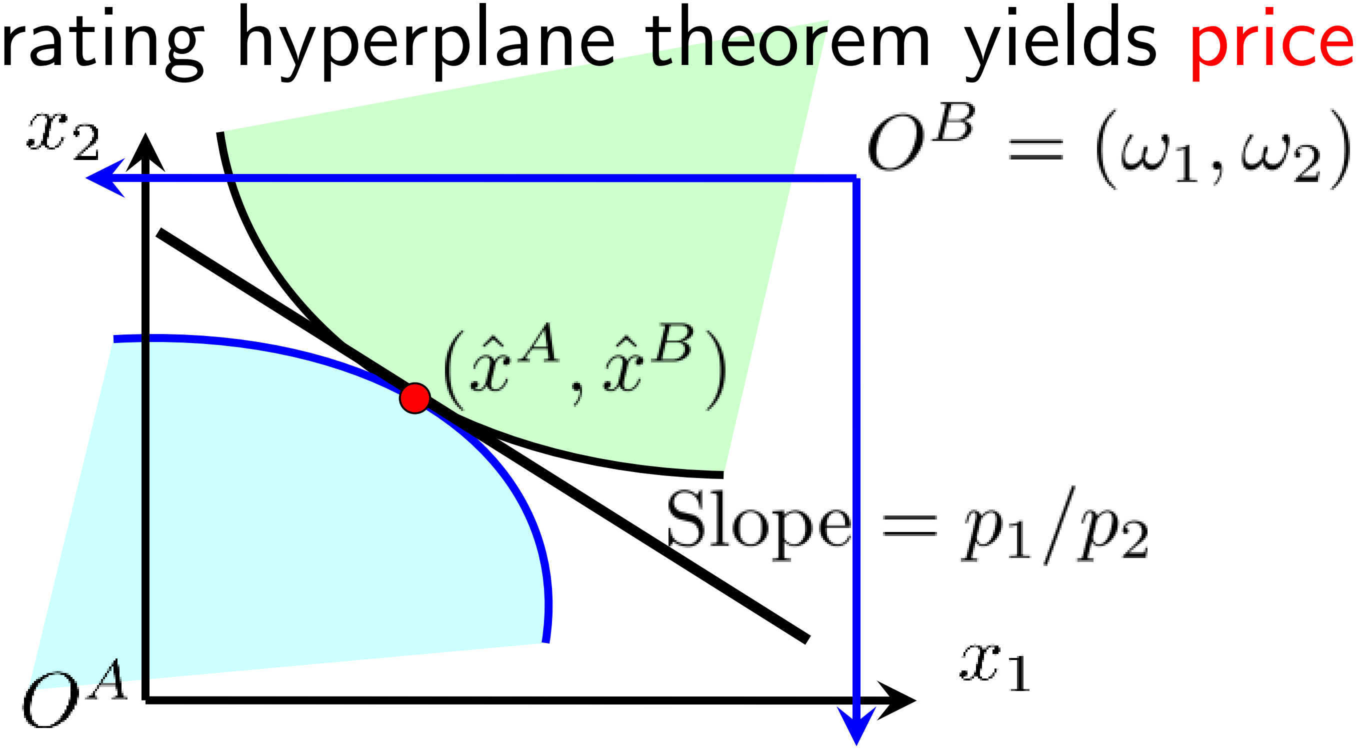
2. Hence, $p \cdot x^h > p \cdot \overline{x}^h$ for at least one, and

- $p \cdot x^h \geq p \cdot \overline{x}^h$ for all others (preferred)
- Thus, $p \cdot \sum_h x^h > p \cdot \sum_h \overline{x}^h = p \cdot \sum_h \omega^h$
- Since $p \geq 0$, at least one j → $\sum_h x_j^h > \sum_h \omega_j^h$
  - Not feasible!

Second Welfare Theorem: PEA → WE

- (2-commodity) For PE allocation $(\hat{x}^A, \hat{x}^B)$

1. Convex preferences imply convex regions
2. Separating hyperplane theorem yields prices



Second Welfare Theorem: PEA → WE

3. Alex and Bev are both optimizing

- For interior Pareto efficient allocation $(\hat{x}^A, \hat{x}^B)$

$$\frac{\frac{\partial U^A}{\partial x_1}(\hat{x}^A)}{\frac{\partial U^A}{\partial x_2}(\hat{x}^A)} = \frac{\frac{\partial U^B}{\partial x_1}(\hat{x}^B)}{\frac{\partial U^B}{\partial x_2}(\hat{x}^B)} \Rightarrow \frac{\partial U^A}{\partial x}(\hat{x}^A) = \theta \cdot \frac{\partial U^B}{\partial x}(\hat{x}^B)$$

- Since we have convex upper contour set

$$X^A = \{x^A | U^A(x^A) \geq U^A(\hat{x}^A)\}$$

- Lemma 1.1-2 yields:

$$U^A(x^A) \geq U^A(\hat{x}^A) \Rightarrow \frac{\partial U^A}{\partial x}(\hat{x}^A) \cdot (x^A - \hat{x}^A) \geq 0$$

Second Welfare Theorem: PEA → WE

$$U^B(x^B) \geq U^B(\hat{x}^B) \Rightarrow \frac{\partial U^B}{\partial x}(\hat{x}^B) \cdot (x^B - \hat{x}^B) \geq 0$$

- Choose $p = \frac{\partial U^B}{\partial x}(\hat{x}^B)$, then $\frac{\partial U^A}{\partial x}(\hat{x}^A) = \theta p$
- And we have:

$$U^A(x^A) \geq U^A(\hat{x}^A) \Rightarrow p \cdot x^A \geq p \cdot \hat{x}^A$$

$$U^B(x^B) \geq U^B(\hat{x}^B) \Rightarrow p \cdot x^B \geq p \cdot \hat{x}^B$$

- In words, weakly "better" allocations are at least as expensive (under this price vector)
  - For $\hat{x}^A, \hat{x}^B$ optimal, need them not affordable…

Second Welfare Theorem: PEA → WE

- Suppose a strictly "better" allocation is feasible
- i.e. $U^A(x^A) > U^A(\hat{x}^A)$ and $p \cdot x^A = p \cdot \hat{x}^A$
- Since U is strictly increasing and continuous,
- Exists $\delta \gg 0$ such that

$$U^A(x^A - \delta) > U^A(\hat{x}^A) \text{ and } p \cdot (x^A - \delta) < p \cdot \hat{x}^A$$

- Contradicting:

$$U^A(x^A) \geq U^A(\hat{x}^A) \Rightarrow p \cdot x^A \geq p \cdot \hat{x}^A$$

  – So, Strictly "better" allocations are not affordable!

Second Welfare Theorem: PEA → WE

- Strictly "better" allocations are not affordable:
- i.e. $U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h, h \in \mathcal{H}$
- So both Alex and Bev are optimizing under p
- Since markets clear at $\hat{x}^A, \hat{x}^B$, it is a WE!
- In fact, to achieve this WE, only need transfers

$$T^h = p \cdot (\hat{x}^h - \omega^h), h \in \mathcal{H}$$

  – Add up to zero (feasible transfer payment), so:
- Budget Constraint is $p \cdot x^h \leq p \cdot \omega^h + T^h, h \in \mathcal{H}$

Proposition 3.1-3: Second Welfare Theorem

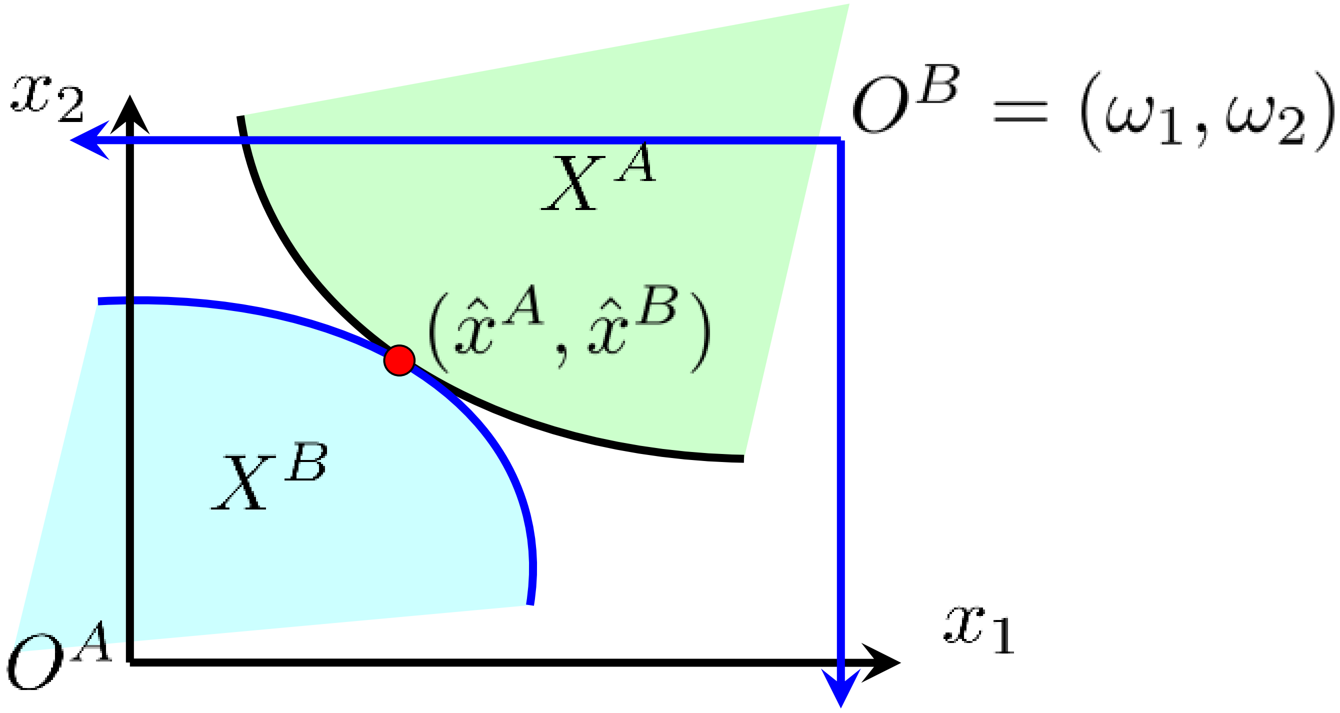
- In an exchange economy with endowment $\{\omega^h\}_{h \in \mathcal{H}}$
- Suppose $U^h(x)$ is continuously differentiable, quasi-concave on $\mathbb{R}^n_+$ and $\frac{\partial U^h}{\partial x^h}(x^h) \gg 0, h \in \mathcal{H}$
- Then any PE allocation $\{\hat{x}^h\}_{h \in \mathcal{H}}$ where $\hat{x}^h \neq 0$
- can be supported by a price vector $p \geq 0$ (as WE)
- Sketch of Proof: (Need not be interior as above!)

1. Constraint Qualification of the PE problem ok
2. Kuhn-Tucker conditions give us (shadow) prices
3. Alex and Bev both maximizing under these prices

Proof of Second Welfare Theorem

- (Proof for 2-player case) PEA $\Rightarrow \hat{x}^A$ solves:

$$\max_{x^A, x^B} \{U^A(x^A) | x^A + x^B \leq \omega, U^B(x^B) \geq U^B(\hat{x}^B)\}$$



Proof of Second Welfare Theorem

$$\max_{x^A, x^B} \{U^A(x^A) | x^A + x^B \leq \omega, U^B(x^B) \geq U^B(\hat{x}^B)\}$$

- Consider the feasible set of this problem:

1. The feasible set has a non-empty interior

- Since $U^B(x)$ is strictly increasing, for small δ,

$$0 < \hat{x}^B < \omega \Rightarrow U^B(\hat{x}^B) < U^B(\omega - \delta) < U^B(\omega)$$

2. The feasible set is convex $(U^B(\cdot)$ quasi-concave$)$
3. Constraint function have non-zero gradient

➢ Constraint Qualifications ok, use Kuhn-Tucker

Proof of Second Welfare Theorem

$$\mathfrak{L} = U^A(x^A) + \nu(\omega - x^A - x^B) + \mu(U^B(x^B) - U^B(\hat{x}^B))$$

- Kuhn-Tucker conditions require: (Inequalities!)

$$\frac{\partial \mathfrak{L}}{\partial x^A} = \frac{\partial U^A}{\partial x^A}(\hat{x}^A) - \nu \leq 0, \quad \hat{x}^A \left[\frac{\partial U^A}{\partial x^A}(\hat{x}^A) - \nu\right] = 0$$

$$\frac{\partial \mathfrak{L}}{\partial x^B} = \mu \frac{\partial U^B}{\partial x^B}(\hat{x}^B) - \nu \leq 0, \quad \hat{x}^B \left[\mu \frac{\partial U^B}{\partial x^B}(\hat{x}^B) - \nu\right] = 0$$

$$\frac{\partial \mathfrak{L}}{\partial \nu} = \omega - \hat{x}^A - \hat{x}^B \geq 0, \quad \nu \left[\omega - \hat{x}^A - \hat{x}^B\right] = 0$$

$$\frac{\partial \mathfrak{L}}{\partial \mu} = U^B(x^B) - U^B(\hat{x}^B) \geq 0, \quad \mu \left[U^B(x^B) - U^B(\hat{x}^B)\right] = 0$$

Proof of Second Welfare Theorem

- Assumed positive MU: $\frac{\partial U^A}{\partial x^A}(\hat{x}^A) \gg 0$

1. $\frac{\partial \mathfrak{L}}{\partial x^A} = \frac{\partial U^A}{\partial x^A}(\hat{x}^A) - \nu \leq 0 \Rightarrow \nu \geq \frac{\partial U^A}{\partial x^A}(\hat{x}^A) \gg 0$
2. $\frac{\partial \mathfrak{L}}{\partial \nu} \geq 0, \nu\left[\omega - \hat{x}^A - \hat{x}^B\right] = 0 \Rightarrow \omega - \hat{x}^A - \hat{x}^B = 0$
3. $\frac{\partial \mathfrak{L}}{\partial x^B} \leq 0, \quad \hat{x}^B\left[\mu \frac{\partial U^B}{\partial x^B}(\hat{x}^B) - \nu\right] = 0$

- Since $\hat{x}^B > 0, \quad \frac{\partial U^B}{\partial x^B}(\hat{x}^B) \gg 0 \Rightarrow \mu > 0$

Proof of Second Welfare Theorem

- Consider Alex's consumer problem with $p = \nu \gg 0$

$$\max_{x^A}\{U^A(x^A) | \nu \cdot x^A \leq \nu \cdot \hat{x}^A\}$$

- FOC: (sufficient since $U^h(\cdot)$ is quasi-concave)

$$\frac{\partial \mathfrak{L}}{\partial x^A} = \frac{\partial U^A}{\partial x^A}(\overline{x}^A) - \lambda^A \nu \leq 0,$$

$$\overline{x}^A \left[\frac{\partial U^A}{\partial x^A}(\overline{x}^A) - \lambda^A \nu\right] = 0$$

- Same for Bev's consumer problem...

Proof of Second Welfare Theorem

- FOC: (sufficient for $U^h(\cdot)$ is quasi-concave)

$$\frac{\partial U^A}{\partial x^A}(\overline{x}^A) - \lambda^A \nu \leq 0,\ \ \overline{x}^A \left[\frac{\partial U^A}{\partial x^A}(\overline{x}^A) - \lambda^A \nu\right] = 0$$

$$\frac{\partial U^B}{\partial x^B}(\overline{x}^B) - \lambda^B \nu \leq 0,\ \ \overline{x}^B \left[\frac{\partial U^B}{\partial x^B}(\overline{x}^B) - \lambda^B \nu\right] = 0$$

- Set, $\lambda^A = 1, \lambda^B = 1/\mu,$
- Then, FOCs are satisfied at $\overline{x}^A = \hat{x}^A, \overline{x}^B = \hat{x}^B$
- At price $p = \nu \gg 0$, neither Alex nor Bev want to trade, so this PE allocation is indeed a WE!

Proof of Second Welfare Theorem

- Define transfers $$T^A = \nu \cdot (\hat{x}^A - \omega^A)$$ $$T^B = \nu \cdot (\hat{x}^B - \omega^B)$$
- With $\omega - \hat{x}^A - \hat{x}^B = \omega^A + \omega^B - \hat{x}^A - \hat{x}^B = 0$
- Alex and Bev's new budget constraints with these transfers are: $$\nu \cdot x^A \leq \nu \cdot \omega^A + T^A = \nu \cdot \hat{x}^A$$ $$\nu \cdot x^B \leq \nu \cdot \omega^B + T^B = \nu \cdot \hat{x}^B$$
- Thus, PE allocation can be support as WE with these transfers. Q.E.D.

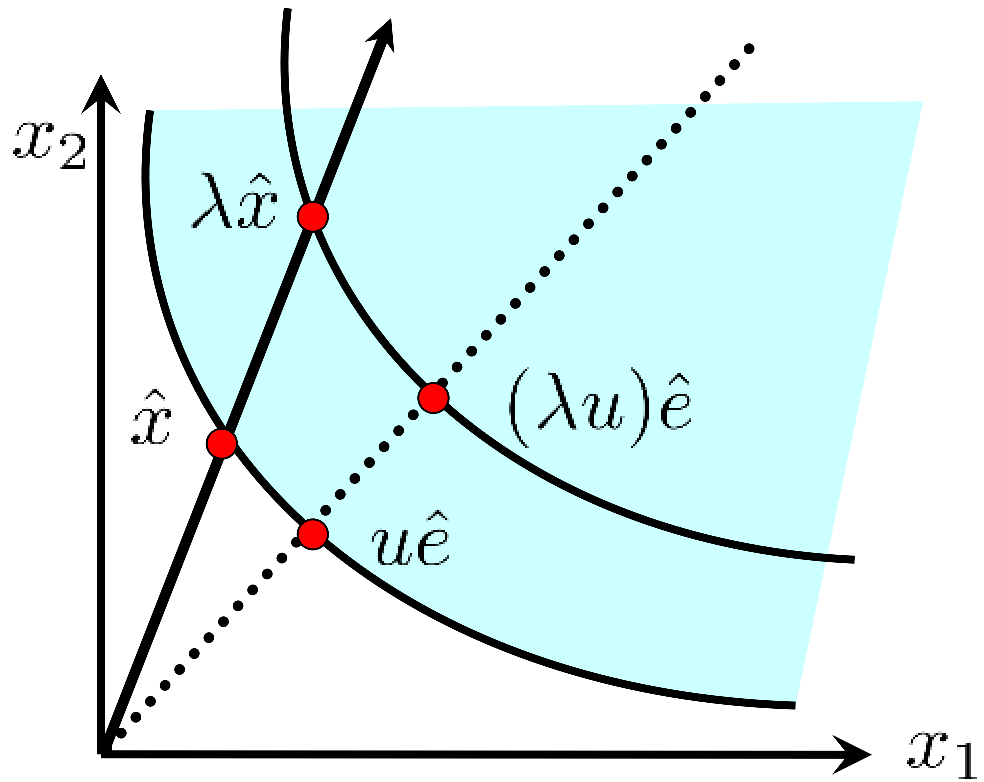
Example: Quasi-Linear Preferences

- Alex has utility function $U^A = x_1^A + \ln x_2^A$
- Bev has utility function $U^B = x_1^B + 2\ln x_2^B$

- Draw the Edgeworth box and find:
- All PE allocations
- Can they be supported as WE?
- What are the supporting price ratios?

Homothetic Preferences: Radial Parallel Pref.

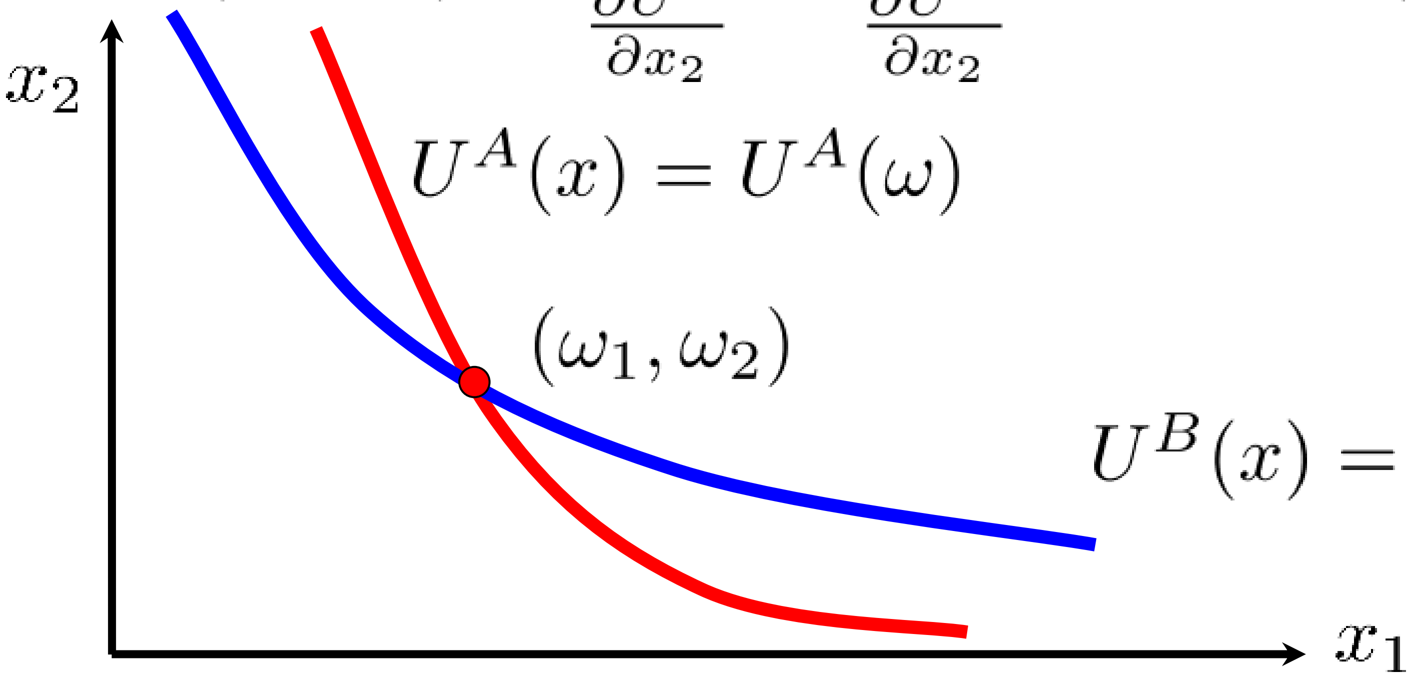
- Consumers have homothetic preferences (CRS)
  - MRS same on each ray, increases as slope of the ray increase



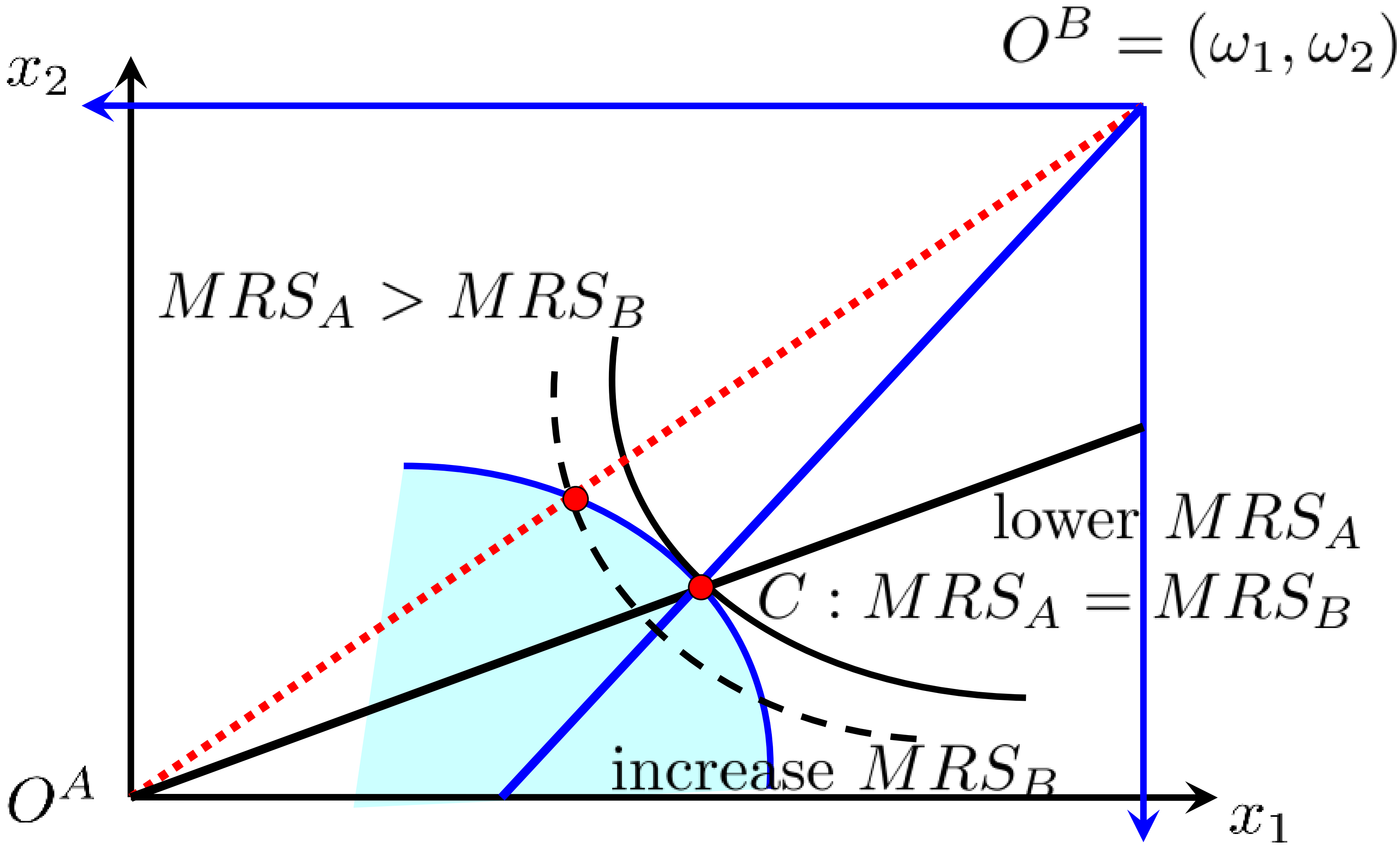
Assumption: Intensity of Preferences

- At aggregate endowment, Alex has a stronger preference for commodity 1 than Bev.

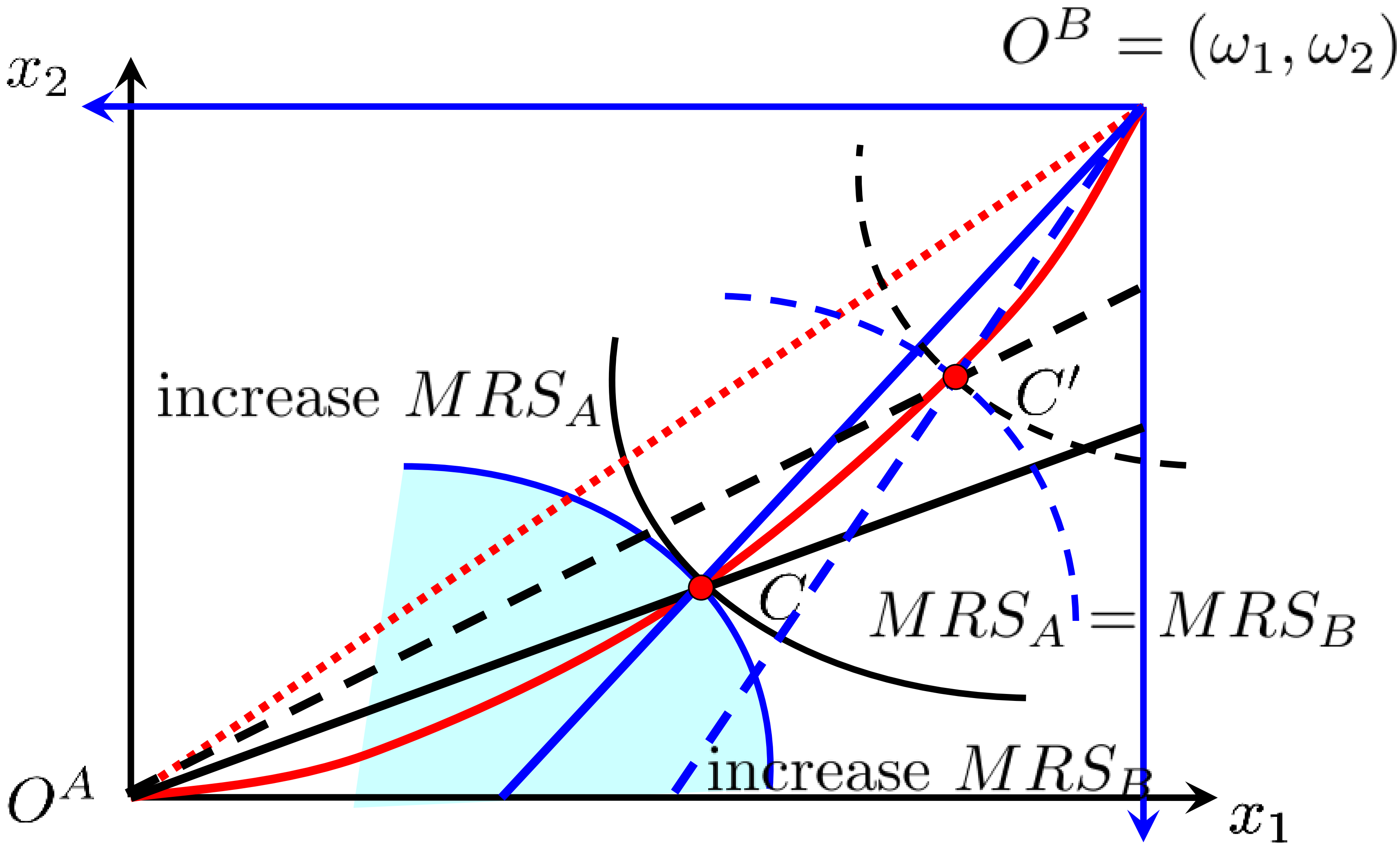
$$MRS_A(\omega_1,\omega_2) = \frac{\frac{\partial U^A}{\partial x_1}}{\frac{\partial U^A}{\partial x_2}} > \frac{\frac{\partial U^B}{\partial x_1}}{\frac{\partial U^B}{\partial x_2}} = MRS_B(\omega_1,\omega_2)$$



PE Allocations with Homothetic Preferences



PE Allocations with Homothetic Preferences



PE Allocations with Homothetic Preferences

- 2x2 Exchange Economy: Alex and Bev have convex and homothetic preferences
- At aggregate endowment, Alex has a stronger preference for commodity 1 than Bev.
- Then, at any interior PE allocation, we have:

$$\frac{x_2^A}{x_1^A} < \frac{\omega_2}{\omega_1} < \frac{x_2^B}{x_1^B}$$

- And, as $U^A(x^A)$ rises, consumption ratio $\frac{x_2^A}{x_1^A}$ and MRS both rise.

Summary of 3.1

- Pareto Efficiency:
  - Can't make one better off without hurting others
- Walrasian Equilibrium: market clearing prices
- First Welfare Theorem: WE is PE
- Second Welfare Theorem: PE allocations can be supported as WE (with transfers)
- Homework: 2008 midterm-Question 3
  - (Optional: 2009 midterm-Part A and Part B)

In-Class Exercise: Quasi-Linear Preferences

- Alex has utility function $U^A = x_1^A + \ln x_2^A$
- Bev has utility function $U^B = x_1^B + 2\ln x_2^B$

- Draw the Edgeworth box and find:
- All PE allocations
- Can they be supported as WE?
- What are the supporting price ratios?

In-Class Homework: Exercise 3.1-1

- Consider a two-person economy in which the aggregate endowment is $(\omega_1, \omega_2) = (100, 200)$
- Both have same quasi-linear utility function

$$U(x^h) = x_1^h + \sqrt{x_2^h}$$

a) Solve for the Walrasian equilibrium price ratio assuming equilibrium consumption of good 1 is positive for both individuals.

b) What is the range of possible equilibrium price ratios in this economy?

In-Class Homework: Exercise 3.1-2

a) If U^A and U^B are strictly increasing, explain why the allocation $\{\hat{x}^A, \hat{x}^B\} = \{\omega^A + \omega^B, 0\}$ is a PE and WE allocation.

- Suppose that $U^A = x_1^A + 10 \ln x_2^A$ and
$$U^B = \ln x_1^B + x_2^B$$
- Aggregate endowment is $(\omega_1, \omega_2) = (20, 10)$

In-Class Homework: Exercise 3.1-2

- Let $U^A = x_1^A + 10 \ln x_2^A$ and $U^B = \ln x_1^B + x_2^B$
- Aggregate endowment is $(\omega_1, \omega_2) = (20, 10)$

b) Show that PEA in the interior of the Edgeworth box can be expressed as $\hat{x}_2^A = f(\hat{x}_1^A)$

c) Suppose that $\omega_2^A = f(\omega_1^A)$. How does the equilibrium price ratio change as ω_1^A increases along the curve?

d) Which allocations on the boundary of the Edgeworth box are PE allocations?