Consumer Choice with N Commodities

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(Lecture 6, Micro Theory I)

From 2 Goods to N Goods...

- More applications of tools learned before...
- Questions we ask: What is needed to...

1. Obtain the compensated law of demand?
2. Have a concave minimized expenditure function?
3. Recover consumer's demand?
4. "Use" a representative agent (in macro)?

Key Problems to Consider

- Revealed Preference: Only assumption needed:
  - Compensated Law of Demand
  - Concave Minimized Expenditure Function
- Indirect Utility Function: (The Maximized Utility)
  - Roy's Identity: Can recover demand function from it
- Homothetic Preferences: (Revealed Preference)
  - Demand is proportional to income
  - Utility function is homogeneous of degree 1
  - Group demand as if one representative agent

Why do we care about this?

- Three separate questions:

1. How general can revealed preference be?
2. How do we back out demand from utility maximization?
3. When can we aggregate group demand with a representative agent (say in macroeconomics)?

- Are these convincing?

Proposition 2.3-1 Compensated Price Change

Consider the dual consumer problem

$$M(p, U^*) = \min_x \{p \cdot x | U(x) \geq U^*\}$$

For x^0 be expenditure minimizing for prices p^0

x^1 be expenditure minimizing at prices p^1

x^0, x^1 satify $U(x) \geq U^*$

$\Rightarrow$ compensated price change is $\Delta p \cdot \Delta x \leq 0$

Proposition 2.3-1 Compensated Price Change

Proof:

$$p^0 \cdot x^0 \le p^0 \cdot x^1, \quad p^1 \cdot x^1 \le p^1 \cdot x^0$$

Since x^0 be expenditure minimizing for prices p^0

x^1 be expenditure minimizing at prices p^1

$$\Rightarrow -p^0 \cdot (x^1 - x^0) \le 0, \quad p^1 \cdot (x^1 - x^0) \le 0$$

$$\Rightarrow \Delta p \cdot \Delta x = (p^1 - p^0) \cdot (x^1 - x^0) \le 0$$

Proposition 2.3-1 Compensated Price Change

- This is true for any pair of price vectors
- For $p^0 = (\overline{p}_1, \cdots, \overline{p}_{j-1}, p_j^0, \overline{p}_{j+1}, \cdots, \overline{p}_n)$
  $p^1 = (\overline{p}_1, \cdots, \overline{p}_{j-1}, p_j^1, \overline{p}_{j+1}, \cdots, \overline{p}_n)$
- We have the (compensated) law of demand:
  $$\Delta p_j \cdot \Delta x_j \leq 0$$
- Note that we did not need differentiability to get this, just revealed preferences!!
- But if differentiable, we have $\dfrac{\partial x_j^c}{\partial p_j} \leq 0$

1st & 2nd Derivatives of Expenditure Function

But what is $\frac{\partial x_j^c}{\partial p_j}$?

Consider the dual problem as a maximization:

$$-M(p, U^*) = \max_x \{-p \cdot x | U(x) \geq U^*\}$$

Lagrangian is $\mathfrak{L} = -p \cdot x + \lambda(U(x) - U^*)$

Envelope Theorem yields $-\frac{\partial M}{\partial p_j} = \frac{\partial \mathfrak{L}}{\partial p_j} = -x_j^c$

$$\Rightarrow \frac{\partial}{\partial p_i}\left(\frac{\partial M}{\partial p_j}\right) = \frac{\partial x_j^c}{\partial p_i}$$

1st & 2nd Derivatives of Expenditure Function

Hence, compensated law of demand yields

$$\frac{\partial x_j^c}{\partial p_j} = \frac{\partial^2 M}{\partial p_j^2} \leq 0$$

$\Rightarrow$ Expenditure function concave for each p_j.

Is the entire Expenditure function concave?

Requires the matrix of second derivatives

$$\left[\frac{\partial^2 M}{\partial p_i \partial p_j}\right] = \left[\frac{\partial x_j^c}{\partial p_i}\right] \text{ to be negative semi-definite}$$

Prop. 2.3-2 Concave Expenditure Function

$$M(p, U^*) \text{ is a concave function over } p.$$

i.e. For any p^0, p^1,

$$M(p^\lambda, U^*) \geq (1-\lambda)M(p^0, U^*) + \lambda M(p^1, U^*)$$

We can show this with only revealed preferences...
(even without assuming differentiability!)

Prop. 2.3-2 Concave Expenditure Function

Proof: For x^λ that solves $M(p^\lambda, U^*)$, (feasible!)

$$M(p^0, U^*) = p^0 \cdot x^0 \leq p^0 \cdot x^\lambda,$$

$$M(p^1, U^*) = p^1 \cdot x^1 \leq p^1 \cdot x^\lambda$$

Since $M(p, U^*)$ minimizes expenditure.

Hence,

$$(1-\lambda)M(p^0, U^*) + \lambda M(p^1, U^*)$$

$$\leq \left[(1-\lambda)p^0 \cdot x^\lambda\right] + \left[\lambda p^1 \cdot x^\lambda\right]$$

$$= p^\lambda \cdot x^\lambda = M(p^\lambda, U^*)$$

What Have We Learned?

- Method of Revealed Preferences
- Used it to obtain:

1. Compensated Price Change
2. Compensated Law of Demand
3. Concave Expenditure Function
   - Special Case assuming differentiability

- Next: How can we get demand from utility?

Indirect Utility Function

Let demand for consumer $U(\cdot)$ with income I, facing price vector p be $x^* = x(p, I)$.

$$V(p, I) = \max_x \{U(x) | p \cdot x \leq I, x \geq 0\}$$
$$= U(x^*(p, I))$$

is maximized $U(x)$, aka **indirect utility function**

Why should we care about this function?

Proposition 2.3-3 Roy's Identity

$$x_j^*(p, I) = -\frac{\frac{\partial V}{\partial p_j}}{\frac{\partial V}{\partial I}}$$

$\boxed{\text{Get this directly from indirect utility function...}}$

Proposition 2.3-3 Roy's Identity

Proof:

$$V(p, I) = \max_x \{U(x) | p \cdot x \le I, x \ge 0\}$$

Lagrangian is $\mathfrak{L}(x, \lambda) = U(x) + \lambda(I - p \cdot x)$

Envelope Theorem yields $\dfrac{\partial V}{\partial I} = \dfrac{\partial \mathfrak{L}}{\partial I}(x^*, \lambda^*) = \lambda^*$

And $\dfrac{\partial V}{\partial p_j} = \dfrac{\partial \mathfrak{L}}{\partial p_j}(x^*, \lambda^*) = -\lambda^* x_j^*(p, I)$

$$\Rightarrow x_j^*(p, I) = -\frac{\frac{\partial V}{\partial p_j}}{\frac{\partial V}{\partial I}}$$

Example: Unknown Utility...

Consider indirect utility function

$$V(p, I) = \prod_{i=1}^{n} \left(\frac{\alpha_i I}{p_i} \right)^{\alpha_i} \quad \text{where} \quad \sum_{i=1}^{n} \alpha_i = 1$$

What's the demand (and original utility) function?

$$\ln V = \ln I - \sum_{i=1}^{n} \alpha_i \ln p_i + \sum_{i=1}^{n} \alpha_i \ln \alpha_i$$

$$\Rightarrow \frac{\partial}{\partial I} \ln V = \frac{1}{V} \frac{\partial V}{\partial I} = \frac{1}{I}, \quad \frac{\partial}{\partial p_i} \ln V = \frac{1}{V} \frac{\partial V}{\partial p_i} = -\frac{\alpha_i}{p_i}$$

Example: Unknown Utility...

$$V(p, I) = \prod_{i=1}^{n} \left(\frac{\alpha_i I}{p_i} \right)^{\alpha_i} \text{ where } \sum_{i=1}^{n} \alpha_i = 1$$

What's the demand (and original utility) function?

$$\ln V = \ln I - \sum_{i=1}^{n} \alpha_i \ln p_i + \sum_{i=1}^{n} \alpha_i \ln \alpha_i$$

$$\Rightarrow \frac{\partial}{\partial I} \ln V = \frac{1}{V} \frac{\partial V}{\partial I} = \frac{1}{I}, \frac{\partial}{\partial p_i} \ln V = \frac{1}{V} \frac{\partial V}{\partial p_i} = -\frac{\alpha_i}{p_i}$$

By Roy's Identity, $x_i^* = -\dfrac{\frac{\partial V}{\partial p_j}}{\frac{\partial V}{\partial I}} = \dfrac{\alpha_i I}{p_i}$

Example: Cobb-Douglas Utility

- Plugging back in

$$U(x) = V = \prod_{i=1}^{n} \left(\frac{\alpha_i I}{p_i} \right)^{\alpha_i} = \prod_{i=1}^{n} (x_i)^{\alpha_i}$$

- What is this utility function?
- Cobb-Douglas!

- Note: This is an example where demand is proportion to income. In fact, we have…

Definition: Homothetic Preferences

Strictly monotonic preference $\succsim$ is **homothetic** if, for any $\theta > 0$ and x^0, x^1 such that $x^0 \succsim x^1$,

$$\theta x^0 \succsim \theta x^1$$

In fact, if $x^0 \sim x^1$,

Then, $\theta x^0 \sim \theta x^1$

Why Do We Care About This?

- Proposition 2.3-4:
  - Demand proportional to income
- Proposition 2.3-5:
  - Homogeneous functions represent homothetic preferences
- Proposition 2.3-6:
  - Homothetic preferences are represented by functions that are homogeneous of degree 1
- Proposition 2.3-7: Representative Agent

Prop. 2.3-4: Demand Proportional to Income

If preferences are homothetic,

and x^* is optimal given income I,

Then θx^* is optimal given income θI.

Proof:

Let x^{**} be optimal given income θI,

Then $x^{**} \succsim \theta x^*$ since θx^* is feasible with θI.

By revealed preferences, $x^* \succsim \dfrac{1}{\theta} x^{**}$ ($\frac{1}{\theta} x^{**}$ feasible)

By homotheticity, $\theta x^* \succsim x^{**}$

Thus, $\theta x^* \sim x^{**}$ (optimal for income θI)

If preferences are represented by $U(\lambda x) = \lambda^k U(x)$,

Then preferences are homothetic.

Proof:

Suppose $x \succsim y$,

Then $U(x) \geq U(y)$.

Since $U(x)$ is homogeneous,

$$U(\lambda x) = \lambda^k U(x) \geq \lambda^k U(y) = U(\lambda y)$$

Thus, $\lambda x \succsim \lambda y$ i.e. Preferences are homothetic.

Prop. 2.3-6: Representation of Homothetic Pref.

If preferences are homothetic,
They can be represented
by a function that is
homogeneous of degree 1.

Proof: $\hat{e} = (1, \cdots, 1)$

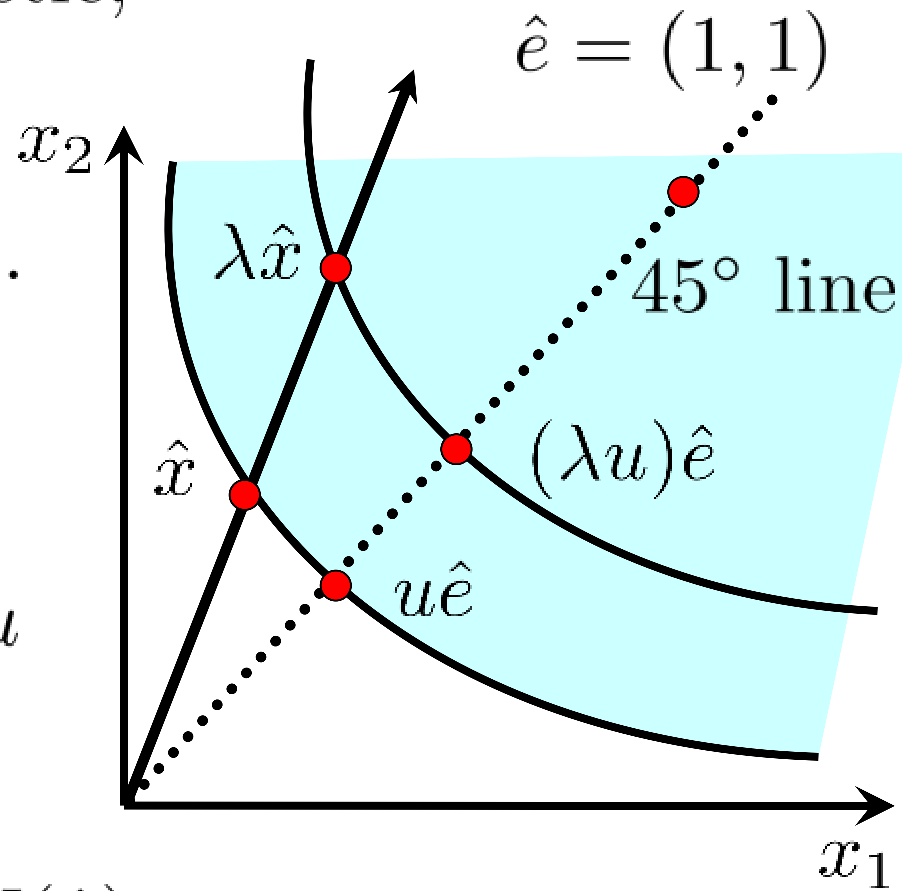
For $\hat{x}$, exists $u\hat{e} \sim \hat{x}$

Utility function $U(x) = u$

By homotheticity,

$$\lambda\hat{x} \sim (\lambda u)\hat{e}$$

Hence, $U(\lambda\hat{x}) = \lambda u = \lambda U(\hat{x})$



Proposition 2.3-7: Representative Preferences

$$x(p,I) = \arg\max_x \left\{ U(x) \middle| p \cdot x \le I \right\},\ U \text{ homothetic}$$

$$\Rightarrow \sum_{h=1}^{H} x(p, I^h) = x(p, I^R),\ \ I^R = \sum_{h=1}^{H} I^h$$

If a group of consumers have the same homothetic preferences,

Then group demand is equal to demand of a representative member holding all the income.

Proposition 2.3-7: Representative Preferences

$$x(p,I) = \arg\max_x \left\{ U(x) \middle| p \cdot x \le I \right\},\ U \text{ homothetic}$$

$$\Rightarrow \sum_{h=1}^{H} x(p, I^h) = x(p, I^R),\ \ I^R = \sum_{h=1}^{H} I^h$$

Proof: $x(p,1) = \arg\max_x \left\{ U(x) \middle| p \cdot x \le 1 \right\}$

$$U \text{ homothetic} \Rightarrow x(p, I^h) = I^h x(p,1)$$

$$\Rightarrow \sum_{h=1}^{H} x(p, I^h) = \sum_{h=1}^{H} I^h x(p,1) = I^R x(p,1)$$

$$= x(p, I^R) \text{ by homotheticity}$$

Summary of 2.3

- Revealed Preference:
  - Compensated Law of Demand
  - Concave Minimized Expenditure Function
- Indirect Utility Function:
  - Roy's Identity: Recovering demand function
- Homothetic Preferences:
  - Demand is proportional to income
  - Utility function is homogeneous of degree 1
  - Group demand as if one representative agent
- Homework: (Optional: Exercise 2.3-3)

Homework: 2008 Midterm Q2– Roy's Identity

1. Draw their income expansion path for two consumers, A and B, with utility functions:

$$u_A(x_1^A, x_2^A) = -\frac{A_1}{x_1^A} - \frac{A_2}{x_2^A} \text{ if } x_1^A \cdot x_2^A > 0,$$

$$u_B(x_1^B, x_2^B) = \min\{2x_1^B, 3x_2^B\}.$$

2. Derive the indirect utility function $V_i(p, I)$
   – Can you use the Roy's Identity to derive each consumer's demand? Why or why not?
3. Derive $x_i^{h^*}(p, I)$, consumer h's demand functions for consumer h and commodity i

In-Class Homework: RPP and Exercise 2.3-1

Consider firm problem $\Pi(p) = \max_{y} \{p \cdot y | y \in \mathcal{Y}^f\}$

For y^0 be profit maximizing for prices p^0

y^1 be profit maximizing at prices p^1

$y^0, y^1 \in \mathcal{Y}^f \quad \Rightarrow \Delta p \cdot \Delta y \leq 0$

- $$U(x) = \times_{j=1}^{n} x_j^{\alpha_j}, \alpha_1 + \cdots + \alpha_n = 1$$

a) Solve for the indirect utility function $V_i(p, I)$

b) Explain why you can "invert" your results to obtain the expenditure function

c) Hence solve for the Expenditure Function

In-Class Homework: Exercise 2.3-2

- Bev has a utility function $U(x) = \sqrt{x_1 x_2} + x_3$

a) Suppose she allocates y towards the purchase of commodity *1* and *2* and purchases x_3 units of commodity *3*. Show that her resulting utility is

$$U^*(x_3, y) = \frac{y}{2\sqrt{p_1 p_2}} + x_3$$

b) Given this preliminary optimization problem has been solved, her budget constraint is $p_3 x_3 + y \leq I$ Solve for her optimizing values of x_3 and y.

– Under what conditions, if any, is she strictly worse off if she is told that she can consume at most 2 of the 3 available commodities?