Budget Constrained Choice with Two Commodities

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(Lecture 5, Micro Theory I)

The Consumer Problem

- We have some powerful tools:
  - Constrained Maximization (Shadow Prices)
  - Envelope Theorem (Changing Environment)
- Can help us understand consumer behavior?

  Such as:
  - "maximizing utility, facing a budget constraint"
  - "minimizing cost, maintaining certain welfare level"

Key Problems to Consider

- Total Price Effect = Sub. Eff. + Income Eff.
- Consumer Problem: How can consumer's Utility Maximization result in demand?
  - Income Effect: How does an increase/decrease in income (budget) affect demand?
- Dual Problem: How is Minimizing Expenditure related to Maximizing Utility?
  - Substitution Effect: How does an increase in commodity price affect compensated demand?

Why do we care about this? Public Policy!

- Taiwan's ministry of defense has to decide whether to buy more fighter jets, or more submarines given a tight budget
- How does the military rank each combination?
- How do they choose which combination to buy?
- How would a price change affect their decision?
- How would a boycott in defense budget affect their decision?

Continuous Demand Function

A Consumer with income I, facing prices p_1, p_2

$$\max_x \{U(x) | p \cdot x \leq I, x \in \mathbb{R}^2_+\}$$

- Assume: LNS (local non-satiation)
  - Then, consumer spends all his/her income!
- $U(x)$ is continuous, strictly quasi-concave on $\mathbb{R}^2_+$
  - There is a unique solution $x^0 = x(p, I)$
- Then, by Prop. 2.2-1, $x(p, I)$ must be continuous.
  - aka Theory of Maximum I (Prop. C.4-1 on p. 581)

- For f continuous, define

$$F(\alpha) = \max_x \left\{ f(x,\alpha) \middle| x \geq 0, \quad x \in X(\alpha) \subset \mathbb{R}^n, \right.$$

$$\left. \alpha \in A \subset \mathbb{R}^m \right\}$$

- If (i) for each α there is a unique

$$x^*(\alpha) = \arg\max_x \left\{ f(x,\alpha) \middle| x \geq 0, x \in X(\alpha), \alpha \in A \right\}$$

- and (ii) $X(\alpha)$ is a compact-valued correspondence that is continuous at α^0
- Then, $x^*(\alpha)$ is continuous at α^0

Appendix C: Prop.C.4-1 Theory of Maximum I

- $U(x)$ is continuous, strictly quasi-concave on $\mathbb{R}^2_+$

$$F(\alpha) = \max_x \{U(x) | p \cdot x \leq I, x \in \mathbb{R}^2_+\}$$

- If (i) for each α there is a unique

$$x^0 = x(p, I)$$

- A Consumer with income I, facing prices p_1, p_2
- Then, $x(p, I)$ must be continuous.

Some Stronger Convenience Assumptions

- Assume:
- $U(x)$ is continuously differentiable on $\mathbb{R}^2_+$
  - FOC is gradient vector of utility (& constraints)
- LNS-plus: $$\frac{\partial U}{\partial x}(x) \gg 0 \text{ for all } x \in \mathbb{R}^2_+$$
  - MU > 0: Preferences are strictly increasing
- No corners: $$\lim_{x_j \to 0} \frac{\partial U}{\partial x_j} = \infty, j = 1, 2$$
  - Always wants to consume some of everything

Indifference Curve Analysis (Lagrangian Ver.)

A Consumer with income I, facing prices p_1, p_2

$$\max_{x} \{U(x) | p \cdot x \leq I, x \in \mathbb{R}^2_+\}$$

Lagrangian is $\mathfrak{L} = U + \lambda(I - p \cdot x)$

$$(FOC) \qquad \frac{\partial \mathfrak{L}}{\partial x_j} = \frac{\partial U}{\partial x_j}(x^*) - \lambda p_j = 0, j = 1, 2$$

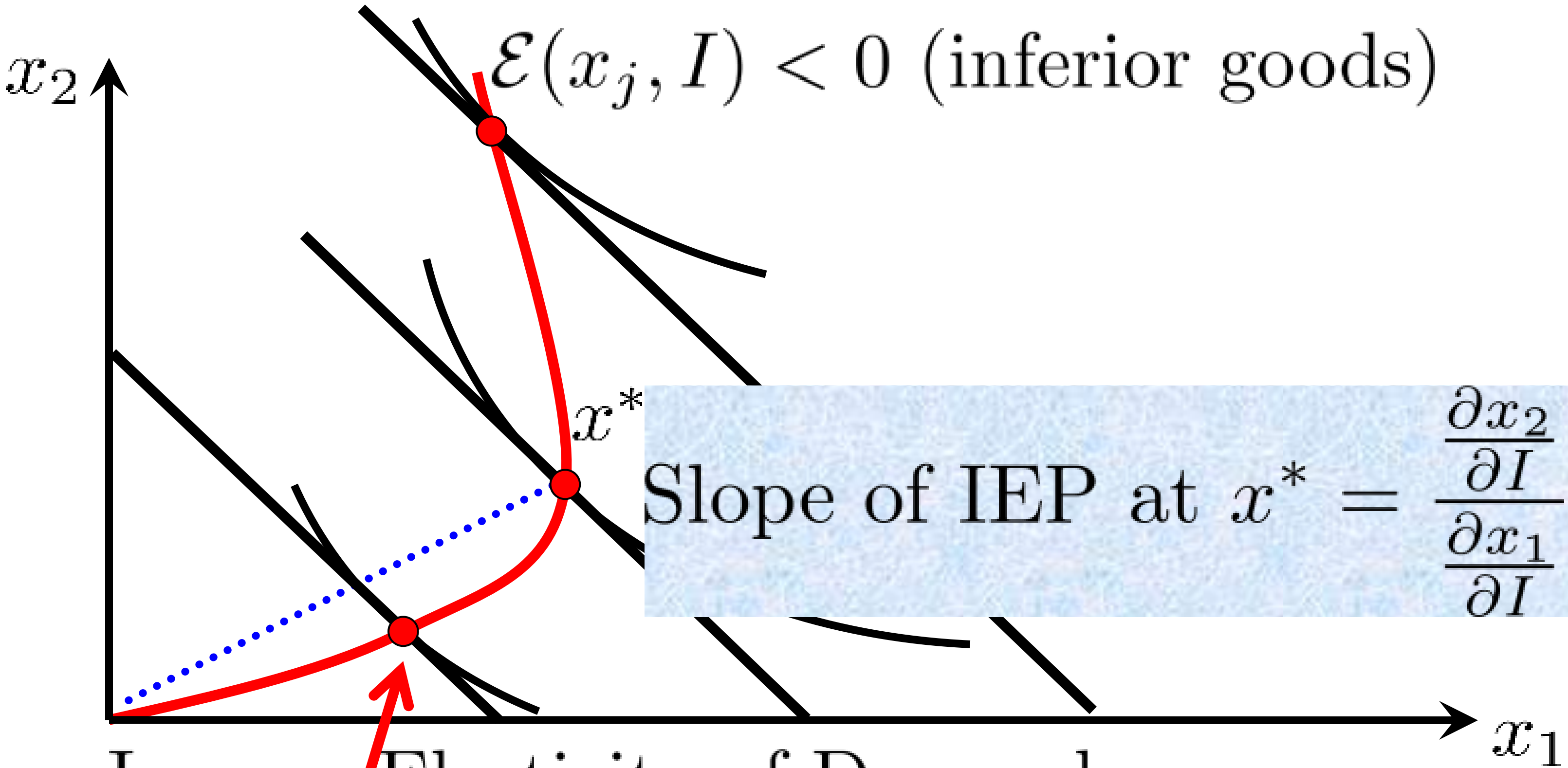
$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \lambda$$

Meaning of FOC

1. Same marginal value for last dollar spent on each commodity $$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \lambda$$
   - Does Taiwan get the same defense MU on fighter jets and submarines?
2. Indifference Curve tangent to Budget Line

$$MRS(x^*) = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{p_1}{p_2}$$

Income Effects



Slope of IEP at $x^* = \dfrac{\frac{\partial x_2}{\partial I}}{\frac{\partial x_1}{\partial I}} > \dfrac{x_2^*}{x_1^*}$

Income Elasticity of Demand

$$\mathcal{E}(x_j, I) = \frac{I}{x_j}\frac{\partial x_j}{\partial I} > 0 \text{ (normal goods)}$$

Income Effects

- If IEP is steeper than the line joining *0* & x^*
- Then, $$\text{Slope of IEP at } x^* = \frac{\frac{\partial x_2}{\partial I}}{\frac{\partial x_1}{\partial I}} > \frac{x_2^*}{x_1^*}$$
- Or, $$\mathcal{E}(x_2, I) = \frac{I}{x_2}\frac{\partial x_2}{\partial I} > \mathcal{E}(x_1, I) = \frac{I}{x_1}\frac{\partial x_1}{\partial I}$$
- Lemma 2.2-2: Expenditure share weighted income elasticity average $= 1$
- So, $\mathcal{E}(x_2, I) > 1 > \mathcal{E}(x_1, I)$

– Expenditure-Share Weighted Average of IE = 1

$$k_1\mathcal{E}(x_1^*, I) + k_2\mathcal{E}(x_2^*, I) = 1$$

- Where $k_j = \frac{p_j x_j^*}{I}$ is the expenditure share of x_j

Proof:

- Budget Constraint $\Rightarrow p_1 \frac{\partial x_1^*}{\partial I} + p_2 \frac{\partial x_2^*}{\partial I} = 1$

$$\Rightarrow \underbrace{\left(\frac{p_1 x_1^*}{I}\right)}_{k_1} \underbrace{\frac{I}{x_1^*}\frac{\partial x_1^*}{\partial I}}_{\mathcal{E}(x_1^*, I)} + \underbrace{\left(\frac{p_2 x_2^*}{I}\right)}_{k_2} \underbrace{\frac{I}{x_2^*}\frac{\partial x_2^*}{\partial I}}_{\mathcal{E}(x_2^*, I)} = 1$$

Three Examples

- Quasi-Linear Convex Preference

$$U(x) = v(x_1) + \alpha x_2$$

- Cobb-Douglas Preferences

$$U(x) = x_1^{\alpha_1} x_2^{\alpha_2}, \alpha_1, \alpha_2 > 0$$

- CES Utility Function

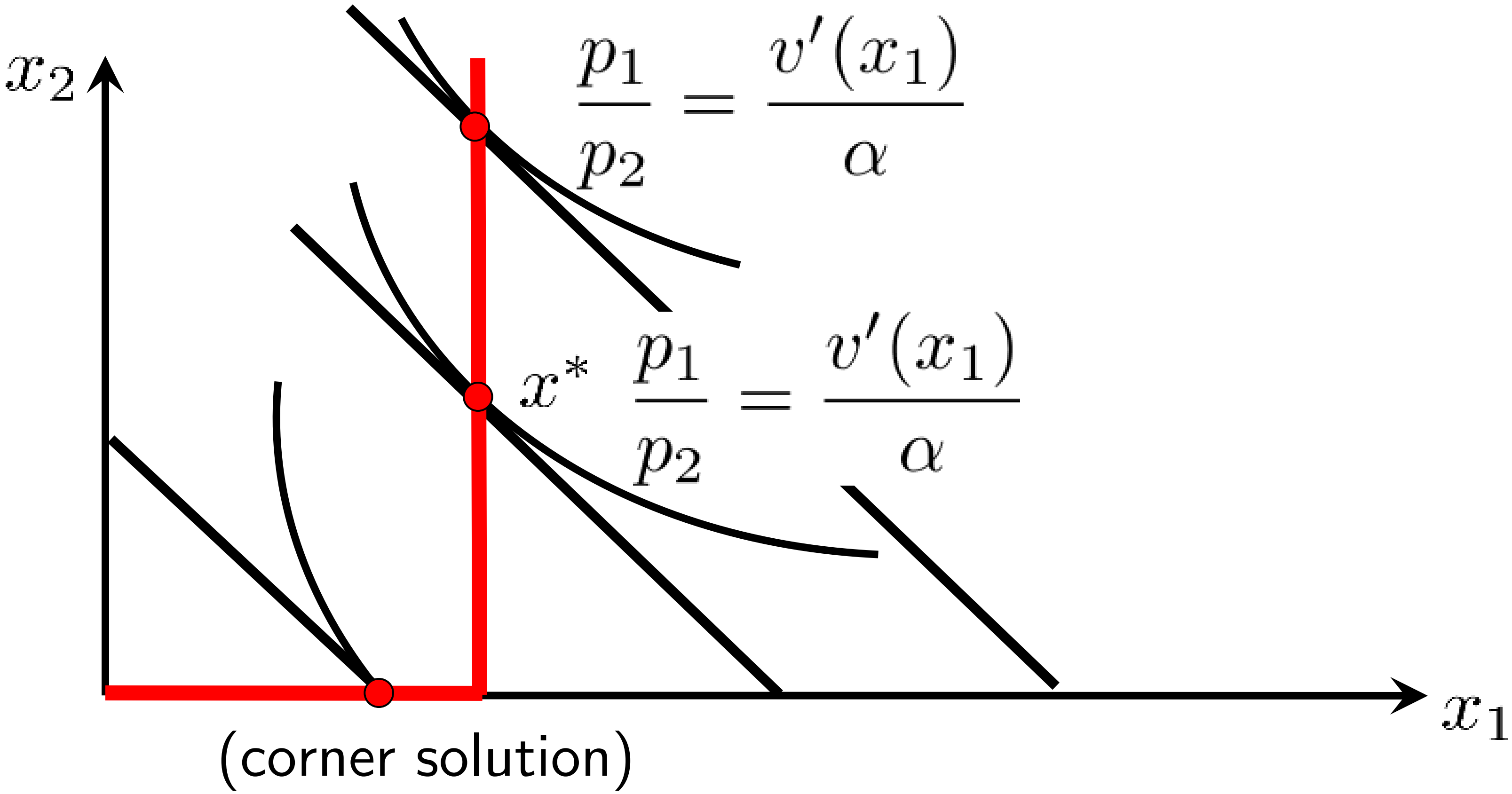
$$U(x) = \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}} \right)^{\frac{1}{1-\frac{1}{\theta}}}$$

Quasi-Linear Convex Utility

$$\max_x \left\{ U(x) = v(x_1) + \alpha x_2 | p_1 x_1 + p_2 x_2 \le I, x \in \mathbb{R}^2_+ \right\}$$

- FOC: $$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \frac{v'(x_1)}{p_1} = \frac{\alpha}{p_2} (= \lambda)$$
- Implication: $\frac{p_1}{p_2} = \frac{v'(x_1)}{\alpha}$ (MRS=price)
- Note that x_2 is irrelevant...
- What does this mean?

Income Effect



- Vertical Income Expansion Path (at interior)

Cobb-Douglas Preferences

$$\max_{x_1,x_2} U(x_1,x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$$
$$\text{s.t. } P_{x_1}\cdot x_1 + P_{x_2}\cdot x_2 \le I = P_{x_1}\cdot\omega_{x_1} + P_{x_2}\cdot\omega_{x_2}$$

$$\mathcal{L} = x_1^{\alpha_1} x_2^{\alpha_2} + \lambda\cdot[I - P_{x_1}\cdot x_1 - P_{x_2}\cdot x_2]$$

FOC: (for interior solutions)

$$\frac{\partial\mathcal{L}}{\partial x_1} = \alpha_1\cdot\frac{x_2^{\alpha_2}}{x_1^{\alpha_2}} - \lambda\cdot P_{x_1} = 0$$
$$\frac{\partial\mathcal{L}}{\partial x_2} = \alpha_2\cdot\frac{x_1^{\alpha_1}}{x_2^{\alpha_1}} - \lambda\cdot P_{x_2} = 0$$
$$\frac{\partial\mathcal{L}}{\partial\lambda} = I - P_{x_1}\cdot x_1 - P_{x_2}\cdot x_2 = 0$$

Cobb-Douglas Preferences

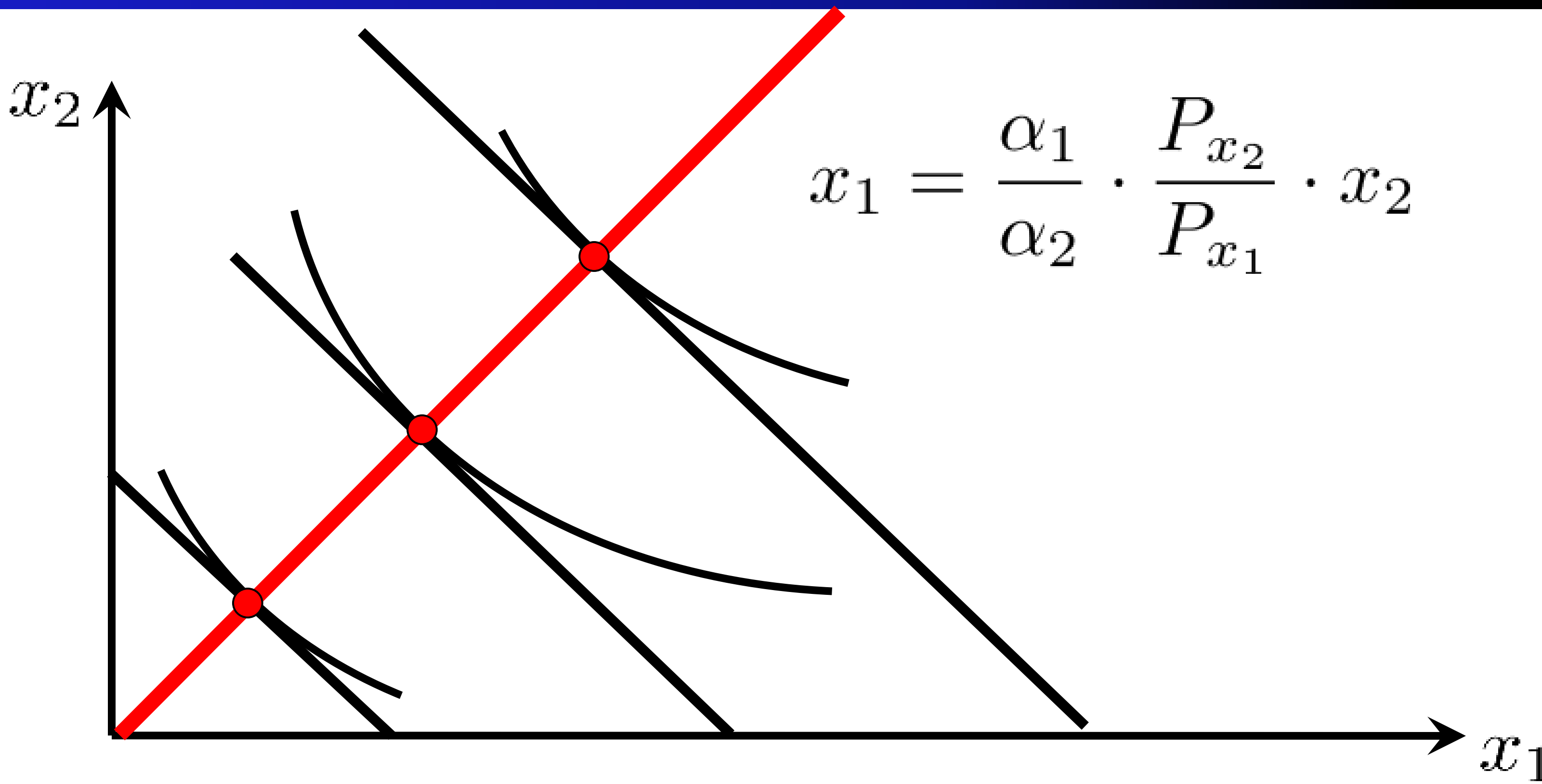
- Meaning of FOC: $MRS = \frac{P_{x_1}}{P_{x_2}}$

$$\frac{P_{x_1}}{P_{x_2}} = \frac{\alpha_1}{\alpha_2} \cdot \frac{x_2}{x_1} \qquad \Rightarrow x_1 = \frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}} \cdot x_2$$

$$\Rightarrow I = P_{x_1} \cdot x_1 + P_{x_2} \cdot x_2 = \frac{\alpha_1 + \alpha_2}{\alpha_2} \cdot P_{x_2} \cdot x_2$$

$$\Rightarrow x_2^* = \frac{\alpha_2}{\alpha_1 + \alpha_2} \cdot \frac{I}{P_{x_2}}, \; x_1^* = \frac{\alpha_1}{\alpha_1 + \alpha_2} \cdot \frac{I}{P_{x_1}}$$

Income Effect



$$x_1 = \frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}} \cdot x_2$$

- Linear Income Expansion Path...

CES Utility Function

$$U(x)=\left(\alpha_1 x_1^{1-\frac{1}{\theta}}+\alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}}$$

$$\mathcal{L}=\left(\alpha_1 x_1^{1-\frac{1}{\theta}}+\alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}}+\lambda\cdot\left[I^A-P_x\cdot x-P_y\cdot y\right]$$

- FOC: (for interior solutions)

$$\frac{\partial\mathcal{L}}{\partial x_1}=\alpha_1 x_1^{-\frac{1}{\theta}}\cdot\left(\alpha_1 x_1^{1-\frac{1}{\theta}}+\alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{\theta-1}}-\lambda\cdot P_{x_1}=0$$

$$\frac{\partial\mathcal{L}}{\partial x_2}=\alpha_2 x_2^{-\frac{1}{\theta}}\cdot\left(\alpha_1 x_1^{1-\frac{1}{\theta}}+\alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{\theta-1}}-\lambda\cdot P_{x_2}=0$$

$$\frac{\partial\mathcal{L}}{\partial\lambda}=I-P_{x_1}\cdot x_1-P_{x_2}\cdot x_2=0$$

CES Utility Function

$$\frac{P_{x_1}}{P_{x_2}} = \frac{\alpha_1}{\alpha_2} \cdot \left(\frac{x_2}{x_1}\right)^{\frac{1}{\theta}} \Rightarrow x_1 = \left(\frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}}\right)^{\theta} \cdot x_2$$

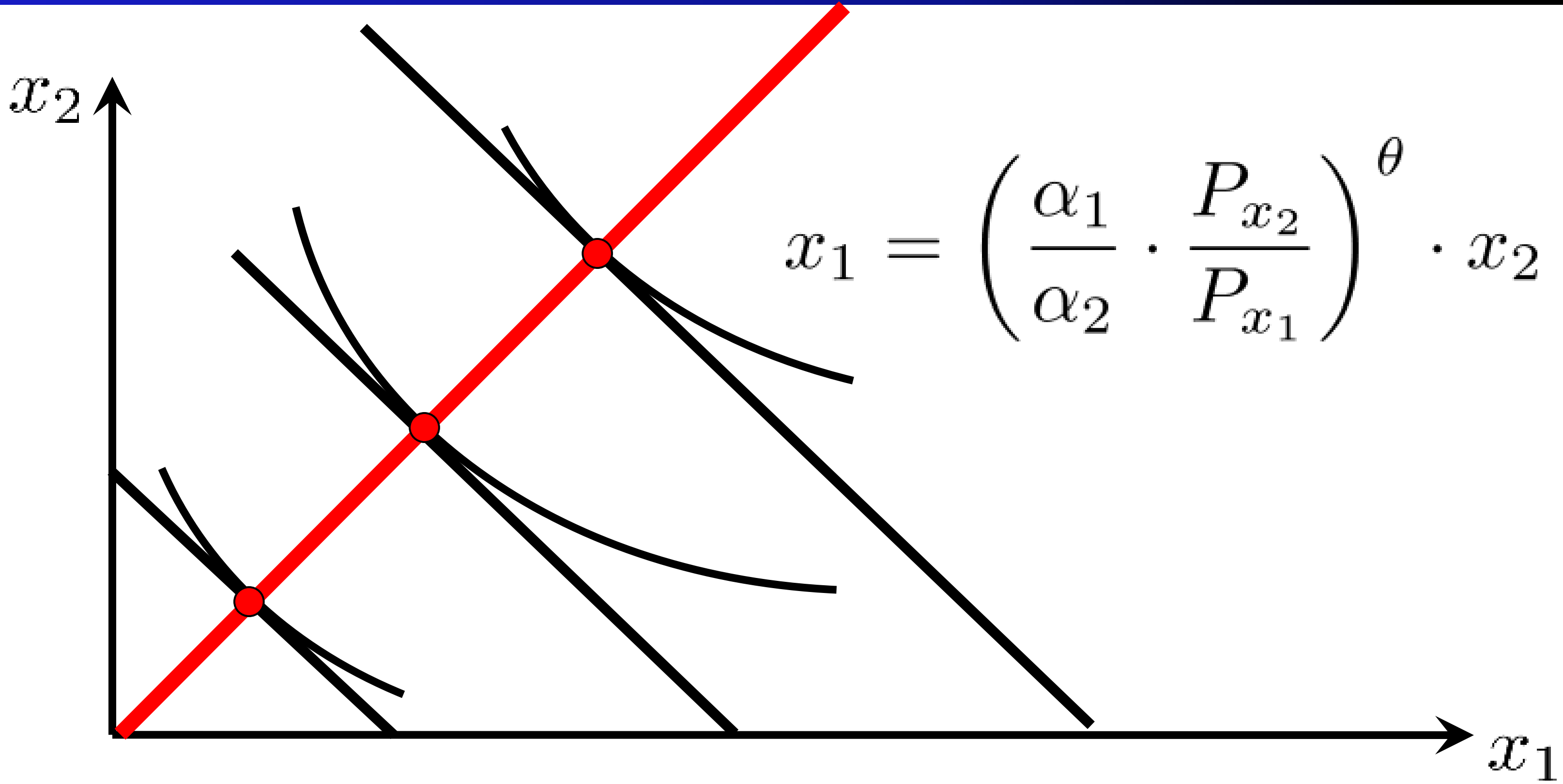
$$\Rightarrow I = P_{x_1} \cdot x_1 + P_{x_2} \cdot x_2$$

$$= \left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\theta} \cdot \left(\frac{P_{x_2}}{P_{x_1}}\right)^{\theta-1}\right] \cdot P_{x_2} \cdot x_2$$

$$\Rightarrow x_2^* = \frac{\alpha_2^{\theta} P_{x_1}^{\theta-1}}{\alpha_1^{\theta} P_{x_2}^{\theta-1} + \alpha_2^{\theta} P_{x_1}^{\theta-1}} \cdot \frac{I}{P_{x_2}},$$

$$x_1^* = \frac{\alpha_1^{\theta} P_{x_1}^{\theta-1}}{\alpha_1^{\theta} P_{x_2}^{\theta-1} + \alpha_2^{\theta} P_{x_1}^{\theta-1}} \cdot \frac{I}{P_{x_1}}$$

Income Effect



$$x_1 = \left(\frac{\alpha_1}{\alpha_2} \cdot \frac{P_{x_2}}{P_{x_1}} \right)^{\theta} \cdot x_2$$

- Linear Income Expansion Path...
- Cobb-Douglas is a special case of CES! $(\theta = 1)$

Dual Problem: Minimizing Expenditure

- Consider the least costly way to achieve $\overline{U}$

$$M(p,\overline{U}) = \min_x \{p \cdot x | U(x) \geq \overline{U}\}$$

- How can you solve this?

$$\mathfrak{L} = -p \cdot x + \lambda(U(x) - \overline{U})$$

$$(FOC) \quad \frac{\partial \mathfrak{L}}{\partial x_j} = -p_j + \lambda \frac{\partial U}{\partial x_j}(x^*) = 0, j = 1,2$$

$$\frac{p_1}{\frac{\partial U}{\partial x_1}} = \frac{p_2}{\frac{\partial U}{\partial x_2}} = \lambda \Rightarrow \text{Solve for } \underline{\underline{x^c(p,\overline{U})}}$$

Compensated Demand

- Can view it as the “sister” (dual) problem of:

$$\max_x \{U(x) | x \geq 0, p \cdot x \leq I\}$$

- Because we have:
- Lemma 2.2-3 Duality Lemma
- LNS holds & $x^* \in \arg\max_x \{U(x) | x \geq 0, p \cdot x \leq I\}$

Max U

- Then,

$$U(x) \geq U(x^*) \Rightarrow p \cdot x \geq p \cdot x^*$$

- So, $x^* \in \arg\min_x \{p \cdot x | x \geq 0, U(x) \geq U(x^*)\}$

min E

Lemma 2.2-3 Duality Lemma

- LNS holds & $x^* \in \arg\max_x \{U(x) | x \geq 0, p \cdot x \leq I\}$

Max U

- Then,

$$U(x) \geq U(x^*) \Rightarrow p \cdot x \geq p \cdot x^*$$

min E

- So, $x^* \in \arg\min_x \{p \cdot x | x \geq 0, U(x) \leq U(x^*)\}$

Proof: Consider $\hat{x}$ such that $p \cdot \hat{x} < I$.

- $N(\hat{x}, \delta) \subset \{x | x \geq 0, p \cdot x \leq I\}$ for some small δ
- LNS means there exists $\hat{\hat{x}}$ such that $\hat{\hat{x}} \succ \hat{x}$, so

$$p \cdot \hat{x} < I \Rightarrow U(\hat{x}) < U(x^*)$$

(Equivalent!)

Expenditure Function and Value Function

- For utility $\overline{U}$ and price vector p, Expenditure Function is $M(p,\overline{U}) = \min_x \{p \cdot x | U(x) \geq U(\overline{x})\}$
- Claim: The Value Function (maximized utility)
  $$V(p, I) = \max_x \{U(x) \big| p \cdot x \leq I\}$$
- is strictly increasing over I (by LNS).
- Then, for any $\overline{U}$, there is a unique income M such that $\overline{U} = V(p, M)$
- Inverting this, we can solve for $M(p,\overline{U})$

Claim: Value Function is Strictly Increasing

- Claim: The Value Function is strictly increasing
$$V(p, I) = \max_x \{U(x) \big| p \cdot x \le I\}$$
- Proof: If not, there exists $I_1 < I_2$ and x_1^*, x_2^*
  - such that $U(x_1^*) = V(p, I_1) \ge V(p, I_2) = U(x_2^*)$
- LNS yields $p \cdot x_1^* = I_1 < I_2$, and there exists $\hat{x}$
  - such that $U(\hat{x}) > U(x_1^*) \ge U(x_2^*)$
- In neighborhood $N(x_1^*, \delta) \subset \{x \big| x \ge 0, p \cdot x \le I_2\}$
- But this means $\hat{x}$ solves $V(p, I_2)$ not x_2^*. (→←)

Dual Problem: Minimizing Expenditure

- In fact, minimizing expenditure yields:

$$\frac{p_1}{\frac{\partial U}{\partial x_1}} = \frac{p_2}{\frac{\partial U}{\partial x_2}} = \lambda$$

- Maximize Utility's FOC yields:

$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \lambda$$

- This close relationship between $x^c(p, \overline{U})$ and $x(p, I)$ indicates why they are "sisters"...

Compensated Demand

$$x^c(p,\overline{U}) \text{ solves } M(p,\overline{U}) = \min_x \{p \cdot x | U(x) \leq \overline{U}\}$$

- By Envelope Theorem:
- Effect of "Compensated" Price Change is
  - aka Substitution Effect…

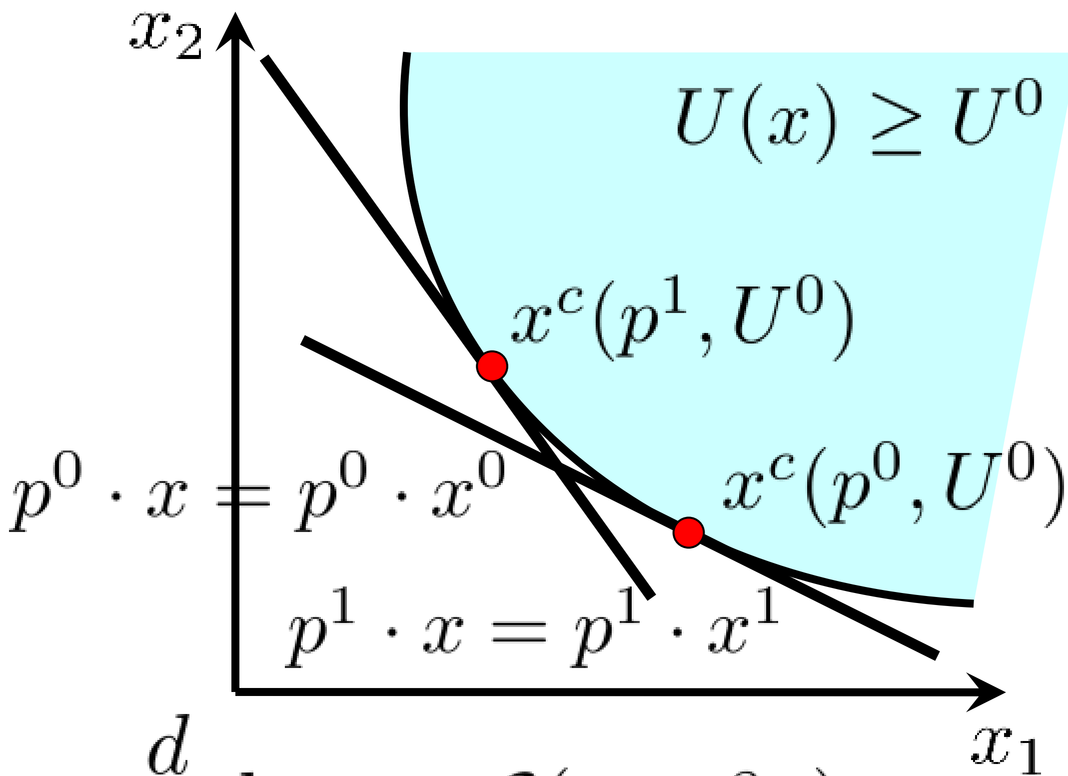
$$\frac{\partial M}{\partial p_j} = x_j^c(p, U^0)$$

  - How much more does Taiwan have to pay if the price of submarines increase (to maintain the same level of defense)?

Elasticity of Substitution (Compensated Demand)

$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, \frac{p_1}{p_2}\right)$$

- The change in consumption ratio in response to a change in prices…
- Note that: (p.502)



$$\mathcal{E}(y,x) = \frac{x}{y}\cdot\frac{dy}{dx} = x\frac{d}{dx}\ln y = \mathcal{E}(\alpha y, \beta x)$$

Lemma 2.2-4 $\sigma = \mathcal{E}\left(x_2^c, p_1\right) - \mathcal{E}\left(x_1^c, p_1\right)$

- Since $\mathcal{E}(y,x) = \frac{x}{y} \cdot \frac{dy}{dx} = x\frac{d}{dx}\ln y = \mathcal{E}(\alpha y, \beta x)$

$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, \frac{p_1}{p_2}\right) = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right)$$

$$= p_1\frac{d}{dp_1}\ln\left(\frac{x_2^c}{x_1^c}\right) = p_1\frac{d}{dp_1}\left(\ln x_2^c - \ln x_1^c\right)$$

$$= p_1\frac{d}{dp_1}\left(\ln x_2^c\right) - p_1\frac{d}{dp_1}\left(\ln x_1^c\right)$$

$$= \mathcal{E}\left(x_2^c, p_1\right) - \mathcal{E}\left(x_1^c, p_1\right)$$

Prop. 2.2-5 ES & Compensated Price Elasticity

- Relation between Elasticity of Substitution and Compensated Own Price Elasticity

1) $$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right) = \frac{\mathcal{E}(x_2^c, p_1)}{k_1}, \quad k_1 = \frac{p_1 x_1}{p \cdot x}$$

$$\left(\frac{\text{compensated cross price elasticity}}{\text{expenditure share}}\right)$$

2) $$\mathcal{E}(x_1^c, p_1) = -(1 - k_1)\sigma$$

Prop. 2.2-5 ES & Compensated Price Elasticity

- On indifference curve, $U\left(x_1^c(p,\overline{U}), x_2^c(p,\overline{U})\right) = \overline{U}$
- Hence, $\frac{\partial U}{\partial x_1}\frac{\partial x_1^c}{\partial p_1} + \frac{\partial U}{\partial x_2}\frac{\partial x_2^c}{\partial p_1} = 0$
- By FOC, $\frac{p_1}{\frac{\partial U}{\partial x_1}} = \frac{p_2}{\frac{\partial U}{\partial x_2}} \Rightarrow p_1\frac{\partial x_1^c}{\partial p_1} + p_2\frac{\partial x_2^c}{\partial p_1} = 0$

$$\mathcal{E}\left(x_1^c, p_1\right) = \frac{p_1}{x_1^c}\frac{\partial x_1^c}{\partial p_1} = -\frac{p_2}{x_1^c}\frac{\partial x_2^c}{\partial p_1}$$

$$= -\left(\frac{p_2 x_2^c}{p_1 x_1^c}\right)\frac{p_1}{x_2^c}\frac{\partial x_2^c}{\partial p_1} = -\frac{k_2}{k_1}\mathcal{E}\left(x_2^c, p_1\right)$$

$$k_j = \frac{p_j x_j^c}{p \cdot x^c}$$

Prop. 2.2-5 ES & Compensated Price Elasticity

- Since $\mathcal{E}\left(x_1^c, p_1\right) = -\dfrac{k_2}{k_1}\mathcal{E}\left(x_2^c, p_1\right)$
- Lemma 2.2-4 becomes:

$$\sigma = \mathcal{E}(x_2^c, p_1) - \mathcal{E}(x_1^c, p_1)$$

$$= \mathcal{E}(x_2^c, p_1) \cdot \left(1 + \frac{k_2}{k_1}\right) = \frac{\mathcal{E}(x_2^c, p_1)}{k_1} \quad \text{...(1)}$$

$$= \mathcal{E}(x_1^c, p_1) \cdot \left(-\frac{k_1}{k_2}\right) \cdot \frac{1}{k_1} = -\frac{\mathcal{E}(x_1^c, p_1)}{k_2}$$

- Hence, $\mathcal{E}(x_1^c, p_1) = -k_2\sigma = -(1 - k_1)\sigma$ ...(2)

Compensated own price elasticity bounded/approx. by ES!

- Verify that $\sigma = \theta$ for CES:
- Since $x_1 = \left(\frac{\alpha_1}{\alpha_2}\frac{p_2}{p_1}\right)^{\theta} \cdot x_2 \;\Rightarrow \frac{x_2}{x_1} = \left(\frac{\alpha_2}{\alpha_1} \cdot \frac{p_1}{p_2}\right)^{\theta}$

$$\Rightarrow \ln\left(\frac{x_2^c}{x_1^c}\right) = \theta(\ln p_1 - \ln p_2 + \ln \alpha_2 - \ln \alpha_1)$$

$$\Rightarrow \sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right) = p_1 \cdot \frac{\partial}{\partial p_1}\left[\ln\left(\frac{x_2^c}{x_1^c}\right)\right]$$

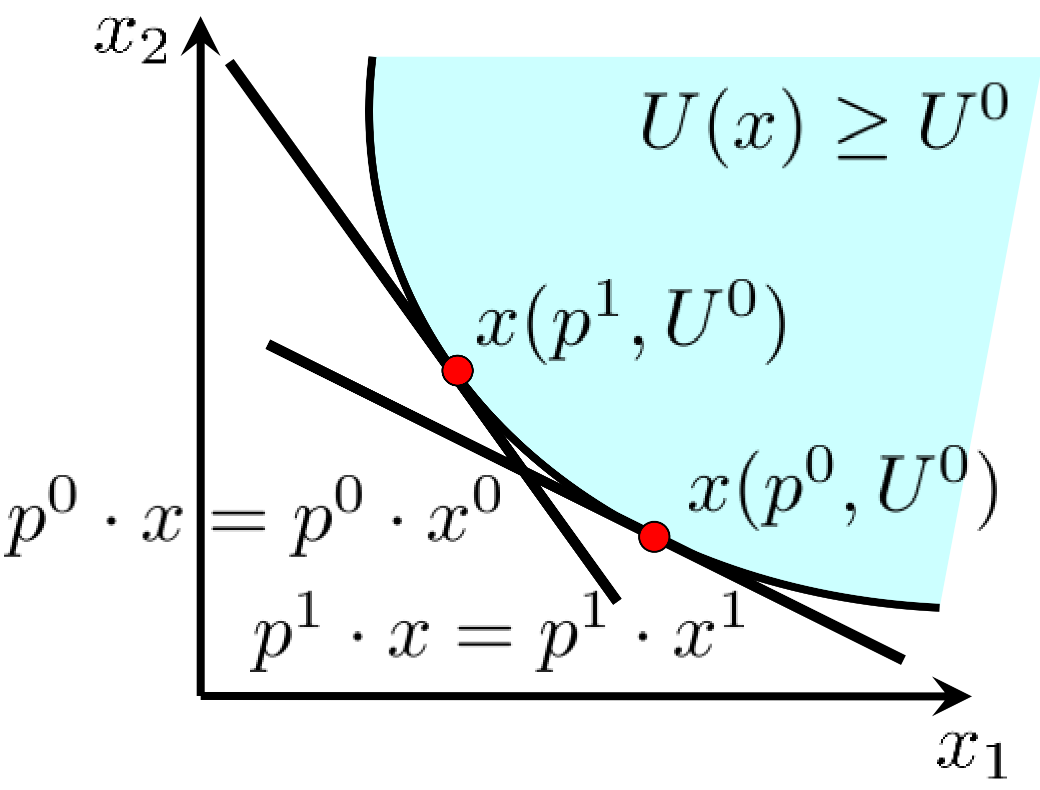
$$= p_1 \cdot \frac{\theta}{p_1} = \theta$$

Summary for Elasticity of Substitution

- 1. $\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, p_1\right)$
- 2. $= \frac{\mathcal{E}(x_2^c, p_1)}{k_1}$

  $= -\frac{\mathcal{E}(x_1^c, p_1)}{1 - k_1}$

  $k_1 = \frac{p_1 x_1}{p \cdot x}$
- 3. $\sigma = \theta$ for CES...



Total Price Effect = Income Ef. + Substit. Ef.

- For $M(p,\overline{U})$ & $x_1(p,I)$
- Compensated Demand:

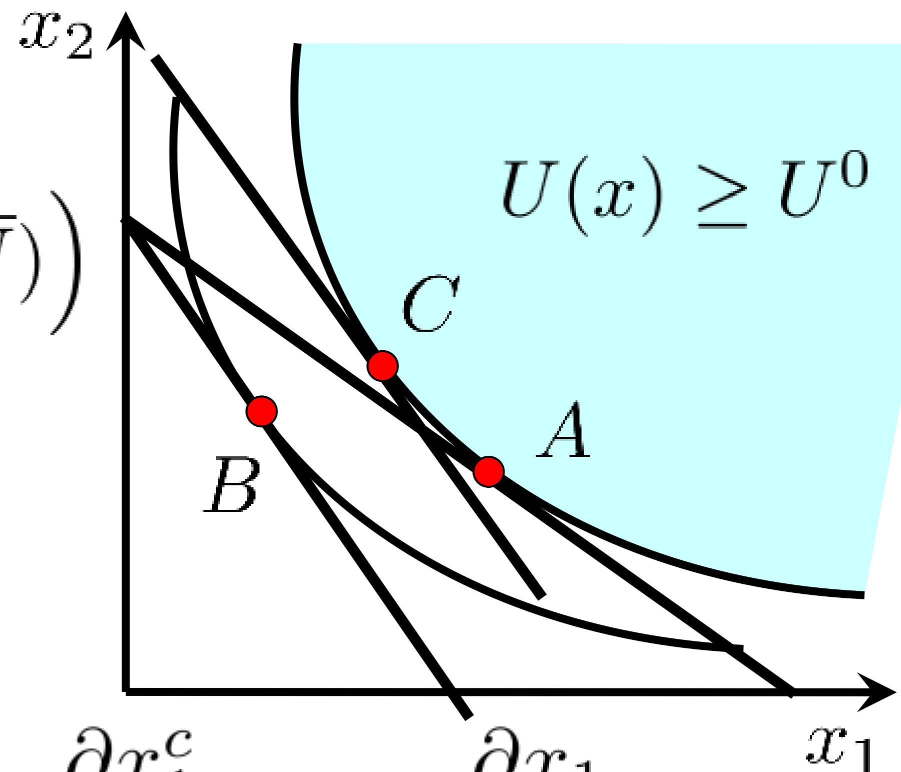
$$x_1^c(p,\overline{U}) = x_1\left(p, M(p,\overline{U})\right)$$

$$\frac{\partial x_1^c}{\partial p_1} = \frac{\partial x_1}{\partial p_1} + \frac{\partial x_1}{\partial I} \cdot \frac{\partial M}{\partial p_1}$$

$$\left(\frac{\partial M}{\partial p_1} = x_1\right)$$

- Slutsky Equation:

$$\underbrace{\frac{\partial x_1}{\partial p_1}}_{A \to B} = \underbrace{\frac{\partial x_1^c}{\partial p_1}}_{A \to C} \underbrace{- x_1 \cdot \frac{\partial x_1}{\partial I}}_{C \to B}$$



Prop. 2.2-6 Decomposition of Own Price Elast.

- Slutsky Equation: $\dfrac{\partial x_1}{\partial p_1} = \dfrac{\partial x_1^c}{\partial p_1} - x_1 \cdot \dfrac{\partial x_1}{\partial I}$
- Elasticity Version:

$$\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1} = \frac{p_1}{x_1}\frac{\partial x_1^c}{\partial p_1} - \frac{p_1 x_1}{I}\frac{I}{x_1} \cdot \frac{\partial x_1}{\partial I}$$

- Or, $\underline{\mathcal{E}(x_1, p_1)} = \underline{\mathcal{E}(x_1^c, p_1)} - \underline{k_1 \cdot \mathcal{E}(x_1, I)}$

$$= \underline{-(1-k_1)\sigma} - \underline{k_1 \cdot \mathcal{E}(x_1, I)}$$

Substitution Effect   Income Effect

– <u>Own price elasticity</u> = weighted average of elasticity of substitution and income elasticity

Summary of 2.2

- Consumer Problem: Maximize Utility
- Income Effect
- Dual Problem: Minimize Expenditure
- Substitution Effect:
  - =Compensated Price Effect
  - Elasticity of Substitution
- Total Price Effect:
  - = Compensated Price Effect + Income Effect
- Homework: Exercise 2.2-4 (Optional: 2.2-5)

In-Class Homework: Exercise 2.2-2

- Show that the price effect on compensated demand is

$$\frac{\partial M}{\partial p_j}(p, U^0) = x_j^c(p, U^0)$$

  – Hint: Convert expenditure minimization into a maximization problem, write down the Lagrangian and use the Envelope Theorem...

In-Class Homework: Exercise 2.2-3

- [Elasticity of Substitution]

a) Show that $\mathcal{E}(y(x), z(x)) = \dfrac{\frac{d}{dx}\ln y}{\frac{d}{dx}\ln z}$.

b) Use this to show that $\mathcal{E}\left(\dfrac{1}{y}, \dfrac{1}{x}\right) = \mathcal{E}(y, x)$

and that $$\mathcal{E}\left(\frac{y_2}{y_1}, x\right) = \mathcal{E}(y_2, x) - \mathcal{E}(y_1, x)$$

c) Use these results to prove Lemma 2.2-4.

In-Class Homework: Exercise 2.2-6

- [Parallel Income Expansion Paths]
- A consumer faces price vector p, has income I and utility function $U(x) = -\alpha_1 e^{-Ax_1} - \alpha_2 e^{-Ax_2}$

a) Show that her optimal consumption bundle satisfies the following: $x_2 - x_1 = a + b \ln \frac{p_1}{p_2}$

b) Depict her Income Expansion Paths.