Refinements of Bayesian Nash Equilibrium

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12/27/2012

(Lecture 11, Micro Theory I)

Market Entry Game w/ Incomplete Information

Example of many BNE; some are less plausible than others:

If Entrant's backing is weak

Agent 2: Incumbent

Agent 1: Entrant		Fight	Share
	Enter	-2, 40	3, 30
	Out	0, 60	0, 60

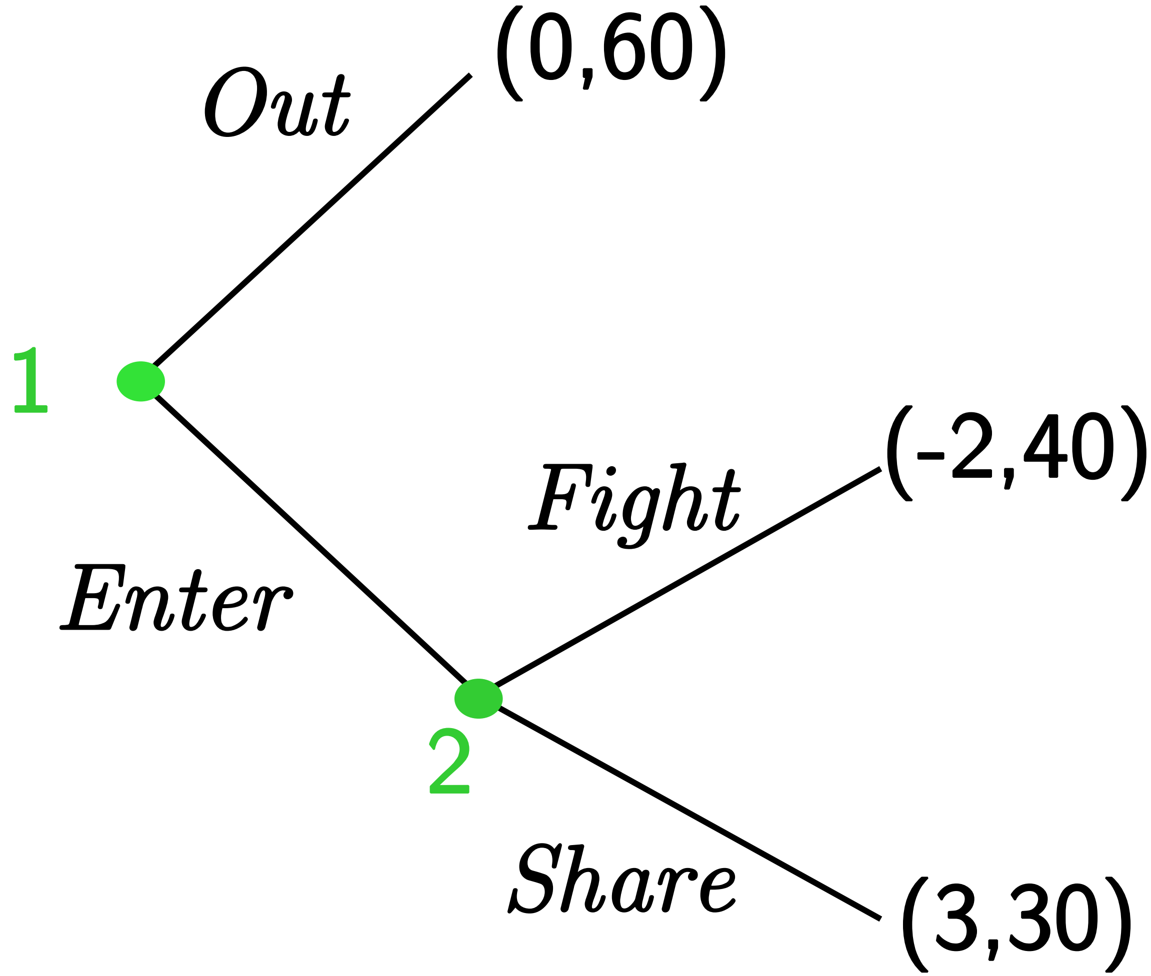
Market Entry Game w/ Incomplete Information

If Entrant's backing is strong

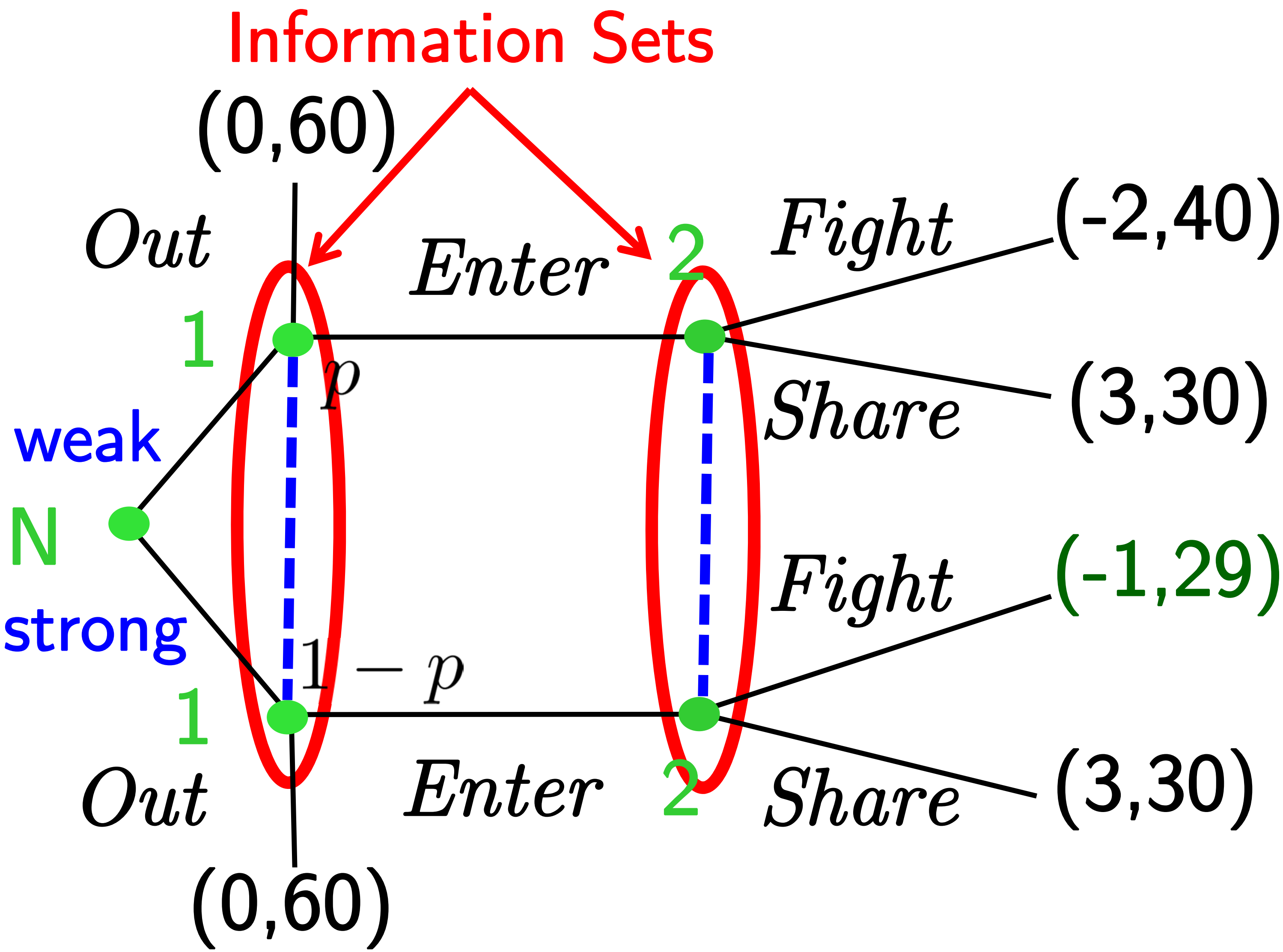
Agent 2: Incumbent

Agent 1: Entrant		Fight	Share
	Enter	-1, 29	3, 30
	Out	0, 60	0, 60

Market Entry Game w/ Incomplete Information



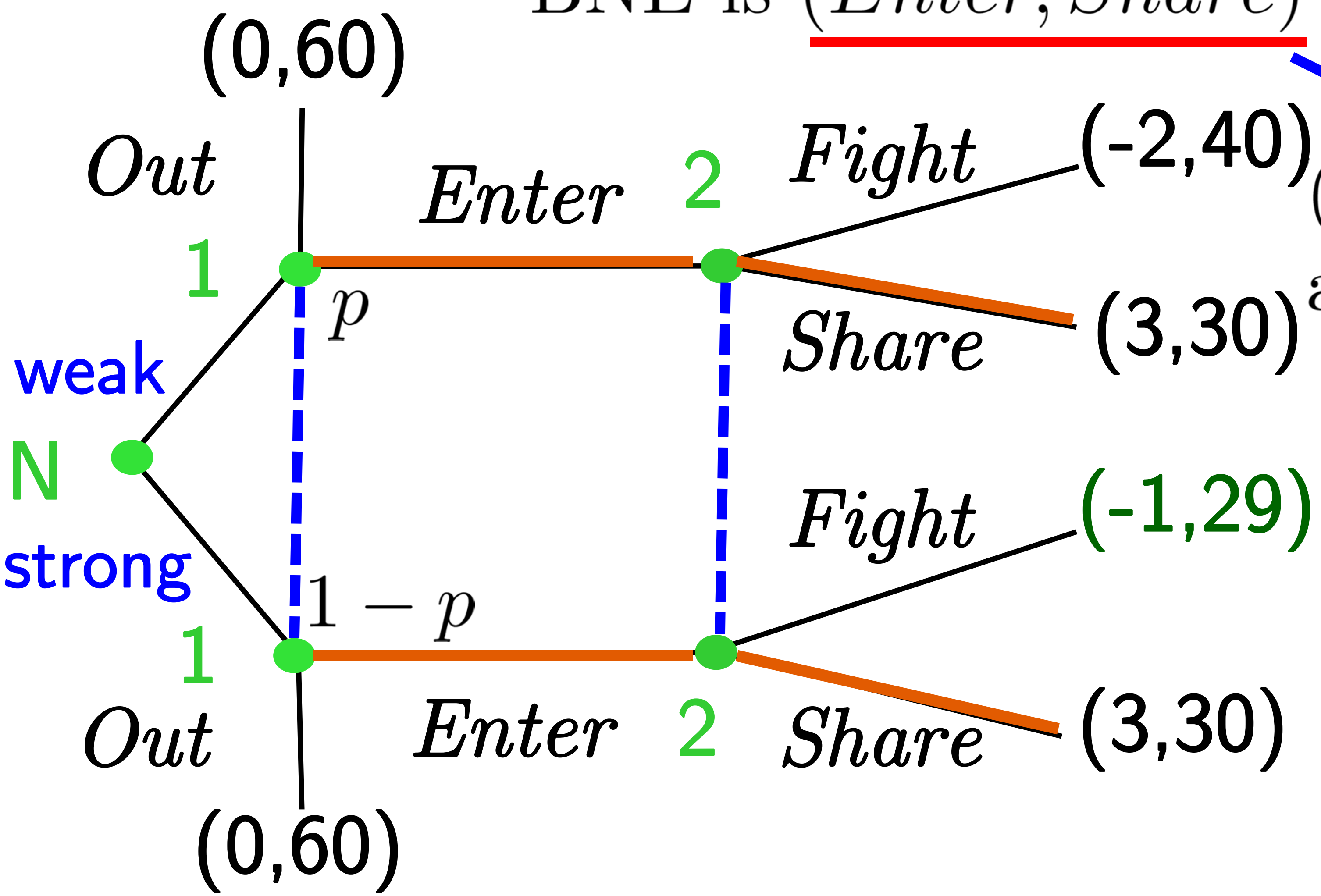
Market Entry Game w/ Incomplete Information



BNE if Player 2 Chooses *Share*

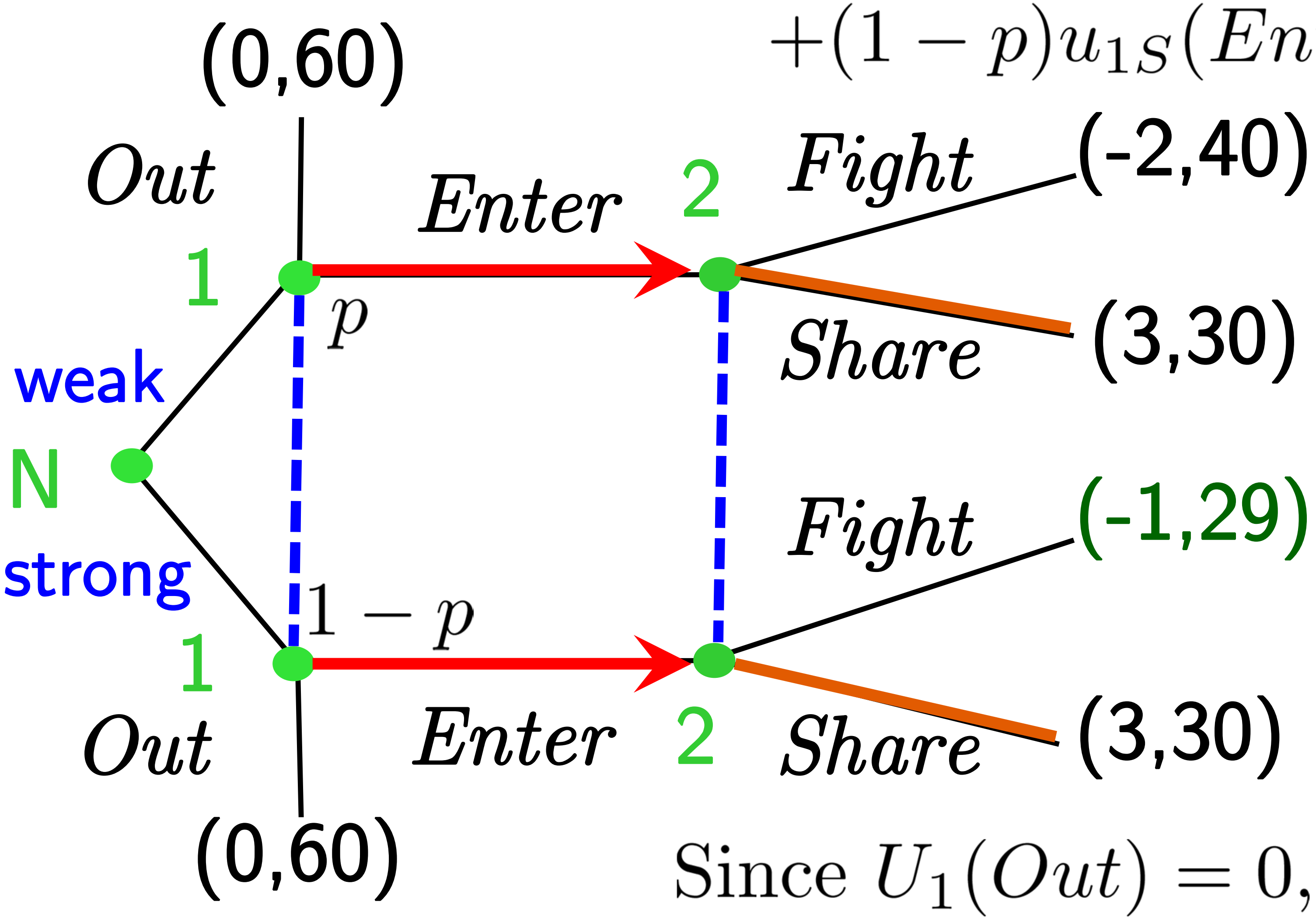
BNE is $(Enter, Share)$

(action of 1, action if $Enter$)



BNE if Player 2 Chooses *Share*: Player 1's BR

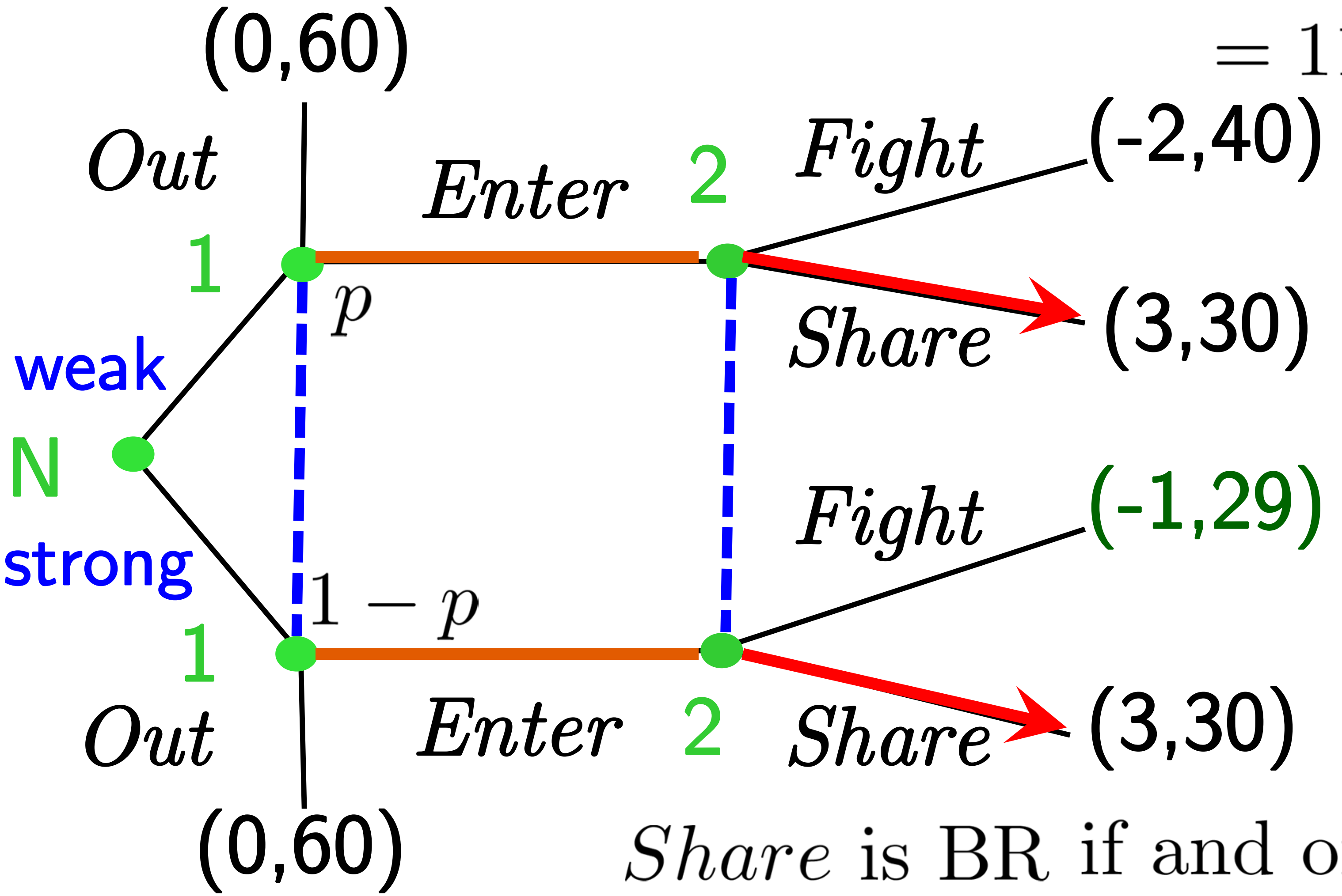
$$U_1(Enter) = pu_{1W}(Enter, Share) + (1-p)u_{1S}(Enter, Share) = 3$$



Since $U_1(Out) = 0$, $Enter$ is BR

BNE if Player 2 Chooses *Share*: Player 2's BR

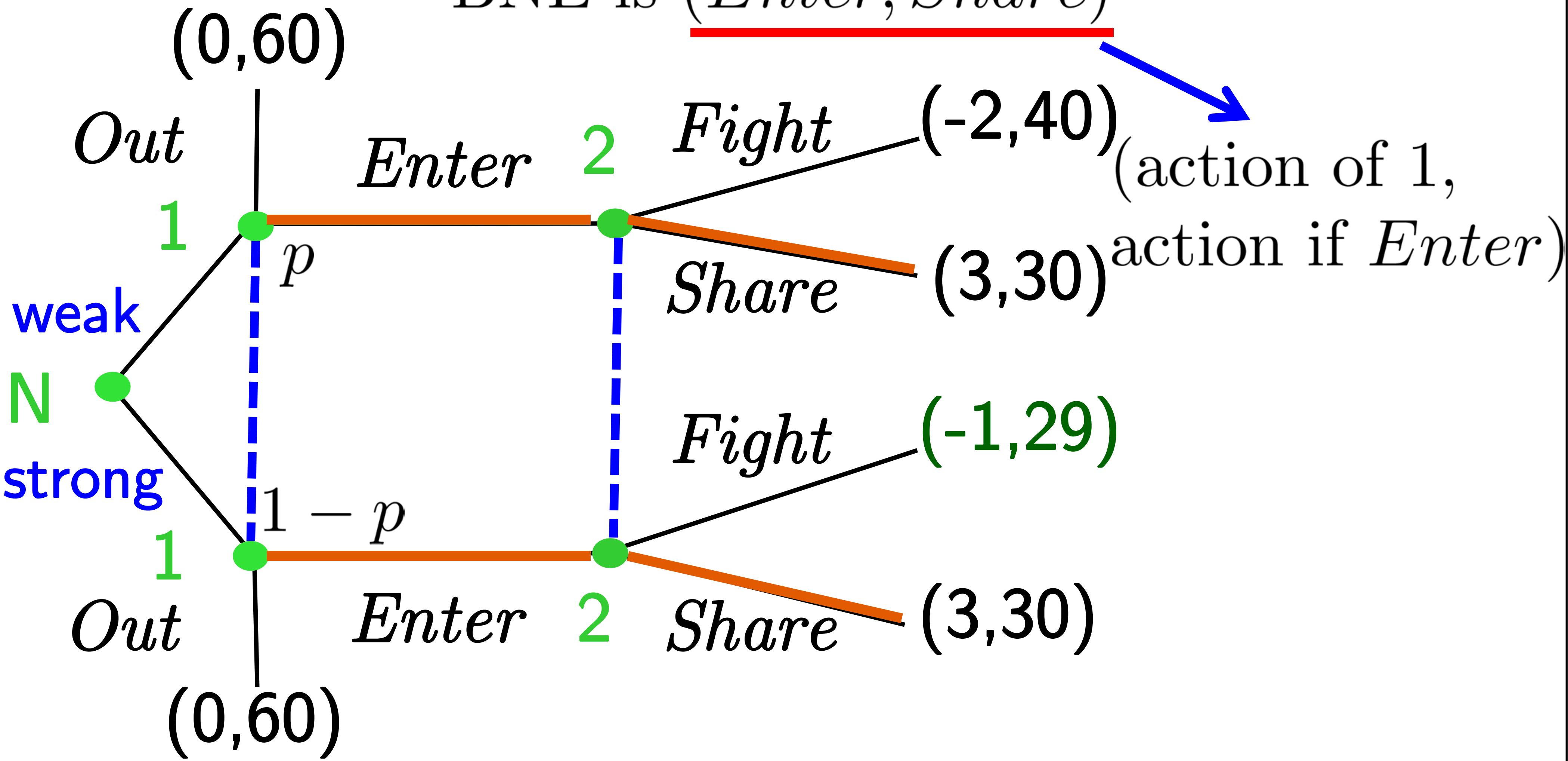
$$U_2(Fight) - U_2(Share) = (29 + 11p) - 30$$
$$= 11p - 1$$



Share is BR if and only if $p \leq 1/11$

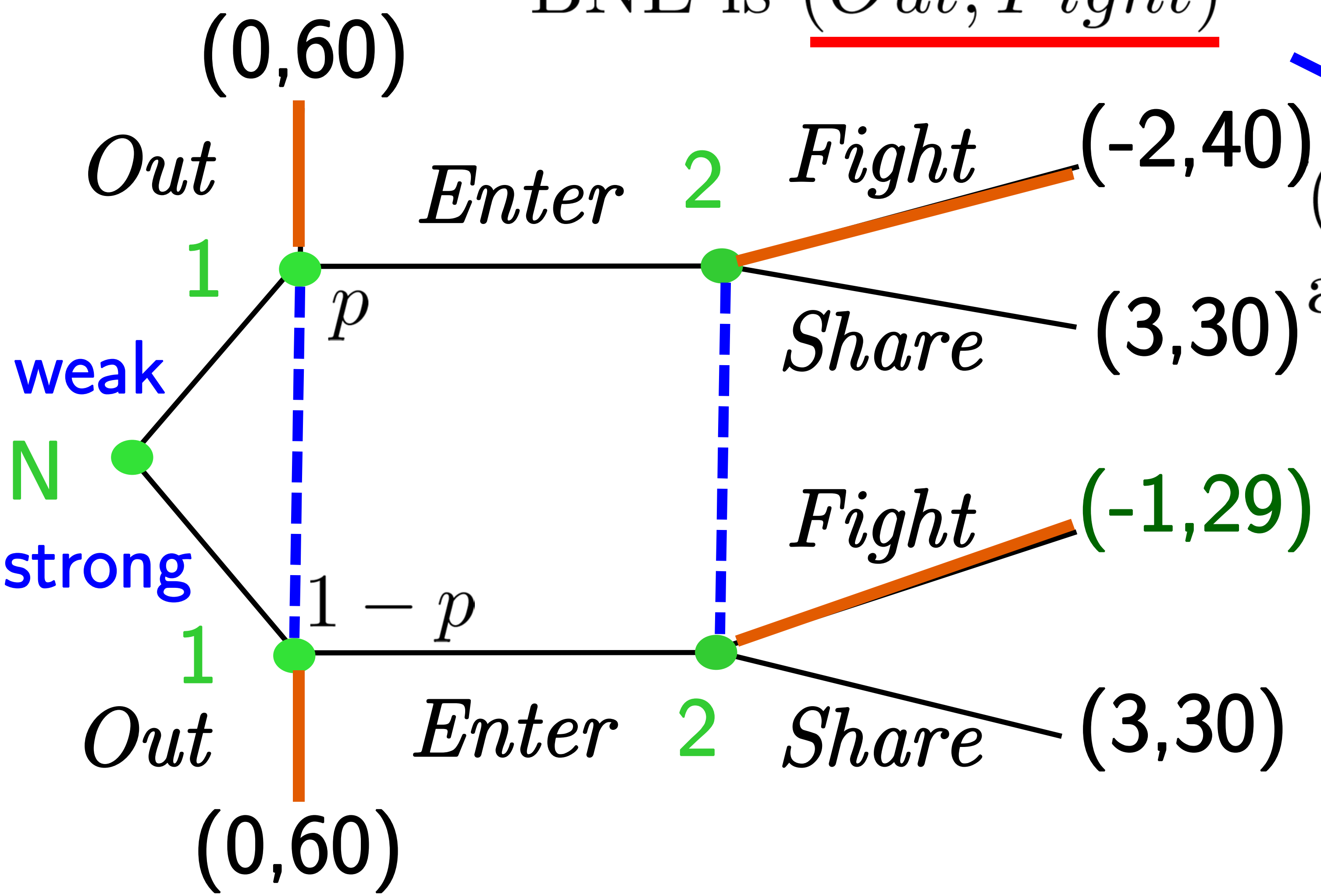
BNE if Player 2 Chooses *Share*

BNE is $(Enter, Share)$



BNE if Player 2 Chooses *Fight*

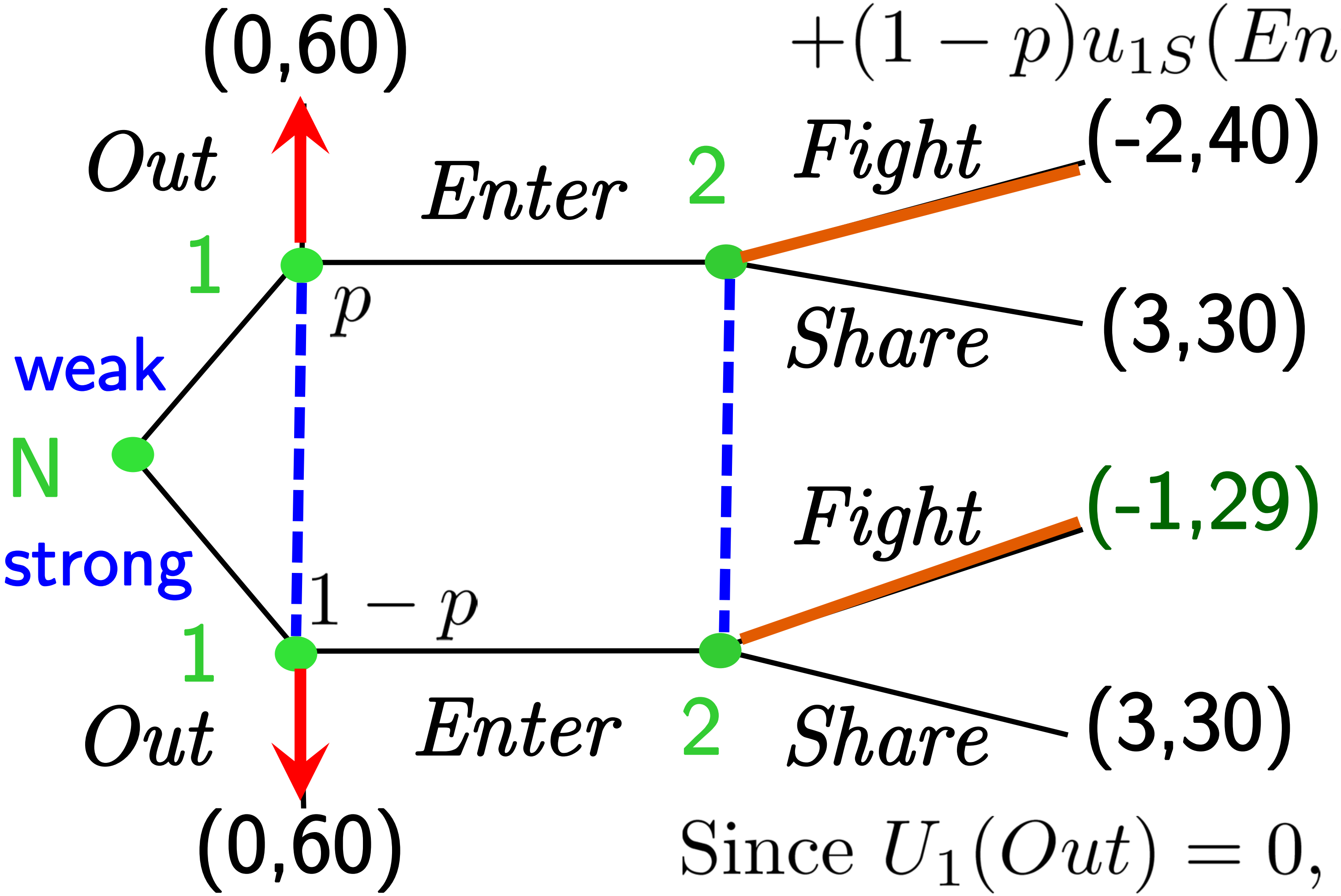
BNE is $(Out, Fight)$



(action of 1, action if *Enter*)

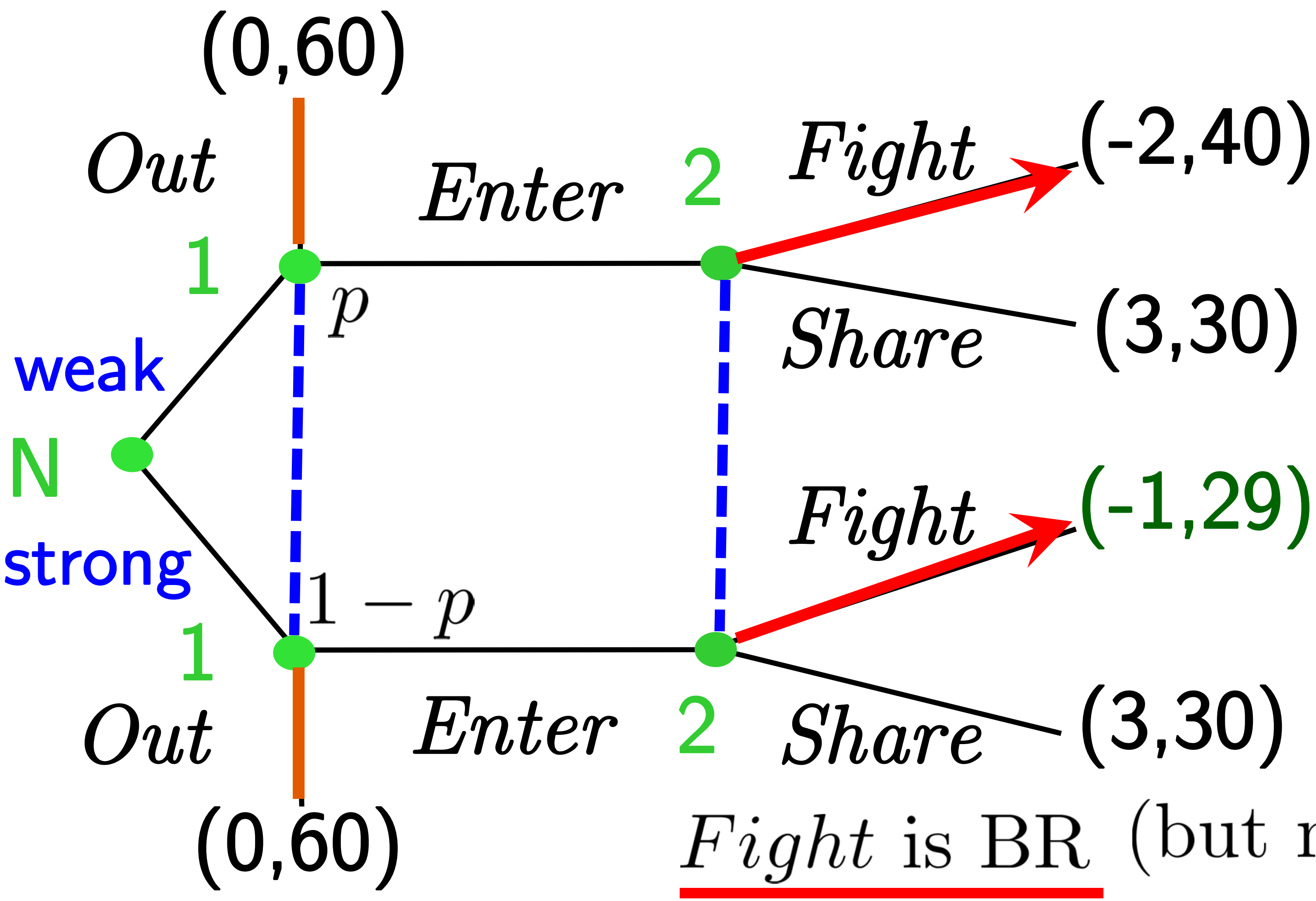
BNE if Player 2 Chooses *Fight*: Player 1's BR

$$U_1(Enter) = pu_{1W}(Enter, Fight) + (1-p)u_{1S}(Enter, Fight) < 0$$



Since $U_1(Out) = 0$, $\underline{Out \text{ is BR}}$

BNE if Player 2 Chooses *Fight*: Player 2's BR

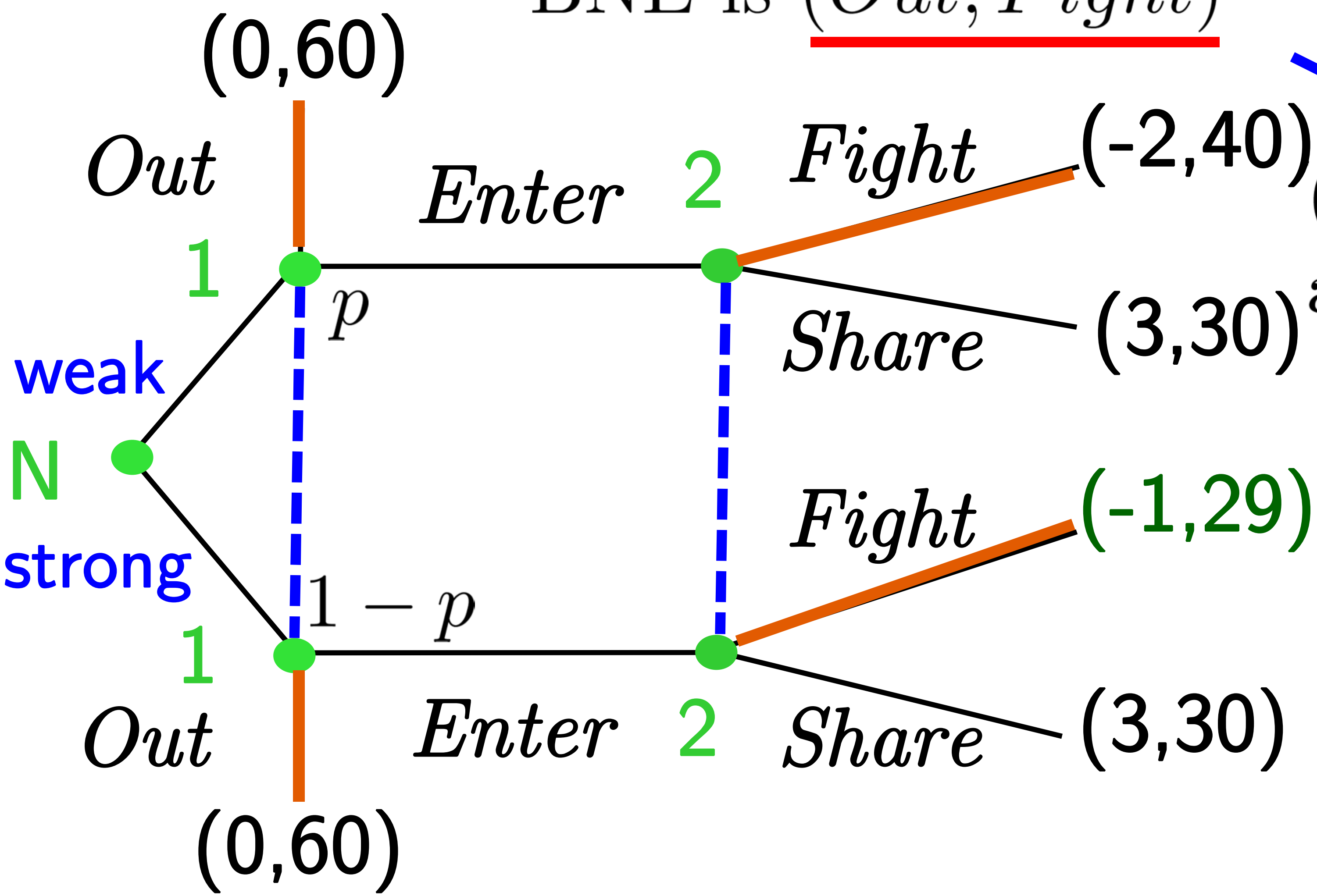


Fight is BR (but never "tested")

BNE if Player 2 Chooses *Fight*

BNE is $(Out, Fight)$

(action of 1, action if *Enter*)



Empty Threats Off the Equilibrium Path

- Not a "Sensible" Equilibrium…
  - If $p \leq 1/11$, Incumbent wouldn't want to $\textit{Fight}$
  - Not SPE when $p=0$
- Problem due to "crazy" beliefs that are:
- Off the Equilibrium Path: nodes that are not reached in equilibrium
  - Not reached = Zero probability (discrete types only)
- On the Equilibrium Path: nodes that are reached in equilibrium

Perfect Bayesian Equilibrium

- A BNE is a Perfect Bayesian Equilibrium (PBE) if at all nodes off the equilibrium path, there are strategies and beliefs consistent with Bayes' Rule such that the strategies (both on and off the equilibrium path) are BR
- When $p<1/11$, $(Out,\ Fight)$ is not a PBE since when $Enter$ occurs (off-equilibrium path), $Fight$ is only a BR if $p \geq 1/11$.

Trembling-Hand Perfect Equilibrium

- To rule out "crazy" equilibrium, can perturb the BNE by making them totally mixed:
  - Consider a game with T stages
- Set of feasible actions at stage t is A_t (finite)
- For the BNE $\overline{\pi} = (\overline{\pi}_1, \cdots, \overline{\pi}_T)$
- Consider a sequence of totally mixed strategies (sequence of trembles) $\{\pi^k\}_{k=1}^{\infty} \to \overline{\pi}$
  - All nodes are reached (and tested in the BNE)
  - No more "crazy" beliefs off the equilibrium path...

Trembling-Hand Perfect Equilibrium

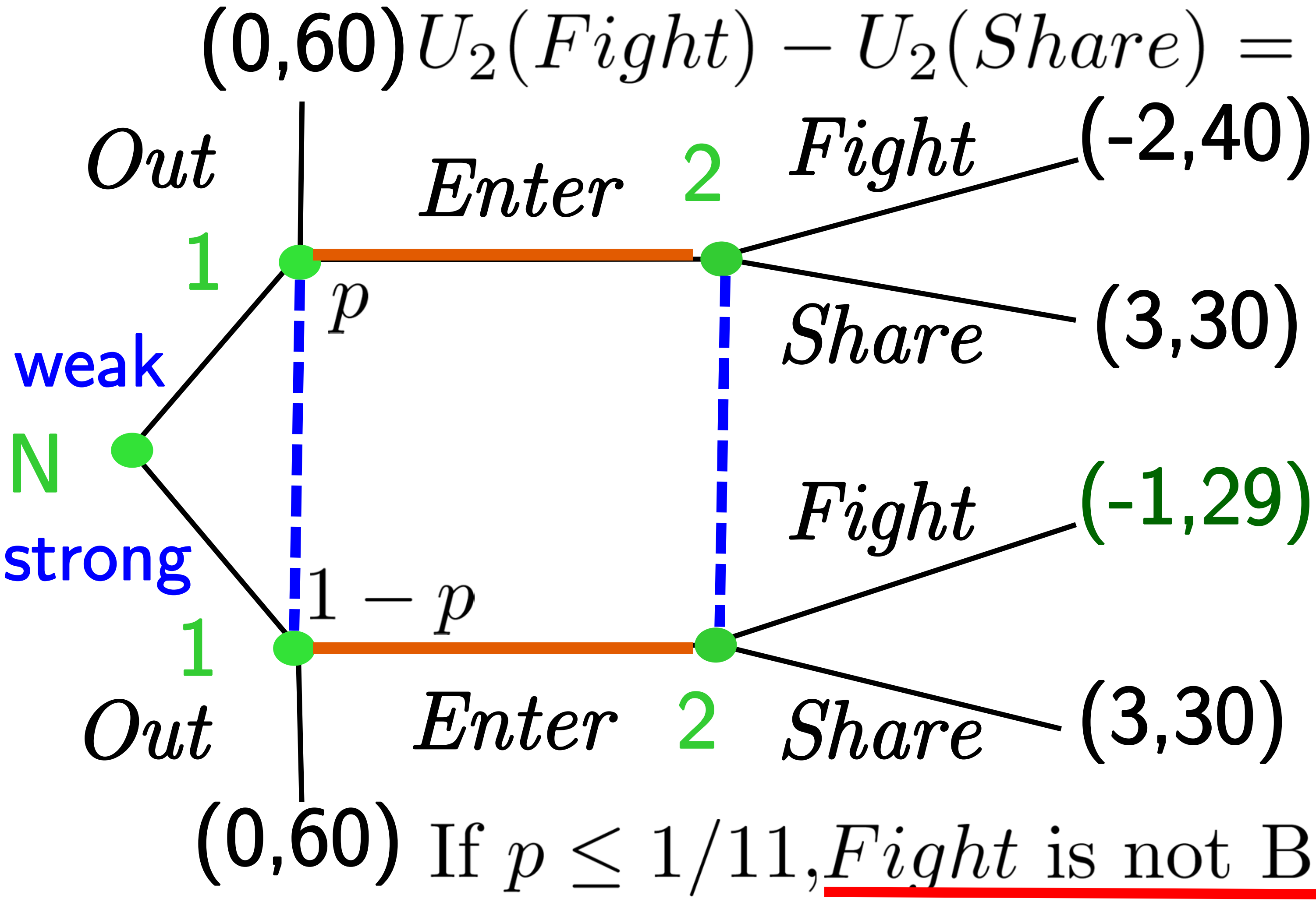
- A BNE $\overline{\pi}$ is Trembling-Hand Perfect (THP) if
- There exists some sequence of totally mixed strategy profiles $\{\pi^k\}_{k=1}^{\infty} \to \overline{\pi}$
  - (Converging to the equilibrium strategies) such that
- For all sufficiently large k, the equilibrium strategies are BR: $\overline{\pi}_i = \arg\max_{\pi_i} U_i(\pi_i, \pi_{-i}^k)$
- Note: If a sequence of Logit-QRE converges to a BNE, would the BNE automatically be THP?
  - QRE solves this by construct (already totally mixed...)

BNE if Player 2 Chooses *Fight*: Not THP

For all $\{\pi^k\}_{k=1}^{\infty}$, $Enter$ with error probability ϵ_1^k

$$U_2(Fight) - U_2(Share) = (29 + 11p) - 30$$

$$= 11p - 1$$

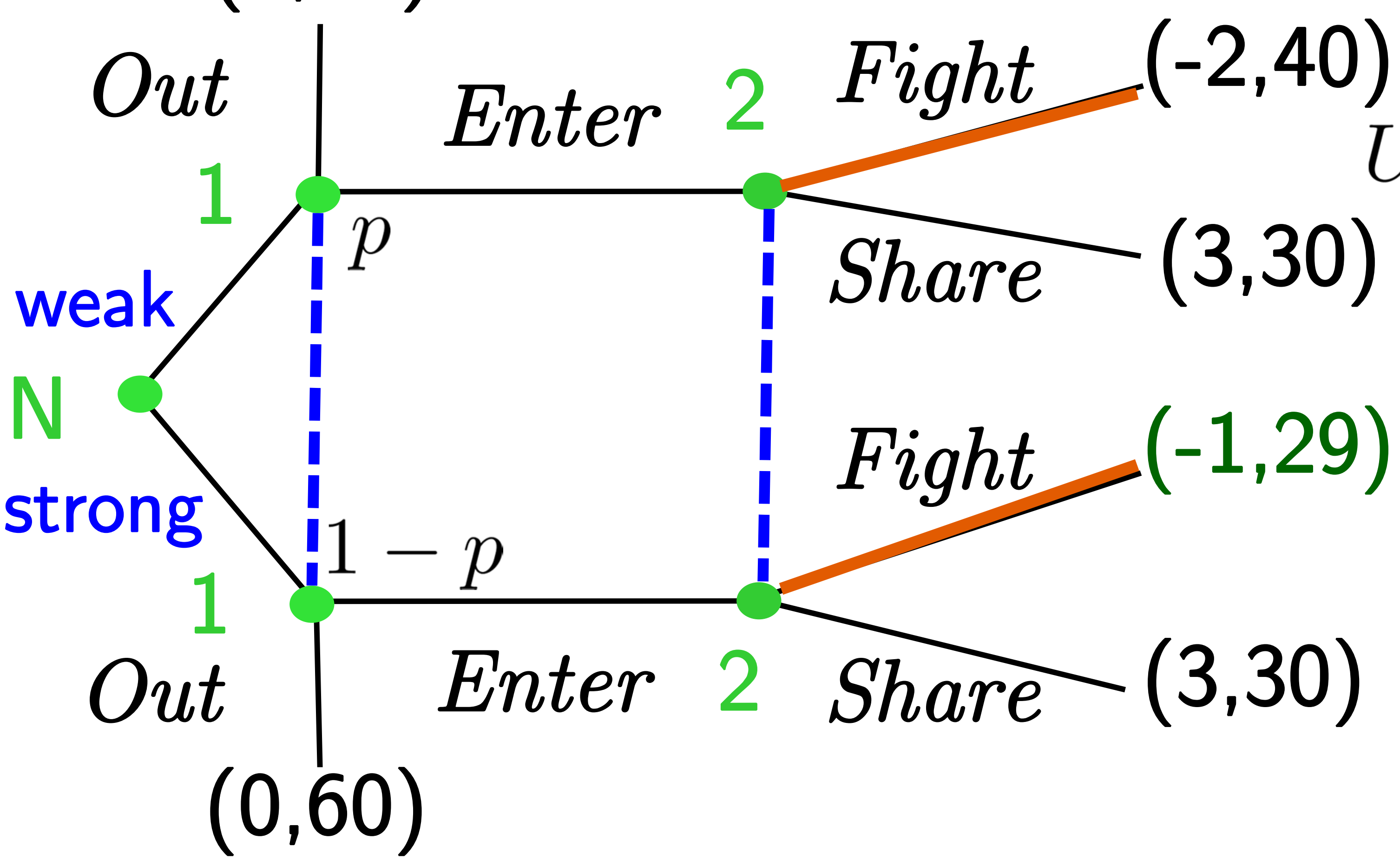


If $p \leq 1/11$, *Fight is not BR when Enter*

BNE if Player 2 Chooses *Share*: Indeed THP

For all $\{\pi^k\}_{k=1}^{\infty}$,

Fight with error probability ϵ_2^k



$U_1(Enter)$

$= 3(1-\epsilon_2^k)$

$-\epsilon_2^k - p\epsilon_2^k$

> 0

for small ϵ_2^k

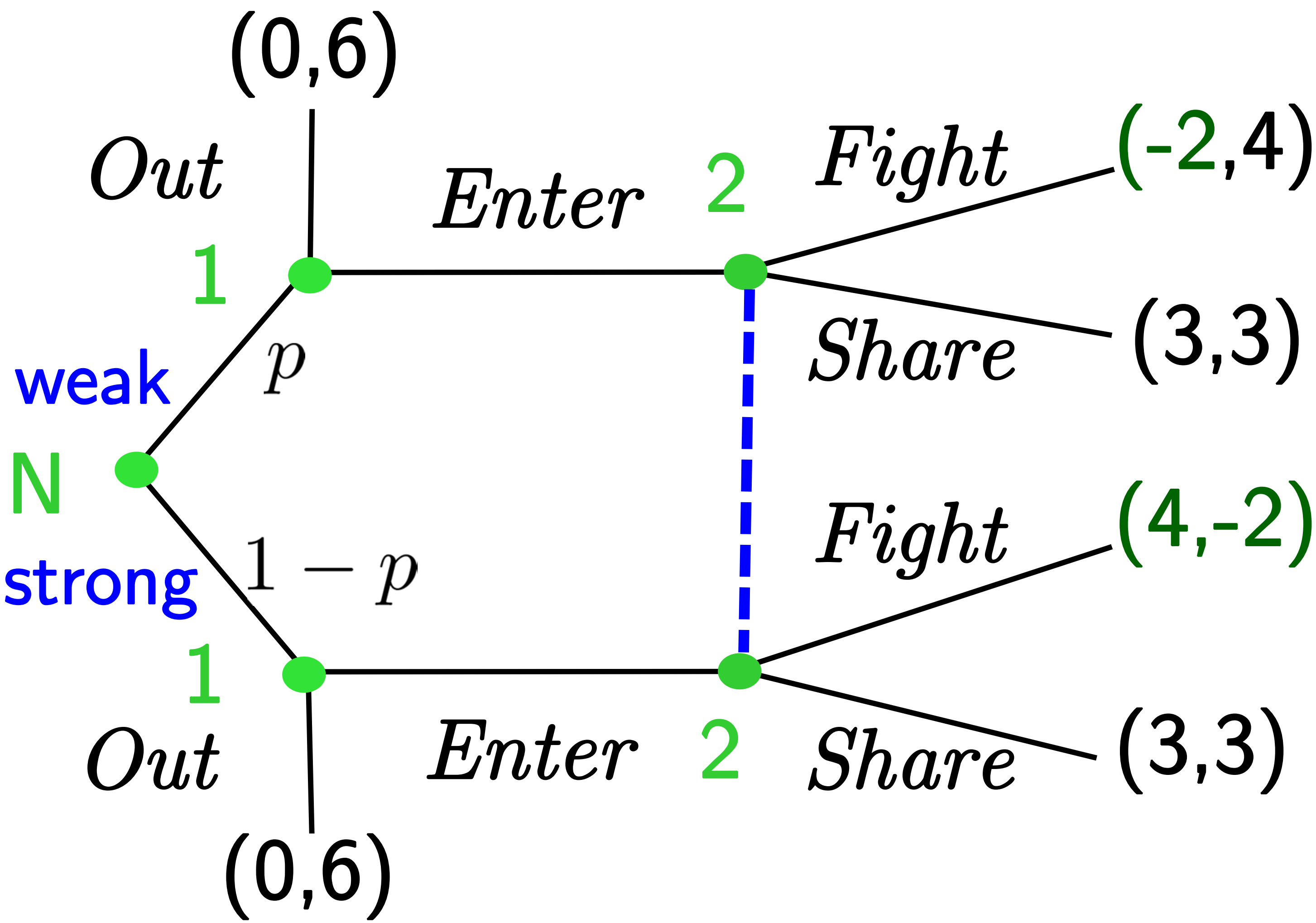
Enter is BR

Sequential Equilibrium

- A BNE is a sequential equilibrium if
- Each strategy at each node is a BR
- When beliefs at each node are the limits of beliefs associated with trembles as the probability of trembles → 0

- Note: THP → SE

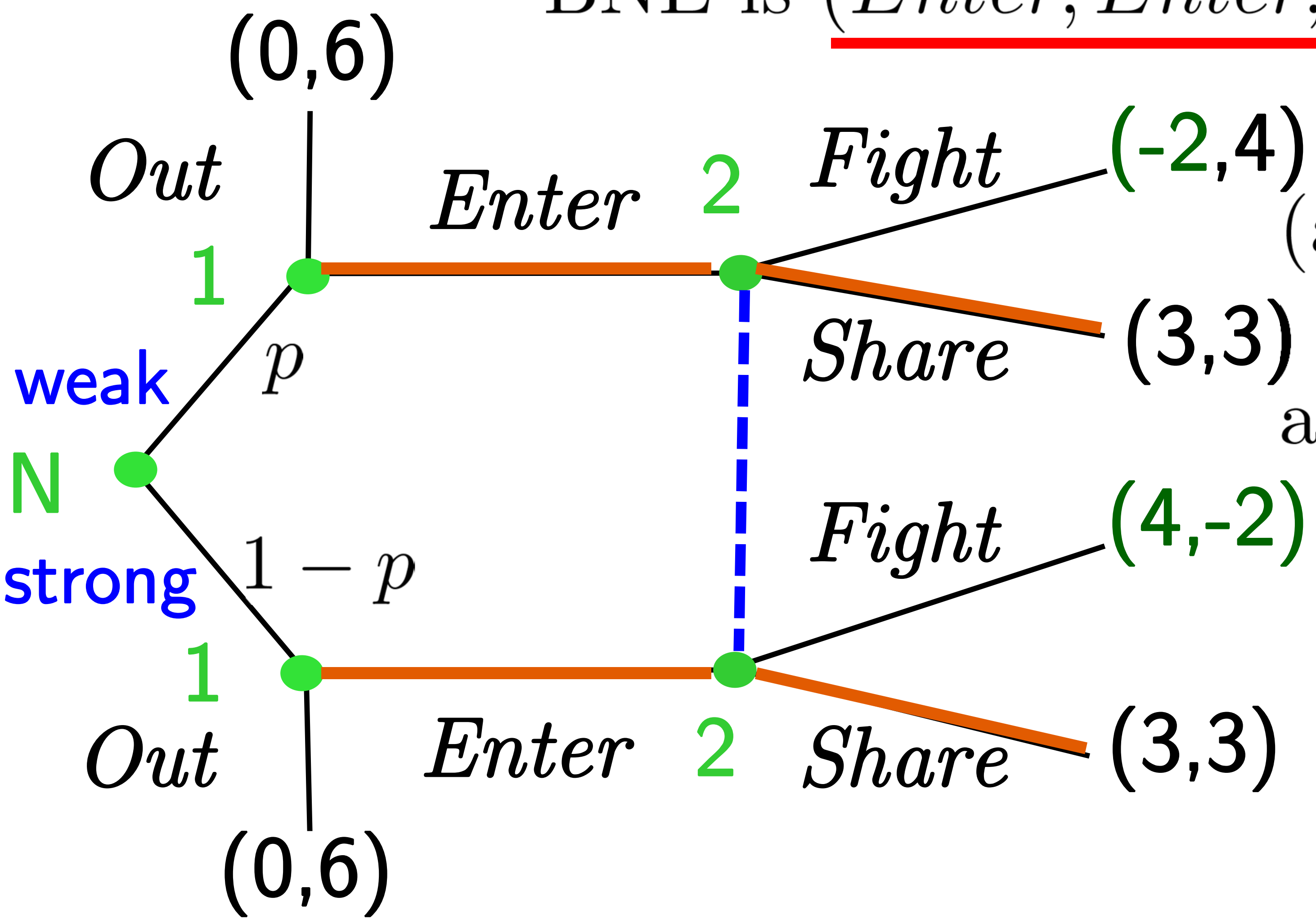
Market Entry Game with Private Information



BNE when $p<5/6$: (*Enter*, *Enter*, *Share*)

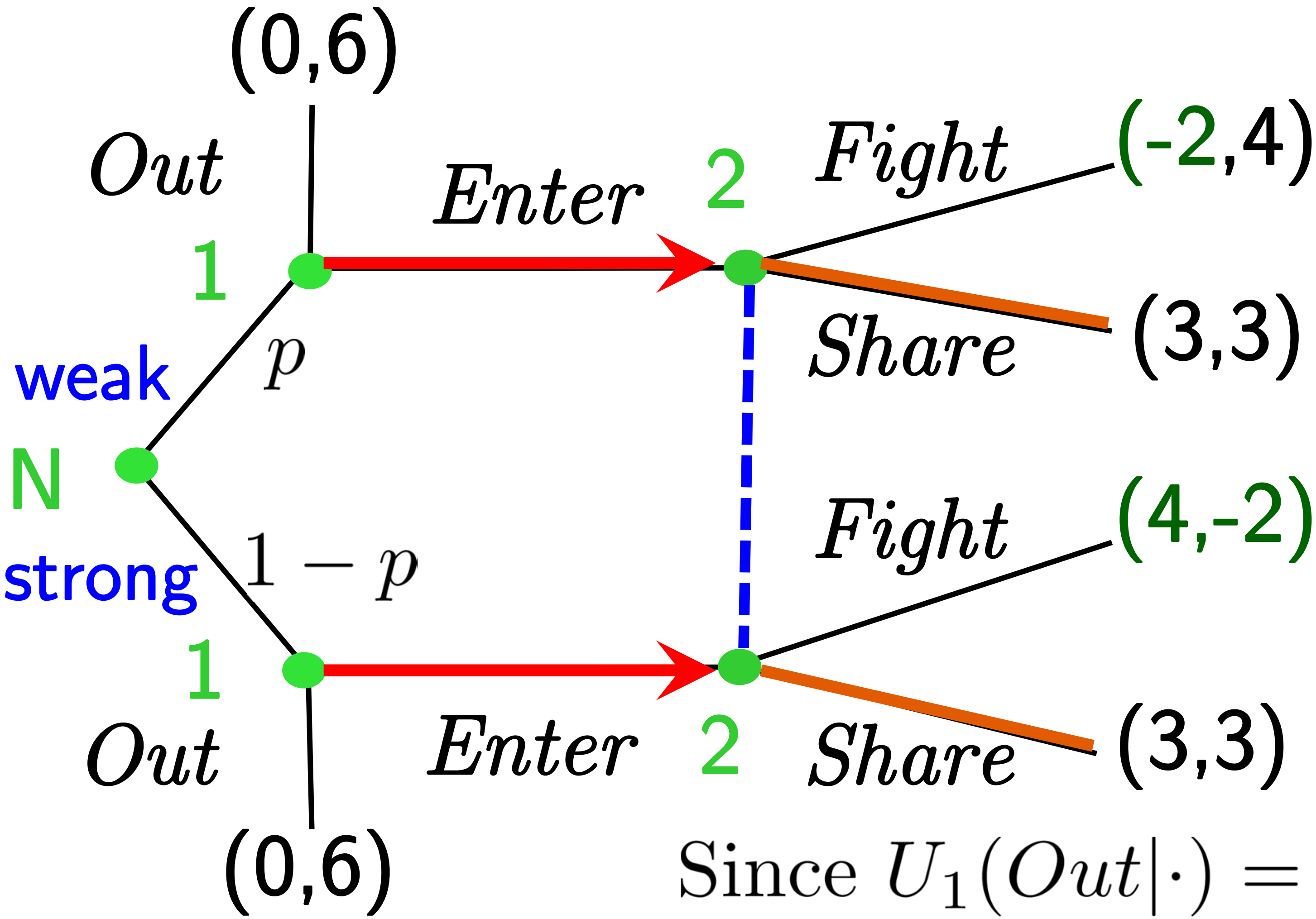
BNE is $(Enter, Enter, Share)$

(action if weak, action if strong, action if $Enter$)



BNE when $p < 5/6$: (*Enter*, *Enter*, *Share*)

$$U_1(Enter|\text{weak}) = U_1(Enter|\text{strong}) = 3$$

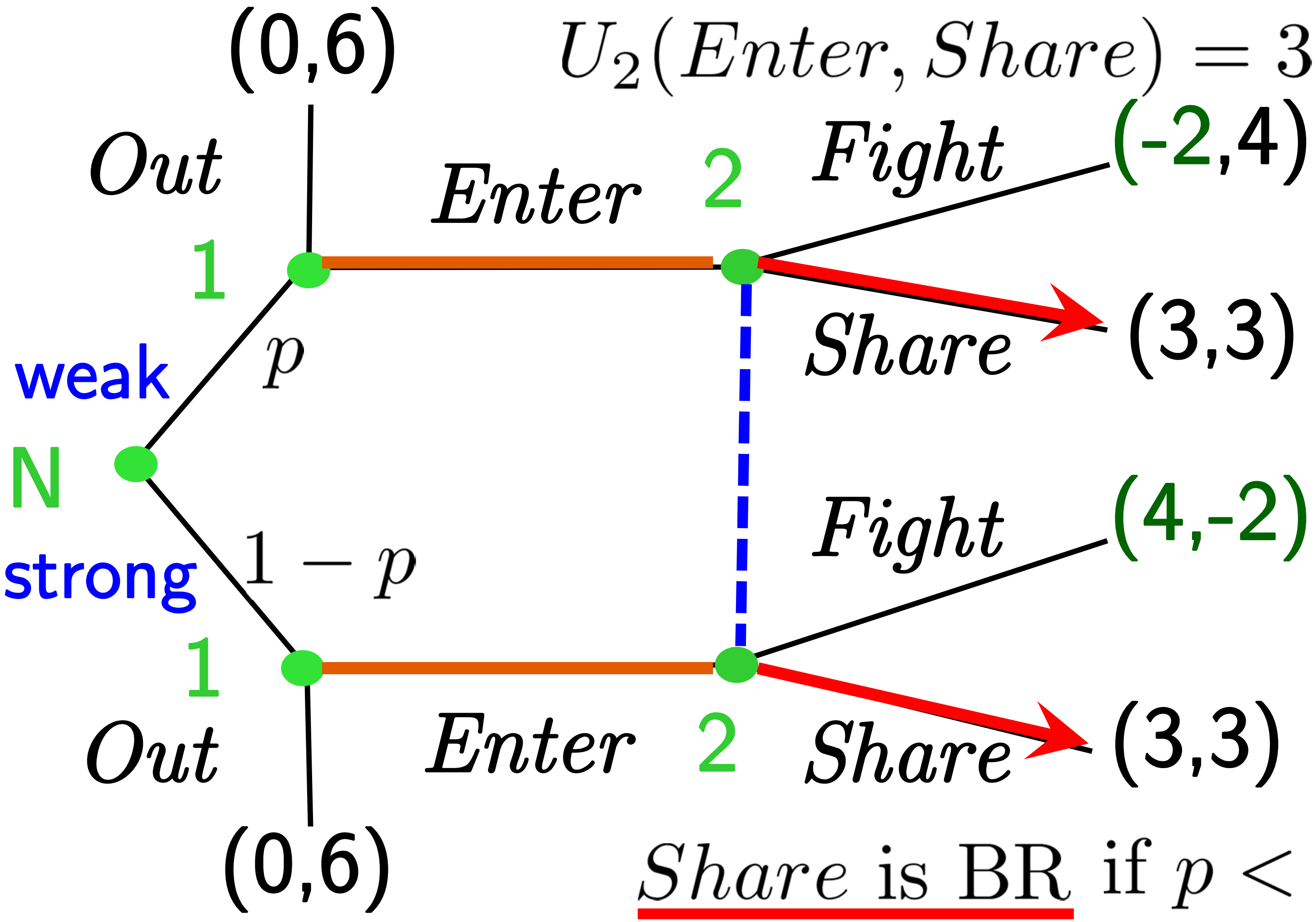


Since $U_1(Out|\cdot) = 0$, $Enter$ is BR

BNE when $p < 5/6$: (*Enter*, *Enter*, *Share*)

$$U_2(Enter, Fight) = 4p - 2(1-p) = 6p - 2$$

$$U_2(Enter, Share) = 3$$

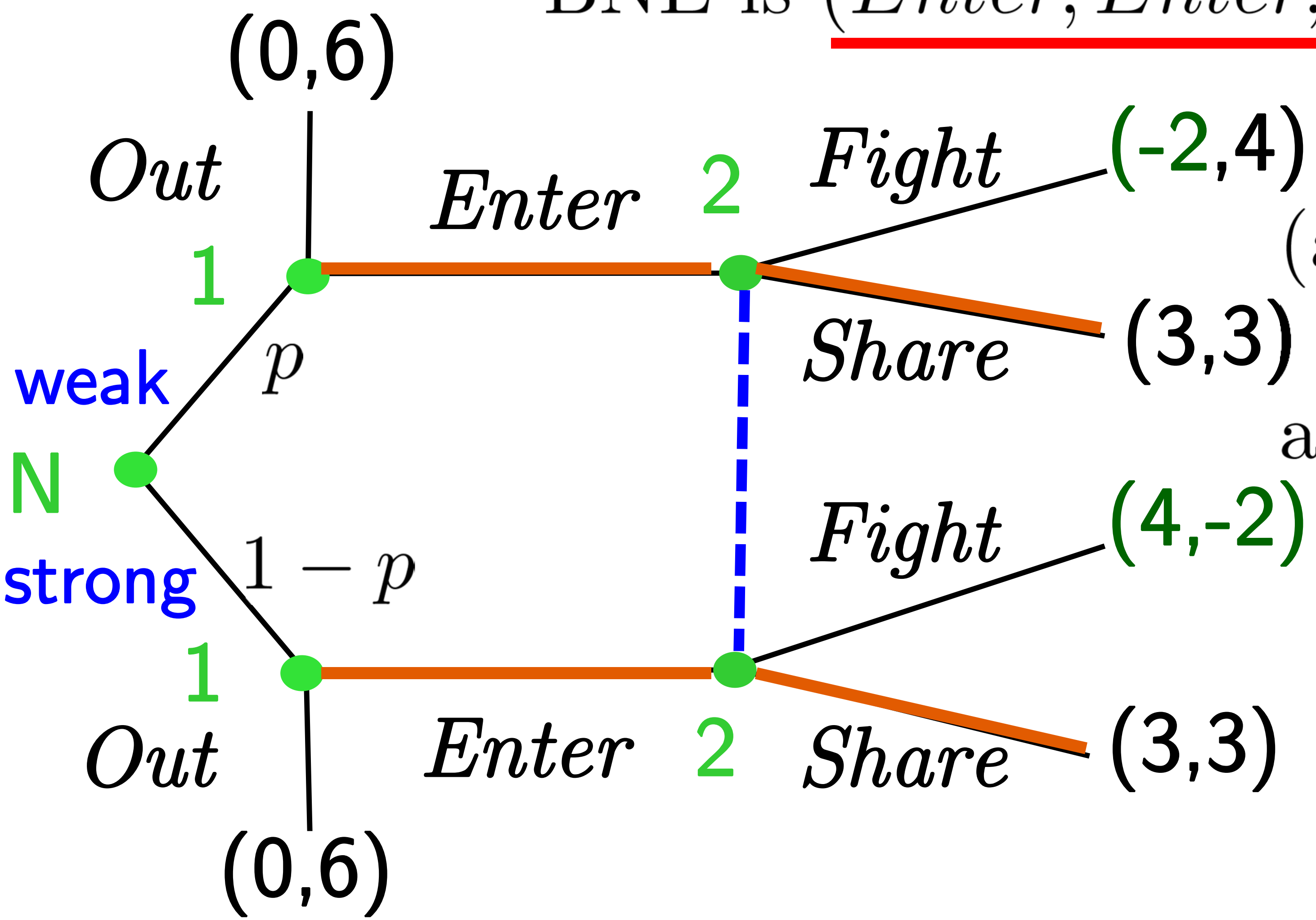


No Information Update!

$Share$ is BR if $p < 5/6$

BNE when $p < 5/6$: (*Enter*, *Enter*, *Share*)

BNE is $(Enter, Enter, Share)$



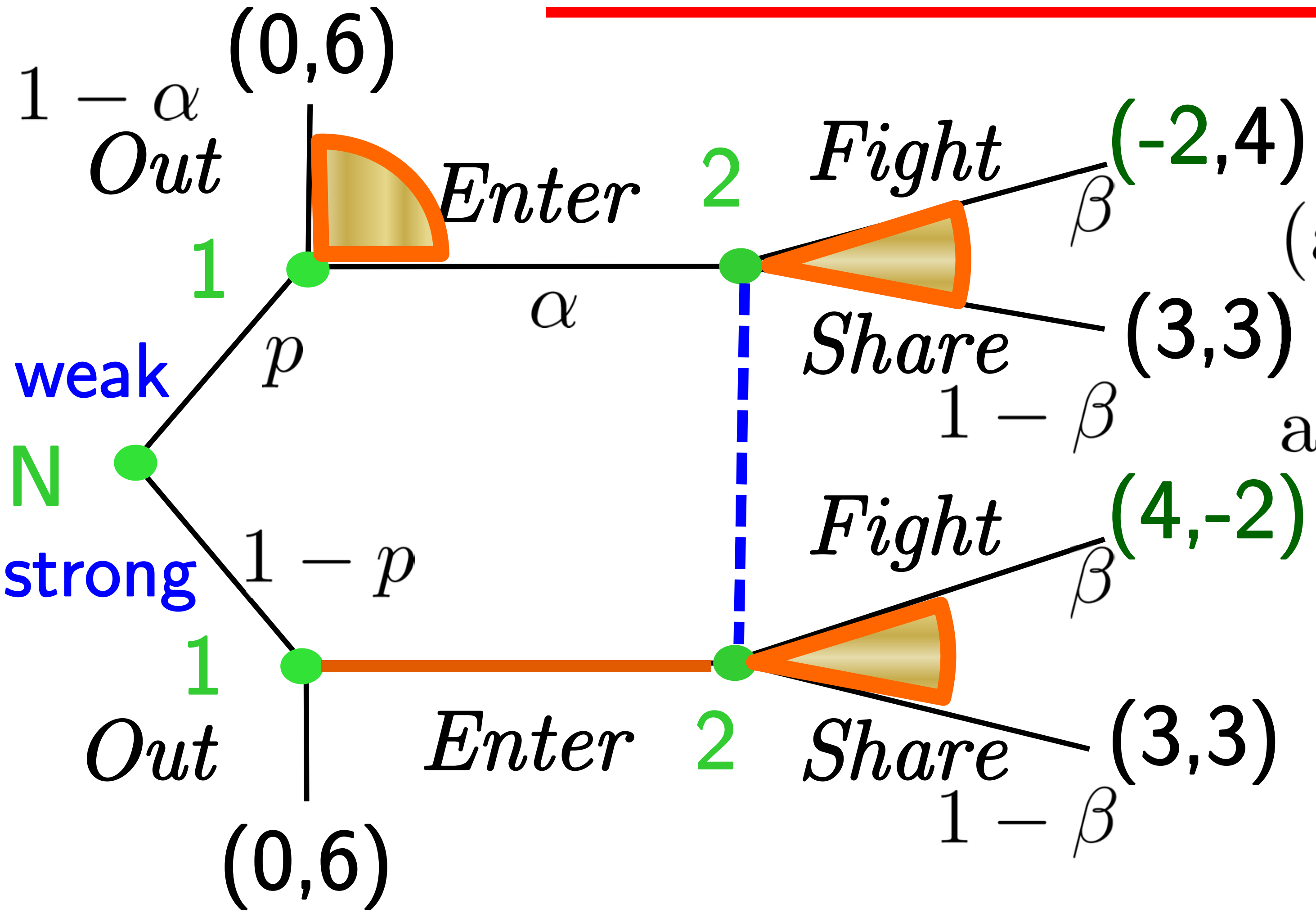
(action if weak, action if strong, action if *Enter*)

But (*Enter*, *Enter*, *Share*) is no longer an equilibrium if $p > 5/6$!!

BNE when $p > 5/6$: (strong *Enter*; Others Mix)

BNE is $(\Pr(Enter) = \alpha, Enter, \Pr(Fight) = \beta)$

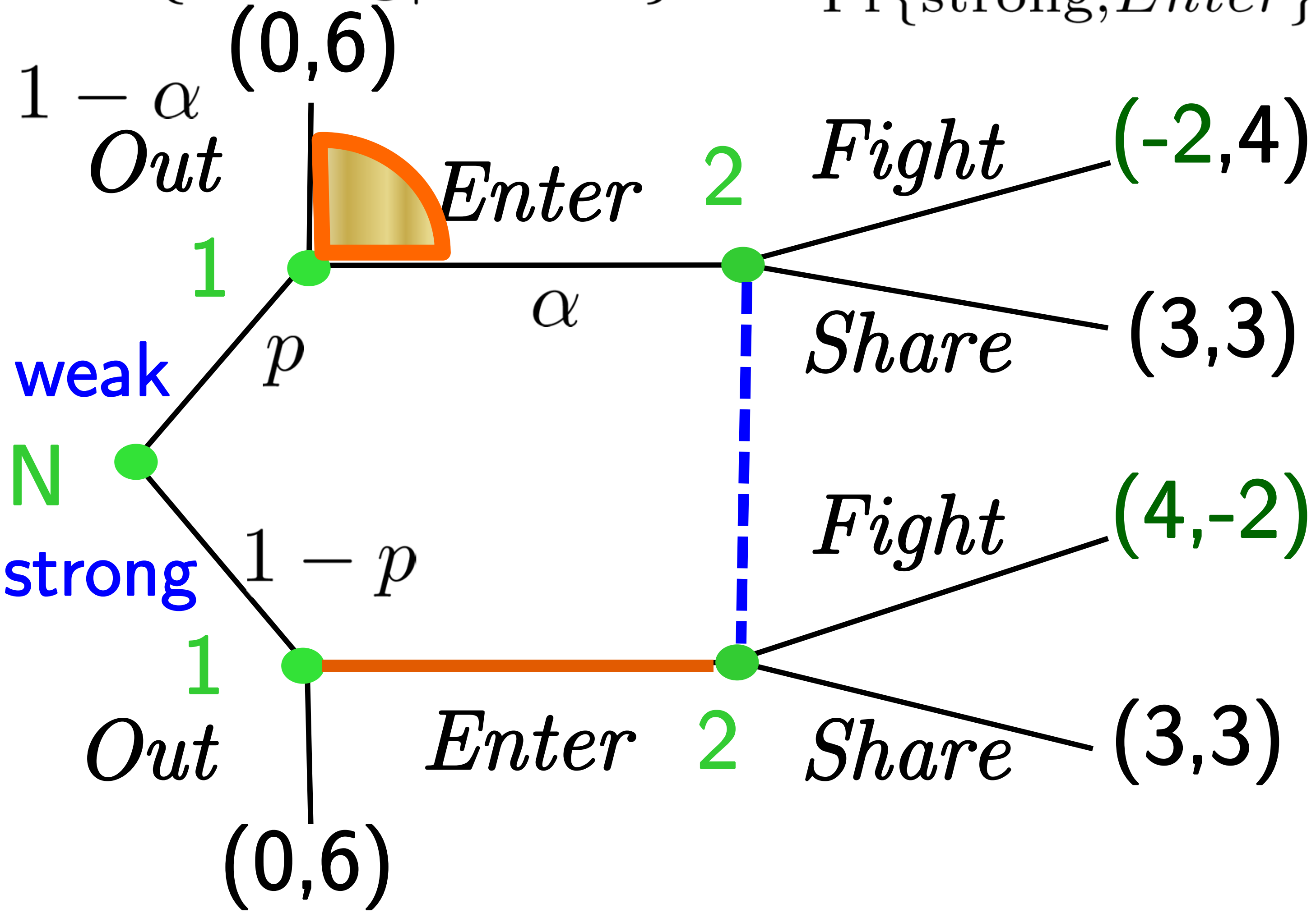
(action if weak, action if strong, action if *Enter*)



BNE when $p > 5/6$: (strong *Enter*; Others Mix)

$$\Pr\{\text{strong}|Enter\} = \frac{\Pr\{\text{strong}, Enter\}}{\Pr\{\text{strong}, Enter\} + \Pr\{\text{weak}, Enter\}}$$

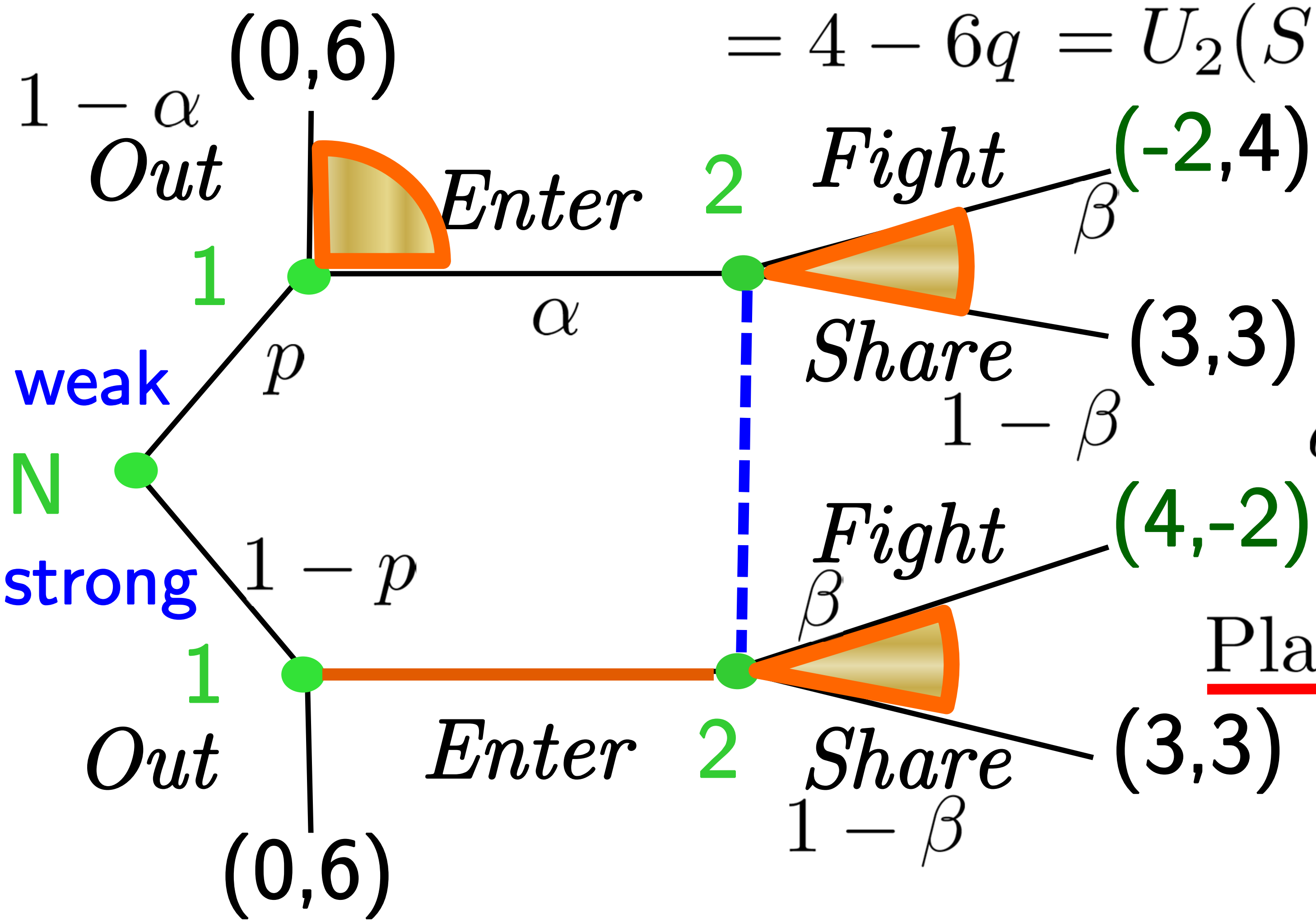
$$= \frac{(1-p)}{p\alpha + (1-p)}$$



BNE when $p > 5/6$: (strong *Enter*; Others Mix)

$$U_2(Fight) = (1-q)\cdot 4 - q \cdot 2$$

$$= 4 - 6q = U_2(Share) = 3$$



if $q = \frac{1}{6}$

$$q = \frac{(1-p)}{p\alpha + (1-p)}$$

Player 2 mixes if

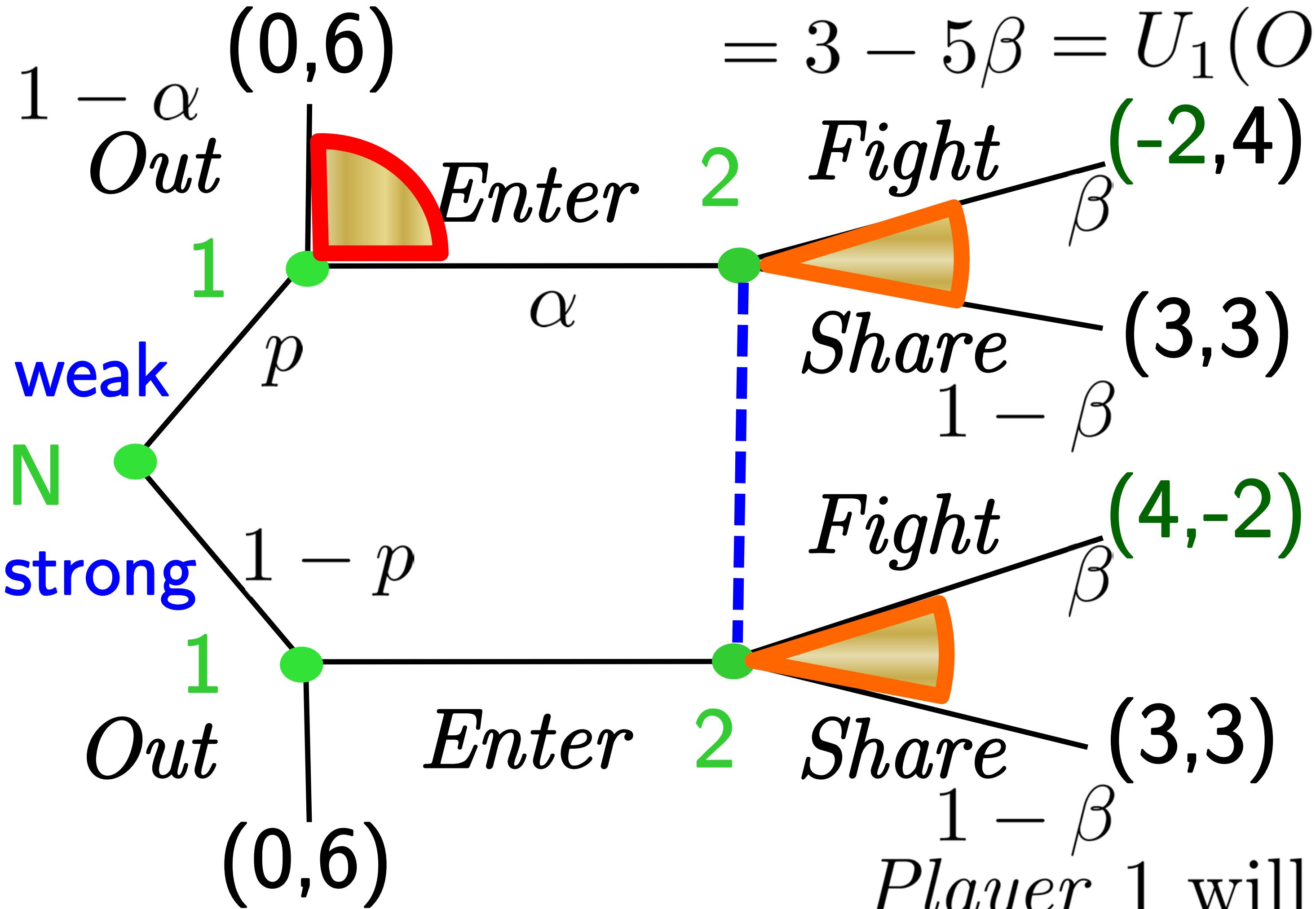
$$\alpha = \frac{5(1-p)}{p}$$

BNE when $p > 5/6$: (strong *Enter*; Others Mix)

$$U_1(Enter|\text{weak}) = \beta \cdot (-2) + (1-\beta) \cdot 3$$

$$= 3 - 5\beta = U_1(Out|\text{weak}) = 0$$

if $\beta = \frac{3}{5}$

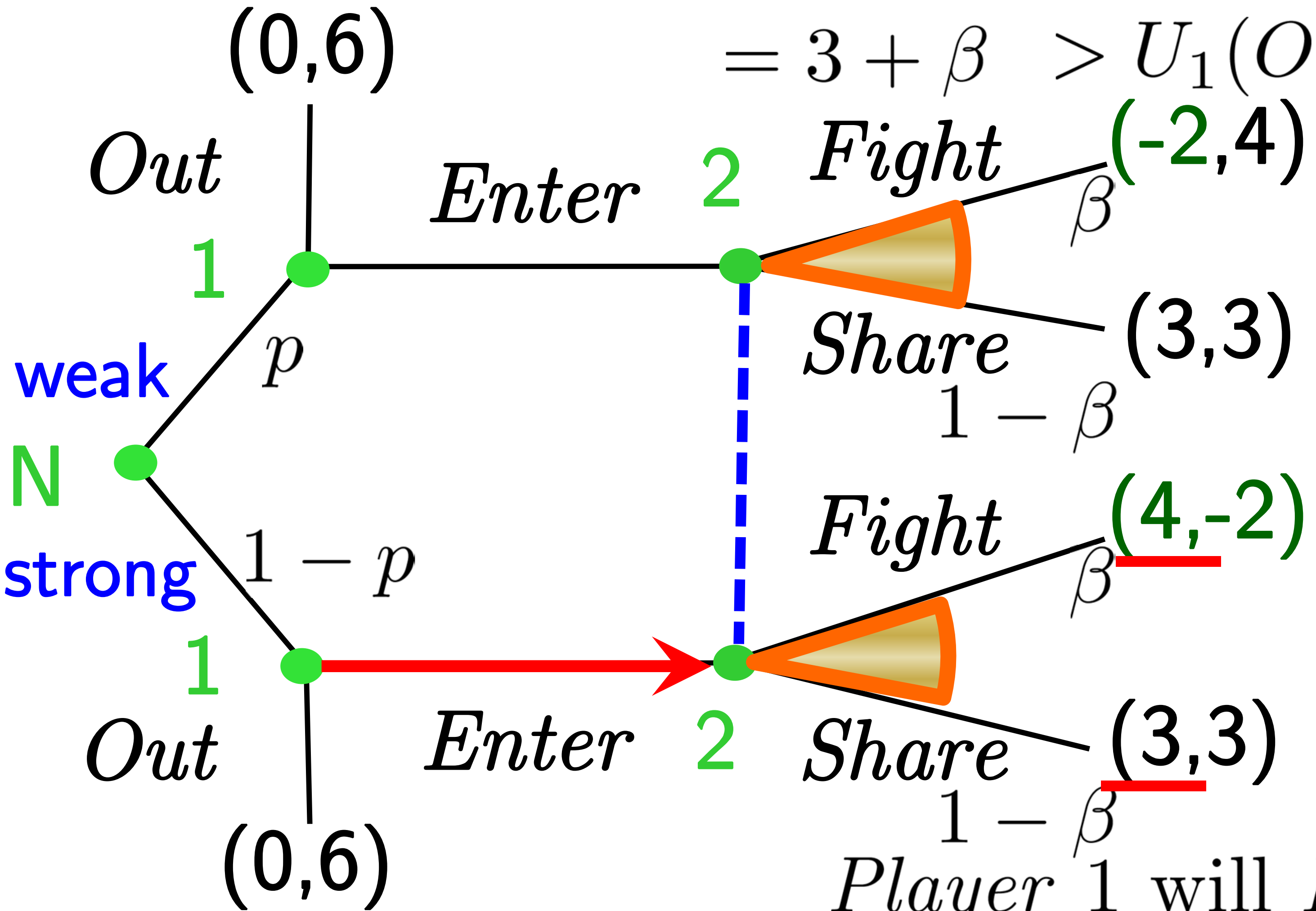


Player 1 will mix if weak

BNE when $p > 5/6$: (strong *Enter*; Others Mix)

If $\beta = \frac{3}{5}$, $U_1(Enter|\text{strong}) = \beta \cdot 4 + (1-\beta)\cdot 3$

$= 3 + \beta > U_1(Out|\text{strong}) = 0$

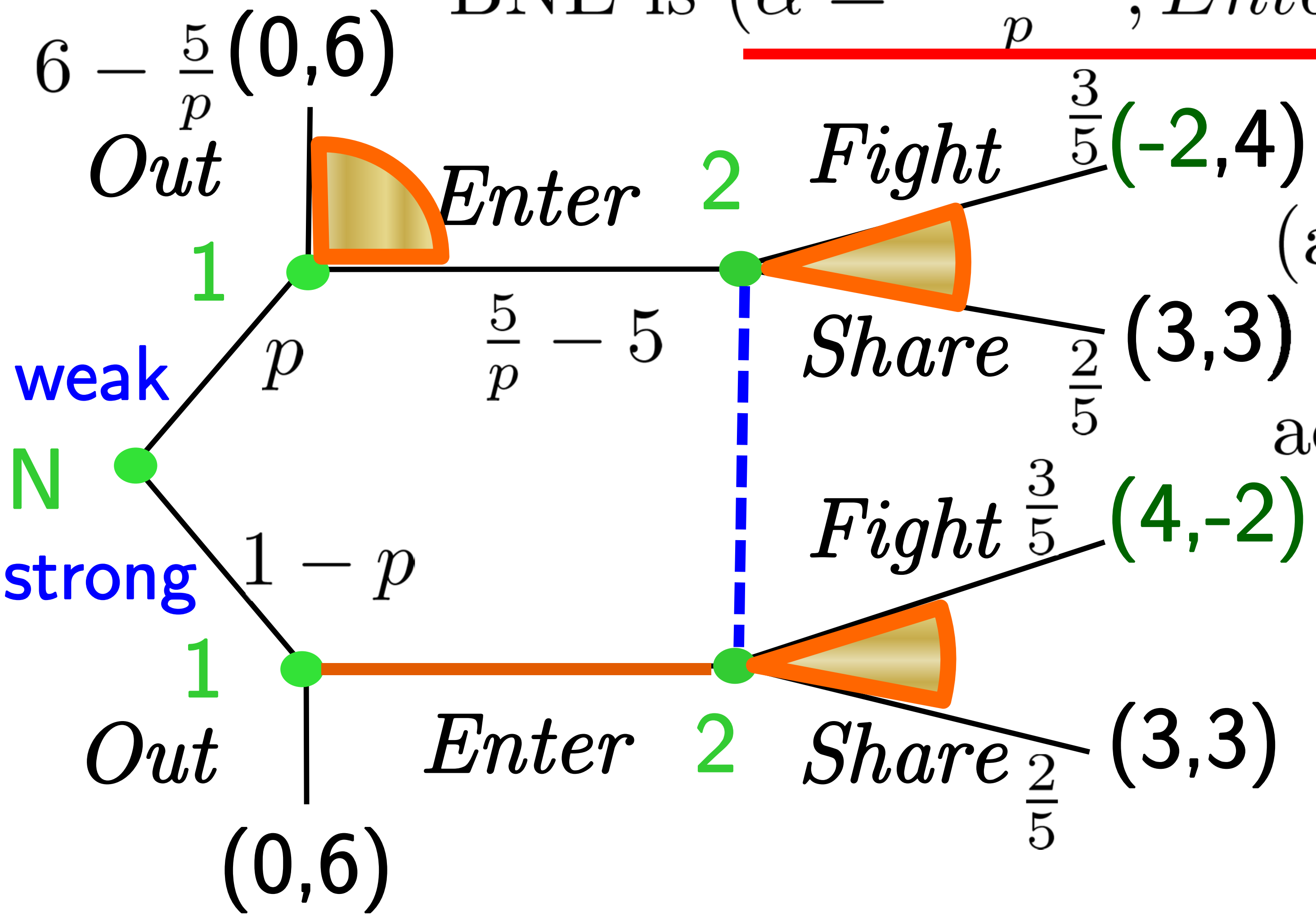


Player 1 will *Enter* if strong

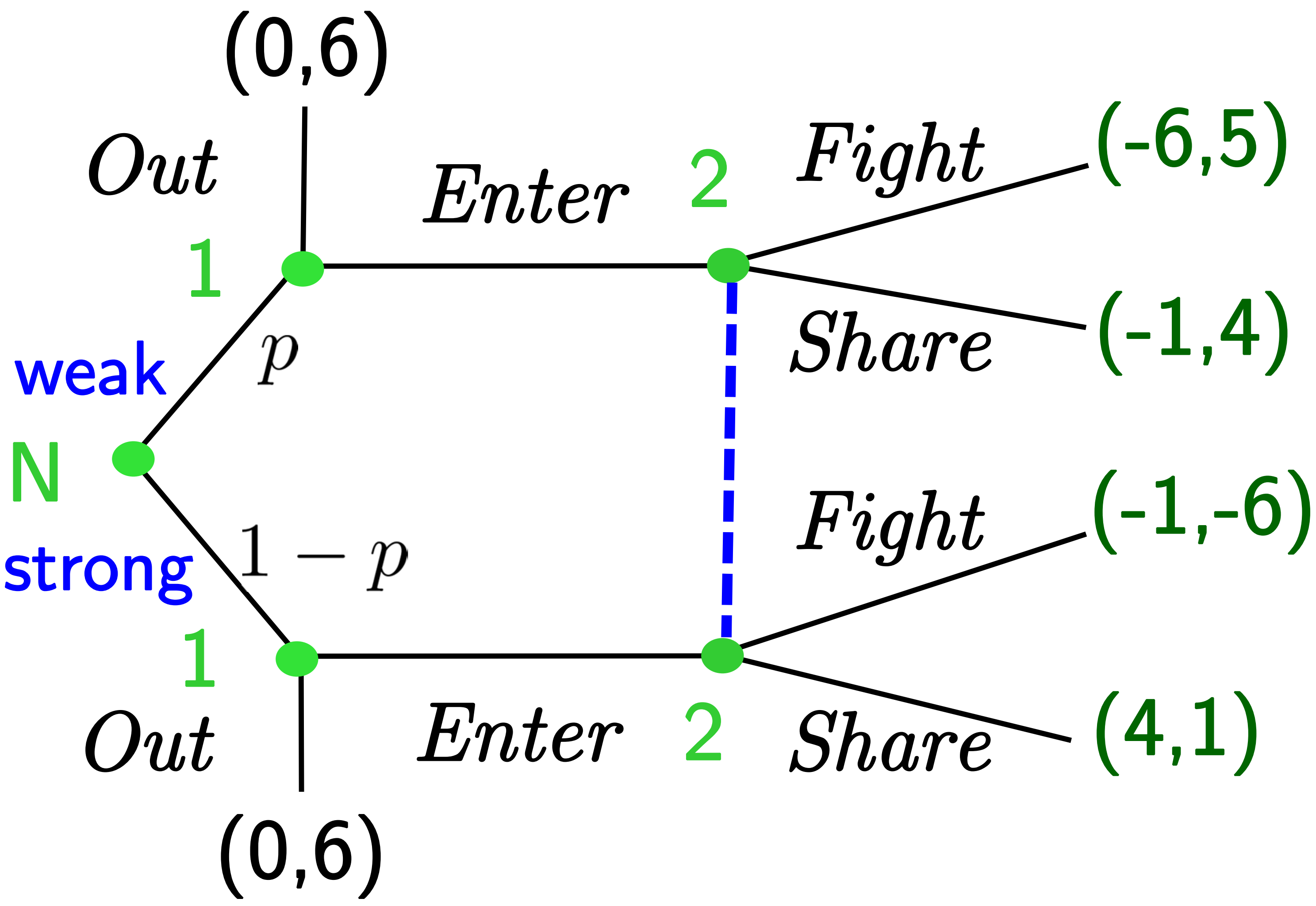
BNE when $p > 5/6$: (strong *Enter*; Others Mix)

BNE is $\left(\alpha = \frac{5(1-p)}{p}, Enter, \beta = \frac{3}{5}\right)$

(action if weak, action if strong, action if *Enter*)



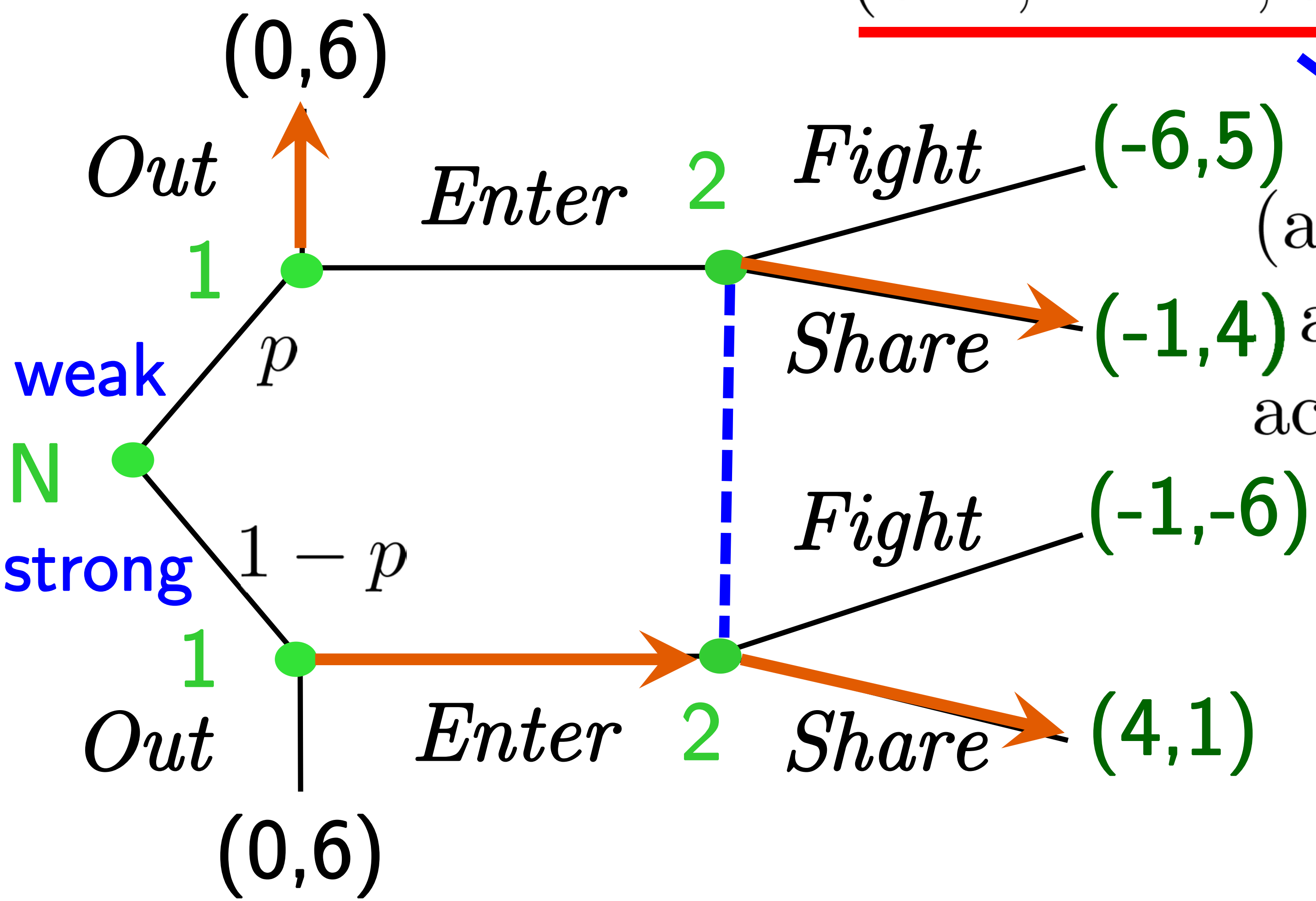
Modified Market Entry Game: New Payoffs...



Separating Equilibrium: strong-*Enter*, weak-*Out*

BNE is $(Out, Enter, Share)$

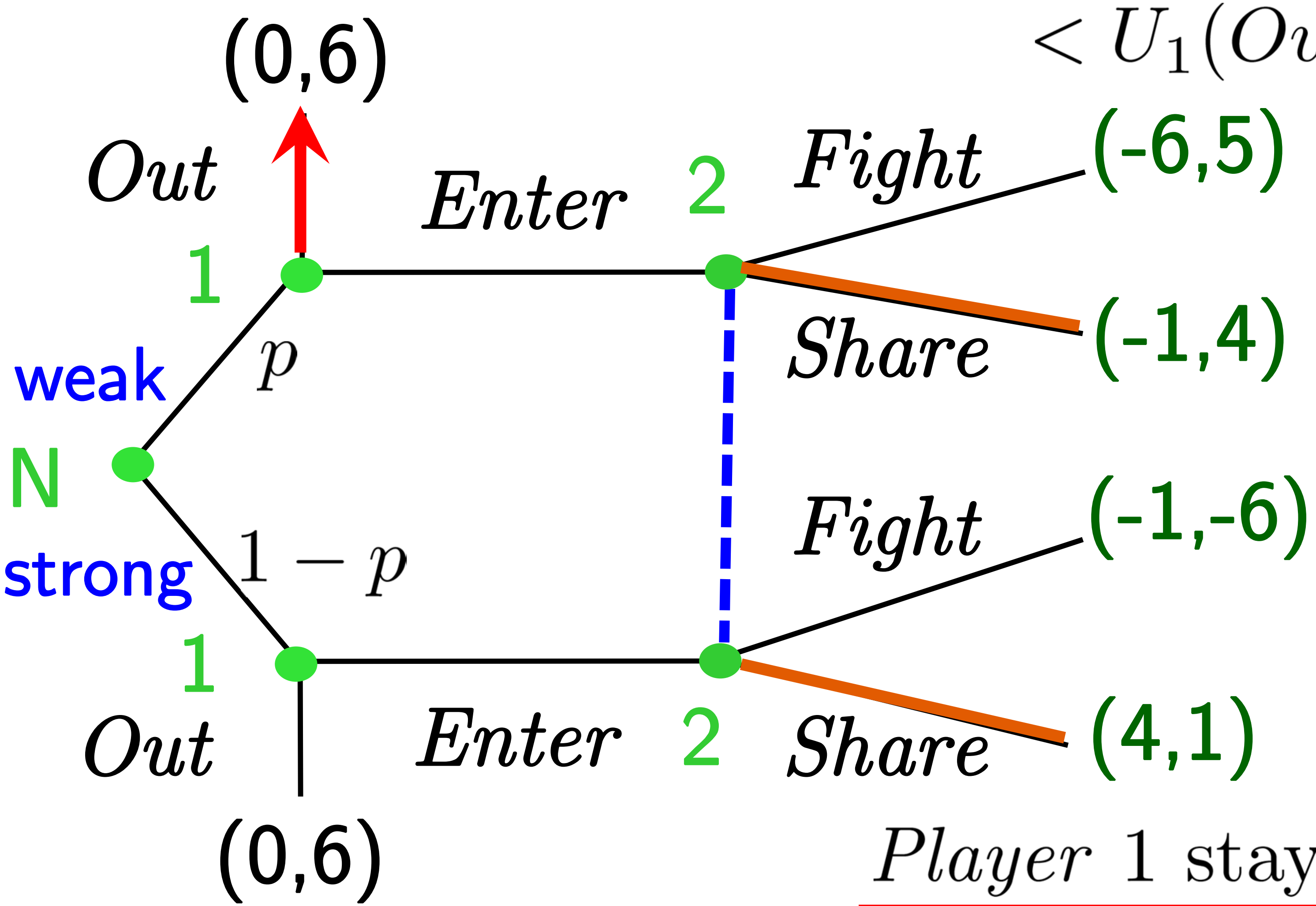
(action if weak, action if strong, action if $Enter$)



Separating Equilibrium: strong-*Enter*, weak-*Out*

$$U_1(Enter|\text{weak}) = -1$$

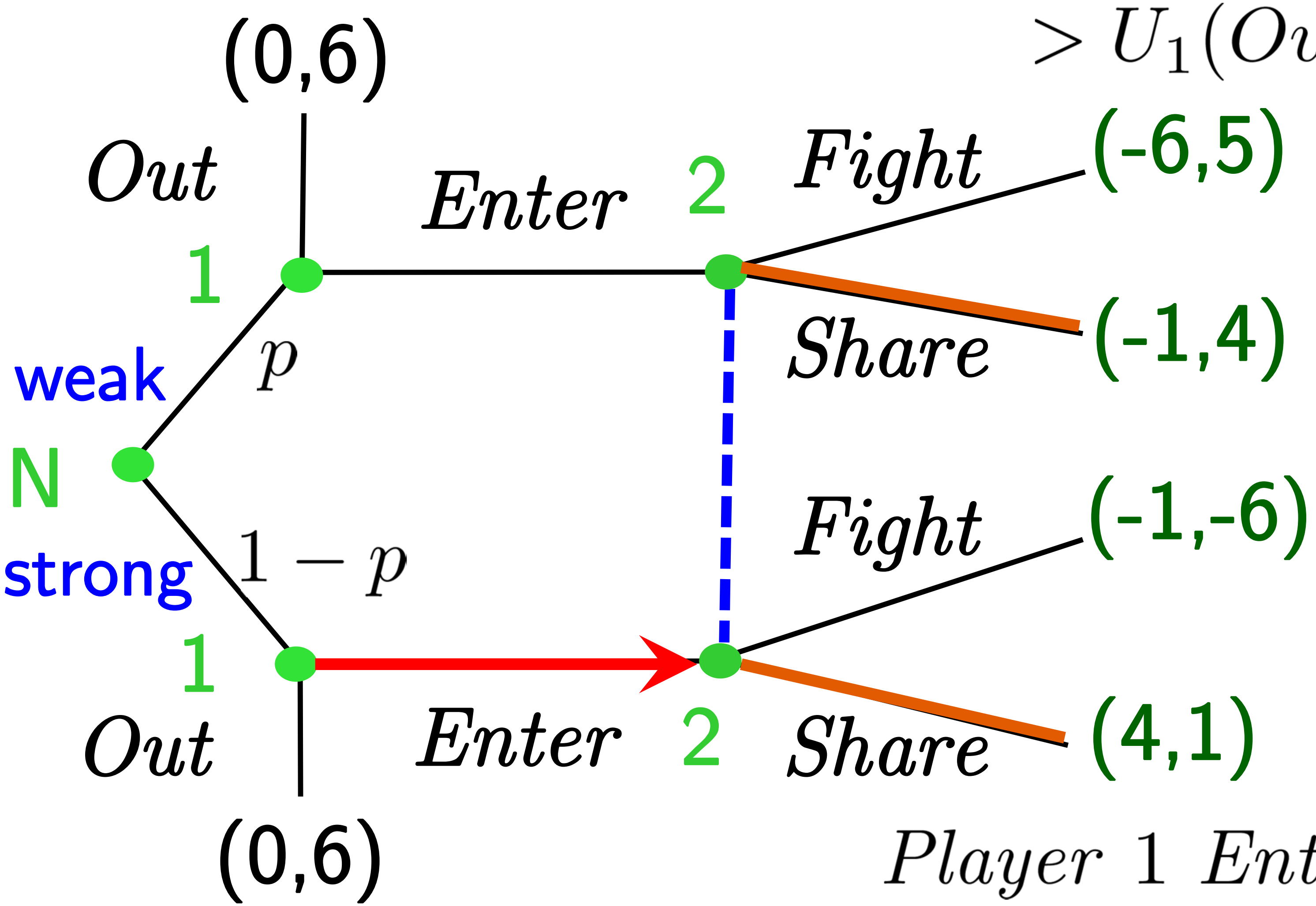
$$< U_1(Out|\text{weak}) = 0$$



Player 1 stays *Out* if weak

Separating Equilibrium: strong-*Enter*, weak-*Out*

$$U_1(Enter|\text{strong}) = 4$$
$$> U_1(Out|\text{strong}) = 0$$

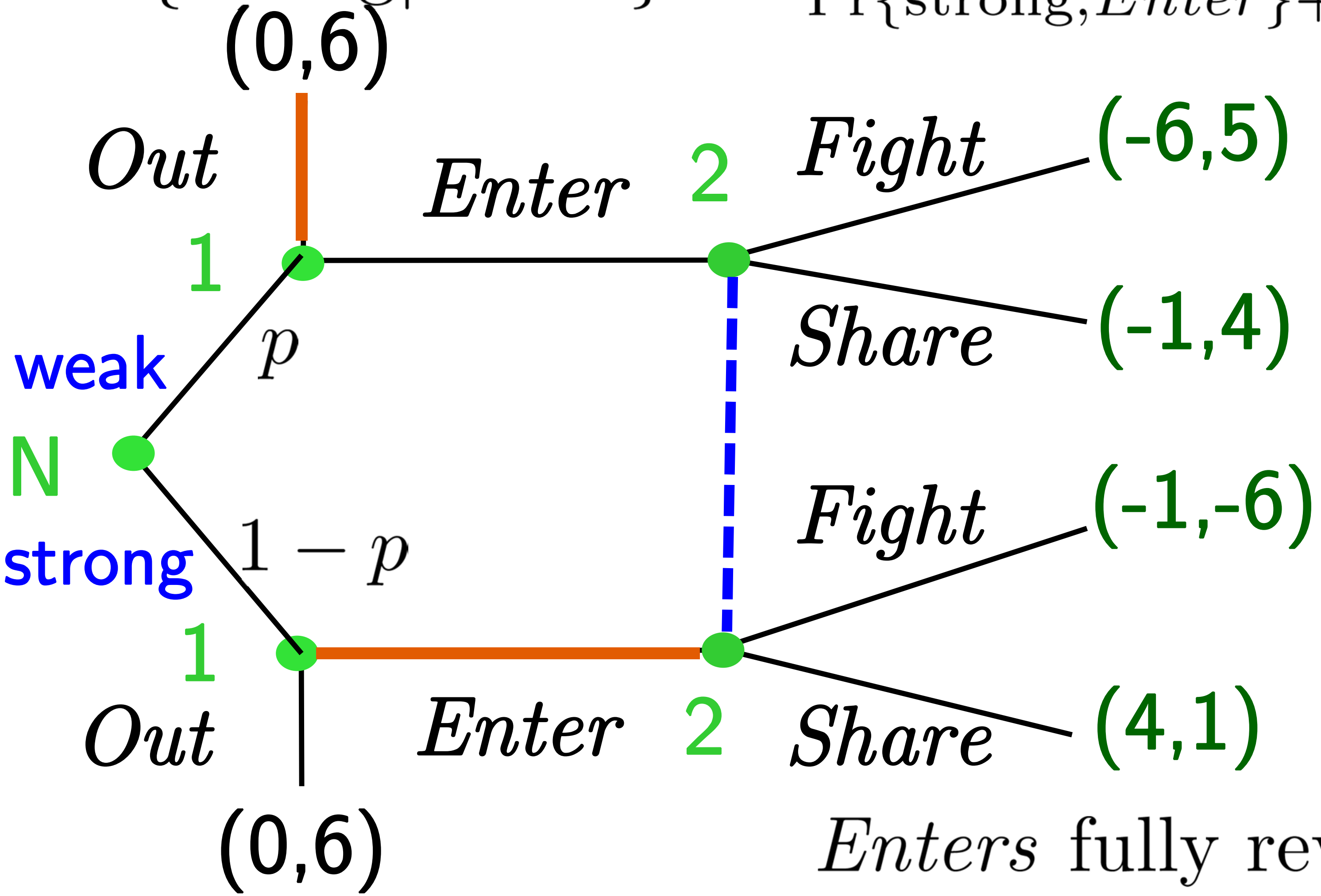


Player 1 *Enters* if strong

Separating Equilibrium: strong-*Enter*, weak-*Out*

$$\Pr\{\text{strong}|Enter\} = \frac{\Pr\{\text{strong}, Enter\}}{\Pr\{\text{strong}, Enter\} + \Pr\{\text{weak}, Enter\}}$$

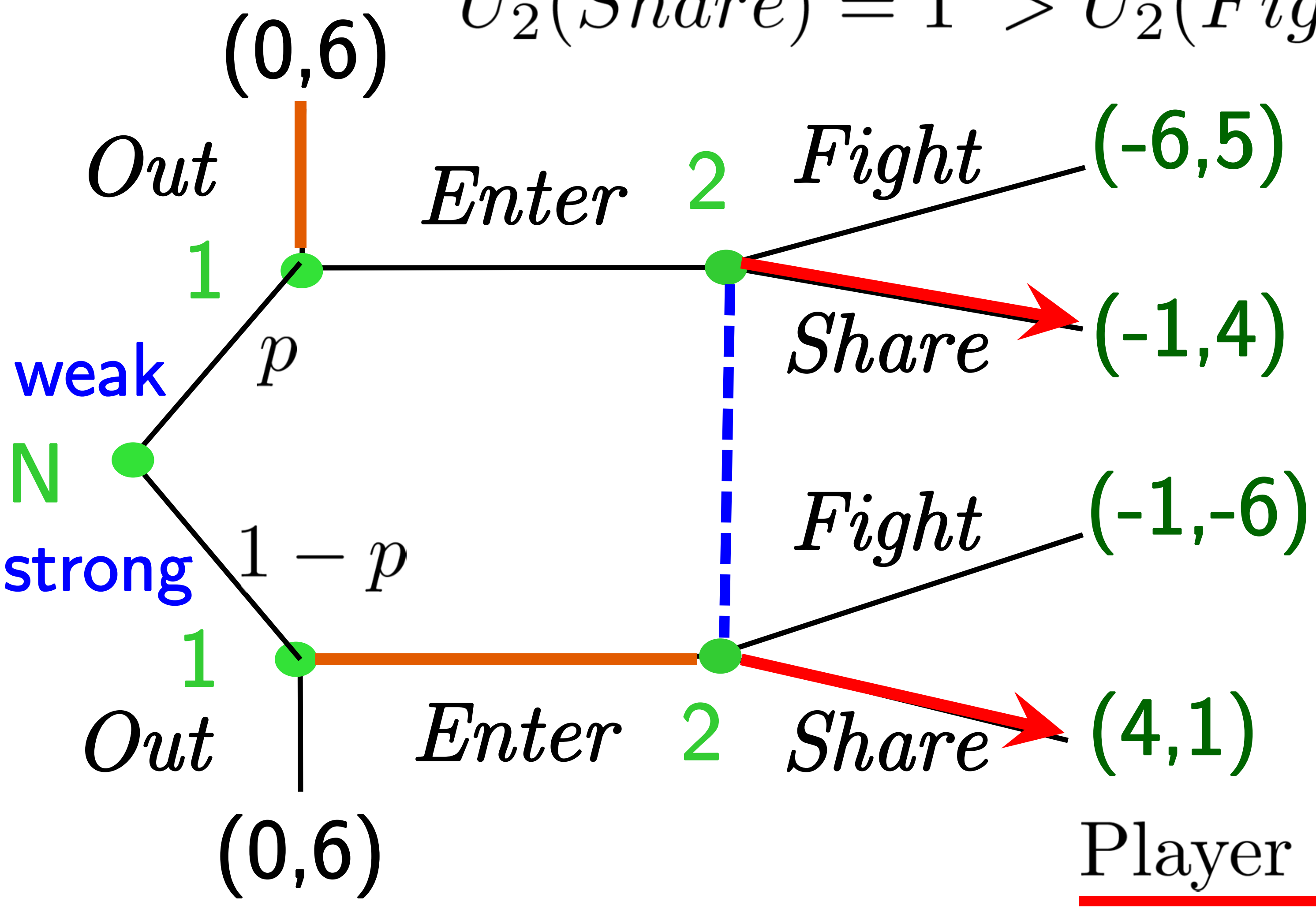
$$= 1$$



Enters fully reveals 1's type!

Separating Equilibrium: strong-*Enter*, weak-*Out*

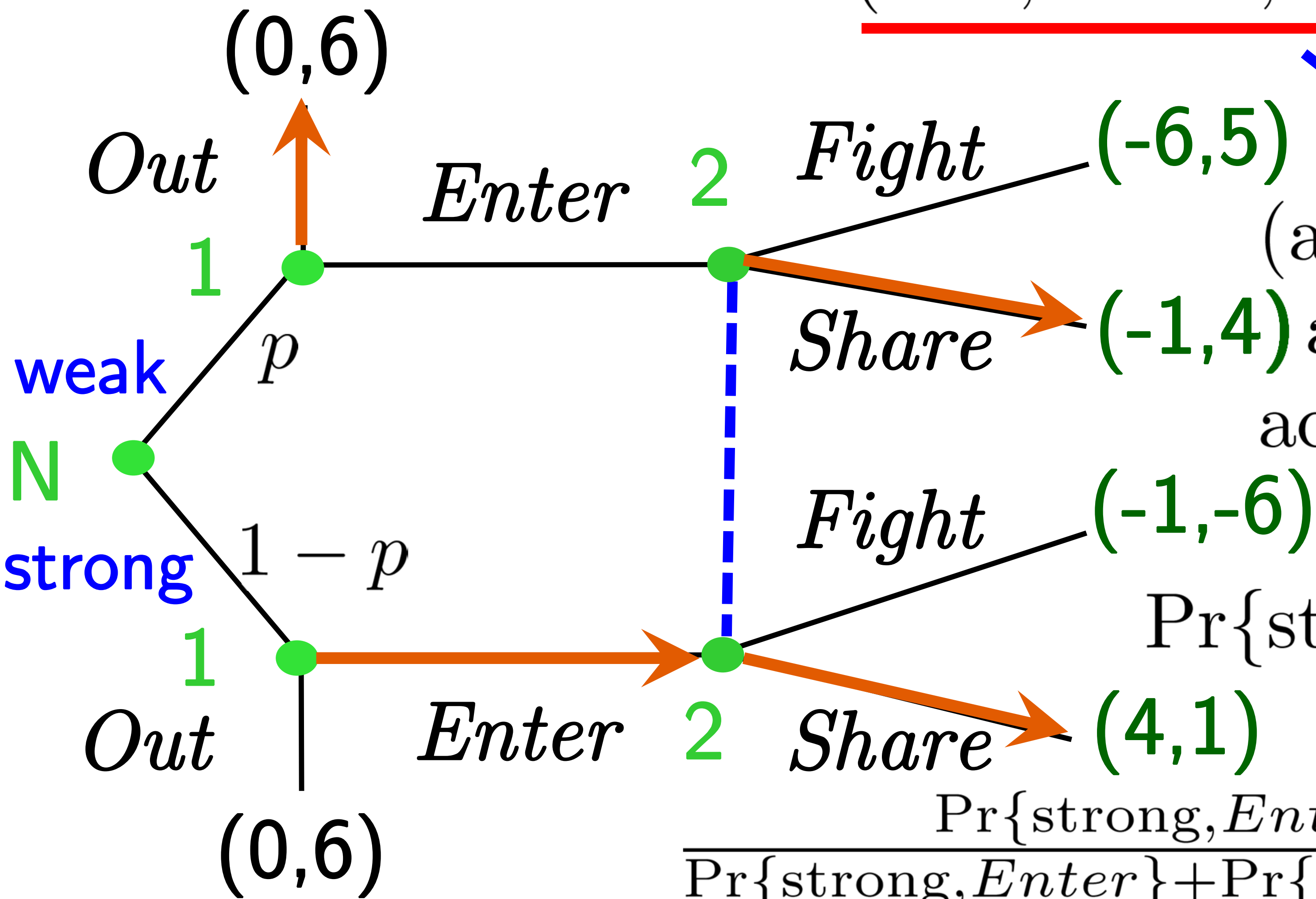
$$U_2(Share) = 1 > U_2(Fight) = -6$$



Player 2 will *Share*

Separating Equilibrium: strong-*Enter*, weak-*Out*

BNE is $(Out, Enter, Share)$



(action if weak, action if strong, action if $Enter$)

$$\Pr\{\text{strong}|Enter\} = \frac{\Pr\{\text{strong}, Enter\}}{\Pr\{\text{strong}, Enter\} + \Pr\{\text{weak}, Enter\}} = 1$$

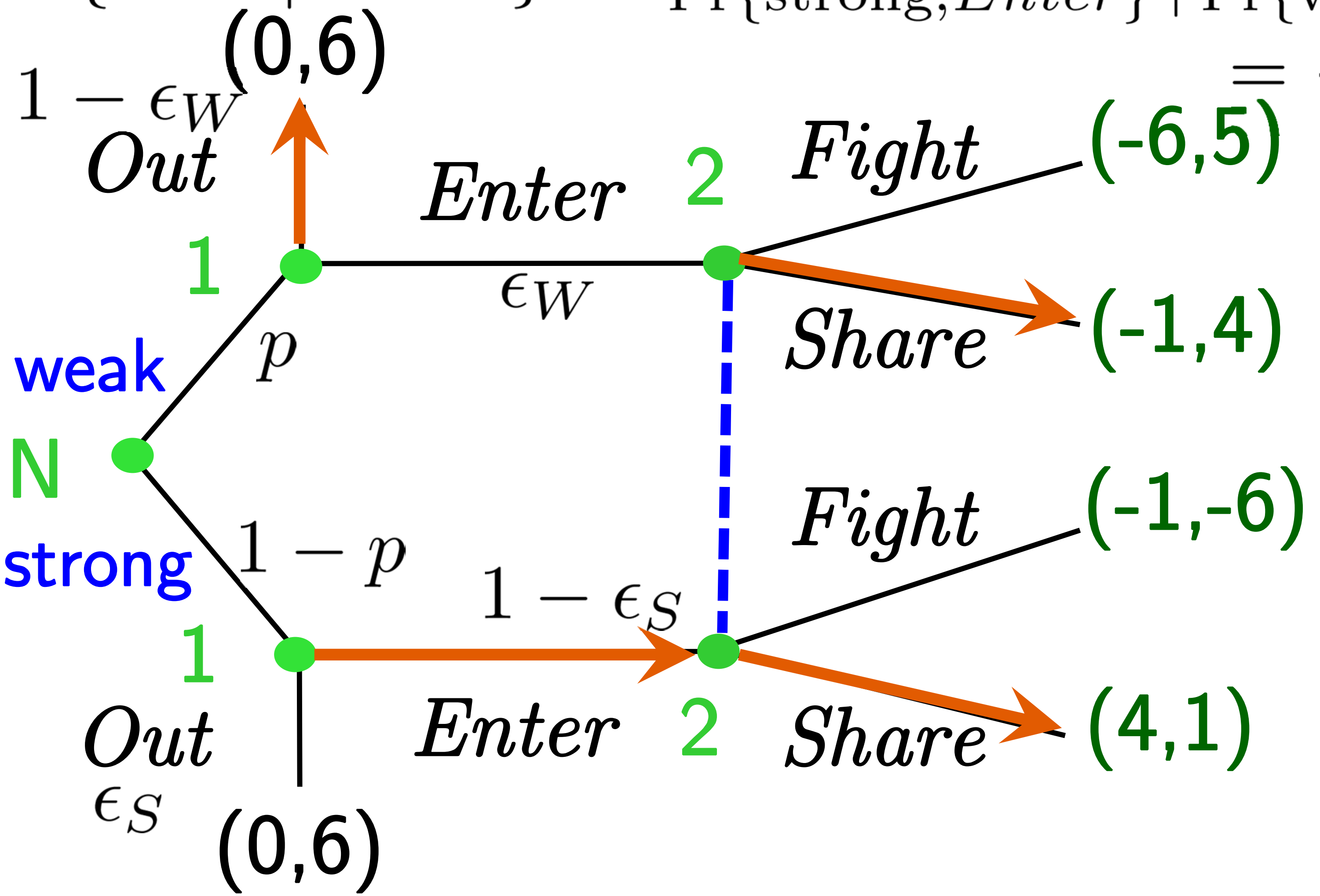
(strong-*Enter*; weak-*Out*) is also Sequential!

$$\Pr\{\text{weak}|Enter\} = \frac{\Pr\{\text{weak}, Enter\}}{\Pr\{\text{strong}, Enter\} + \Pr\{\text{weak}, Enter\}}$$

$$= \frac{p\epsilon_W}{(1-p)(1-\epsilon_S) + p\epsilon_W}$$

$$\to 0$$

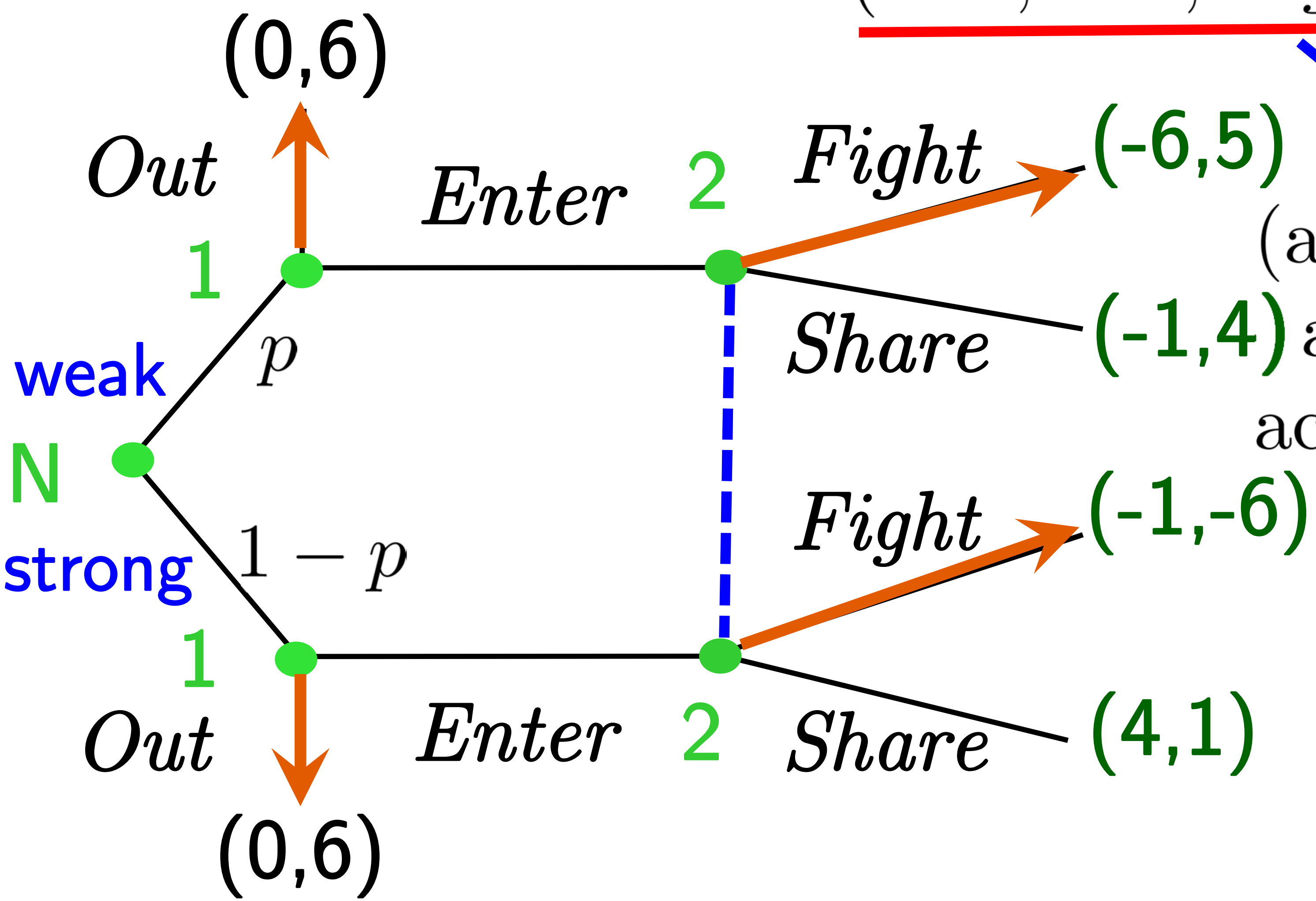
as $\epsilon_W \to 0$



Pooling Equilibrium: (Out, Out, $Fight$)

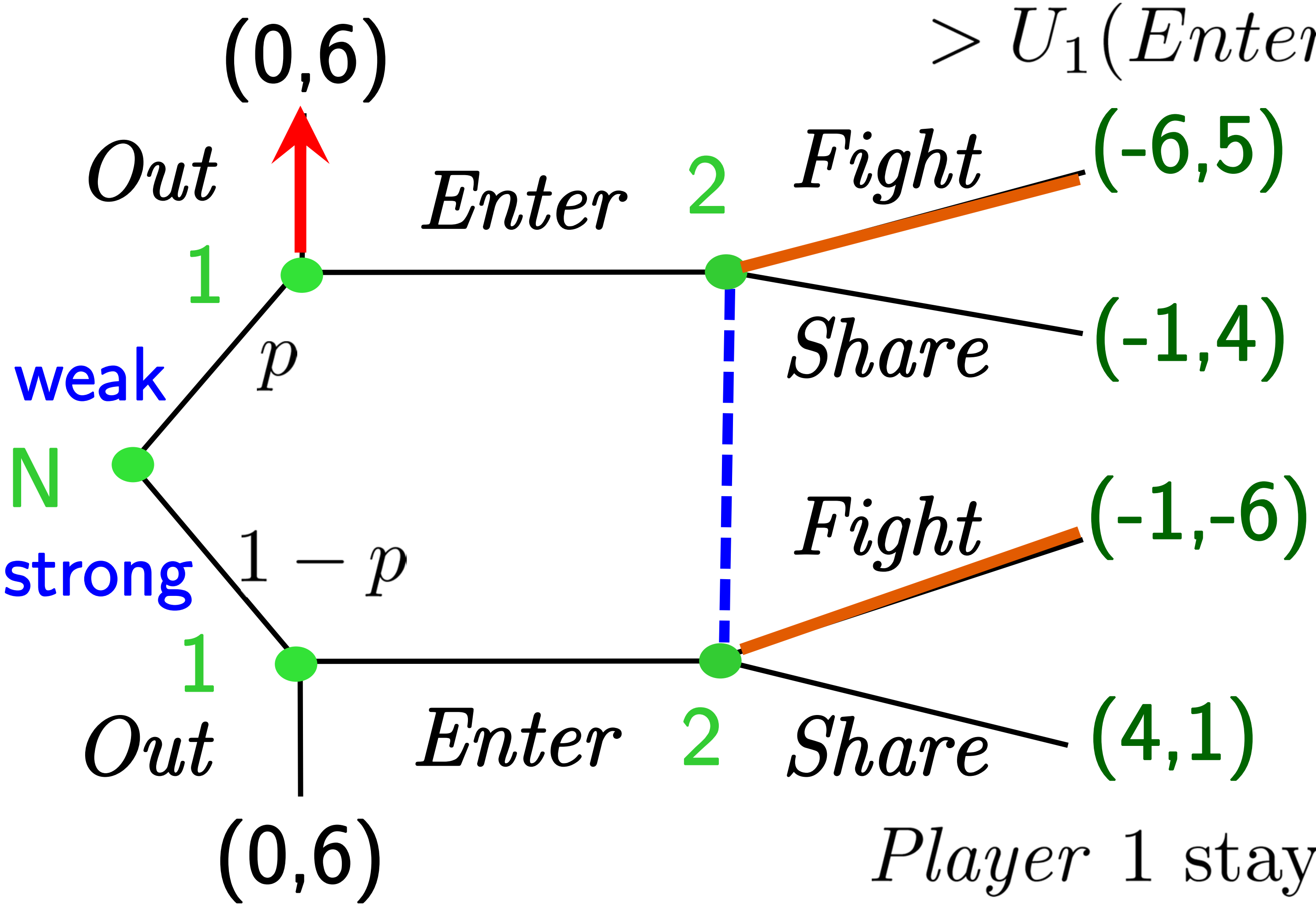
BNE is $(Out, Out, Fight)$

(action if weak, action if strong, action if $Enter$)



Pooling Equilibrium: (Out, Out, $Fight$)

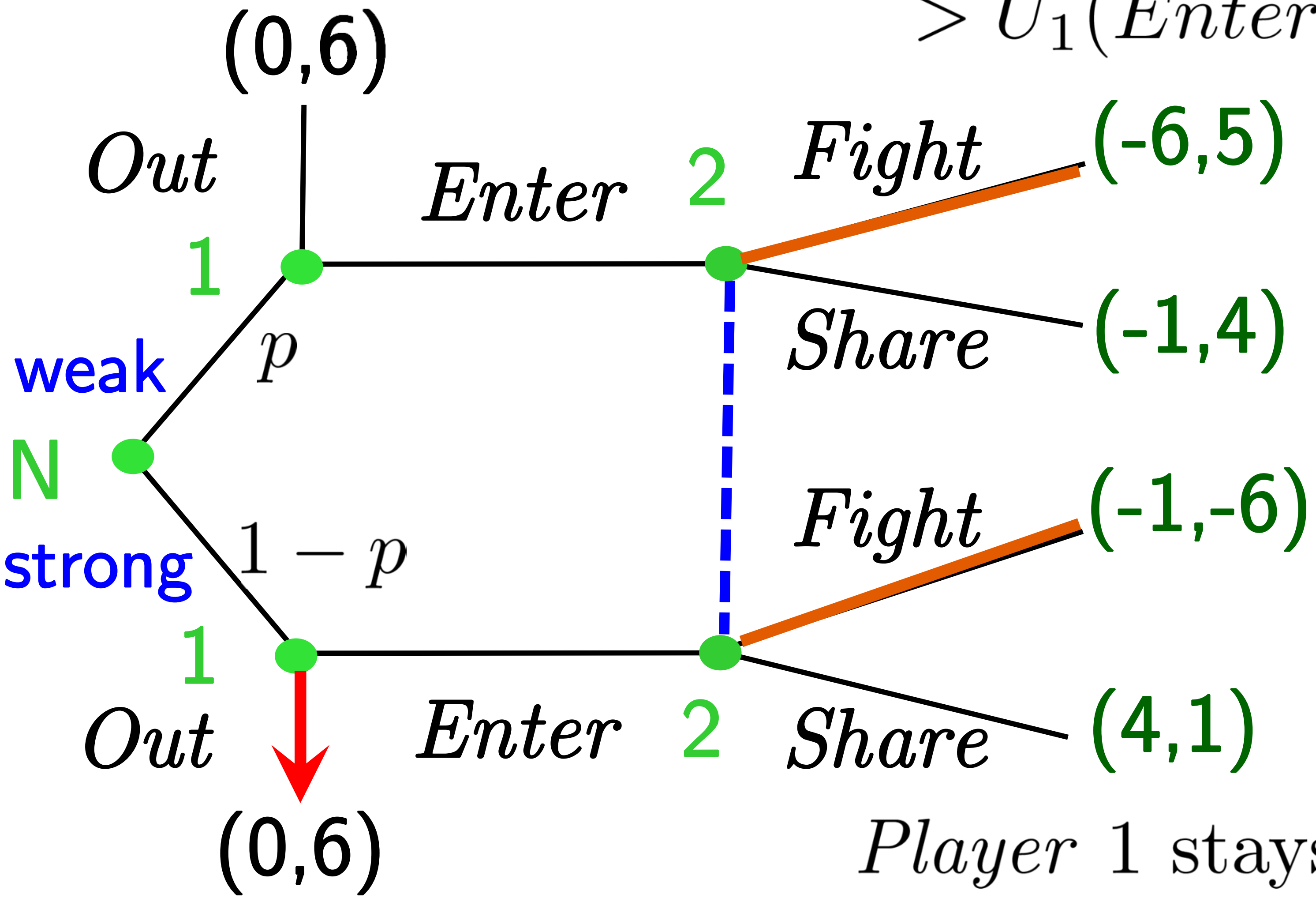
$$U_1(Out|\text{weak}) = 0$$
$$> U_1(Enter|\text{weak}) = -6$$



$Player$ 1 stays Out if weak

Pooling Equilibrium: (*Out*, *Out*, *Fight*)

$$U_1(Out|\text{strong}) = 0$$
$$> U_1(Enter|\text{strong}) = -1$$

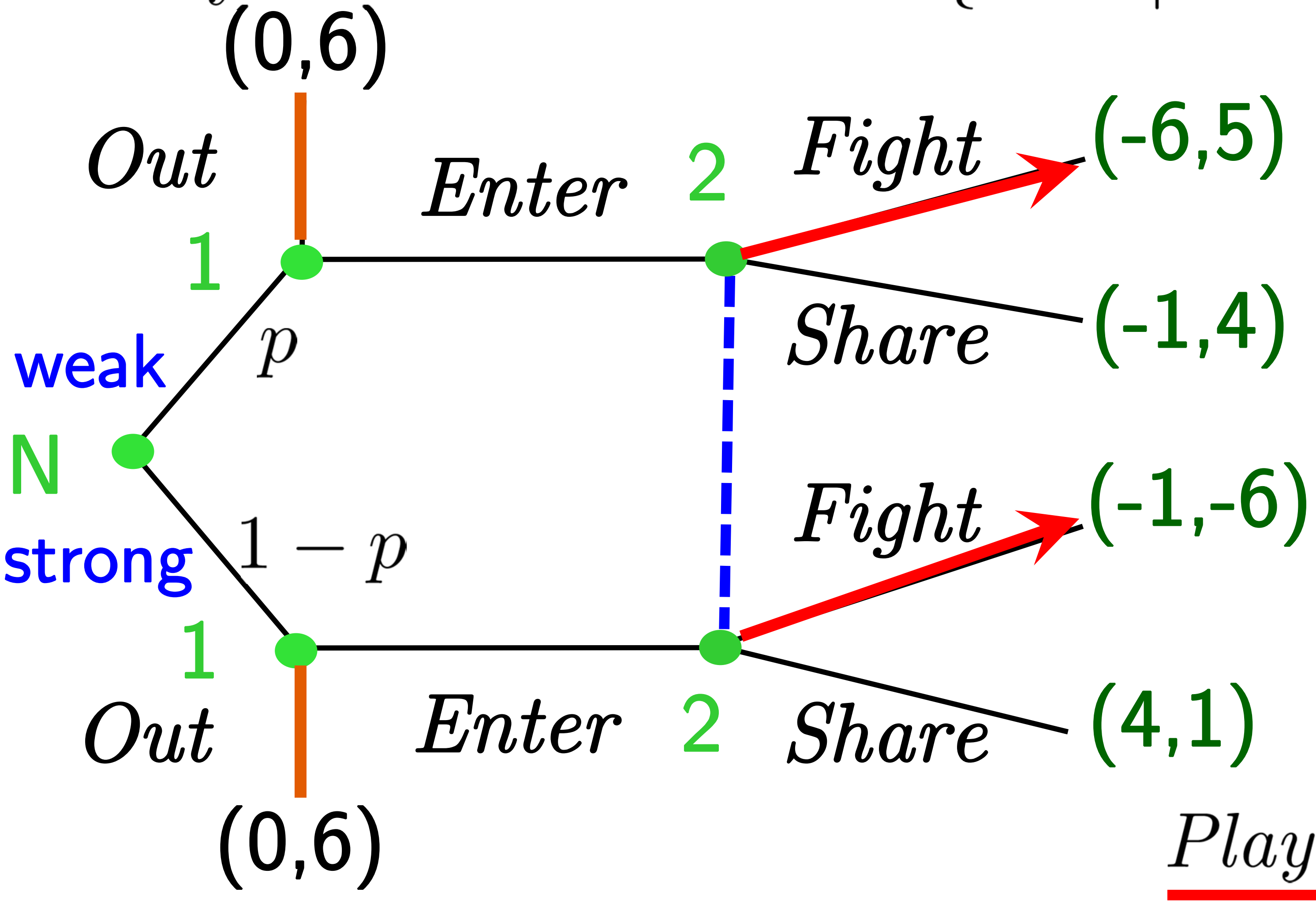


Player 1 stays *Out* if strong

Pooling Equilibrium: (*Out*, *Out*, *Fight*)

If Player 2 believes that $\Pr\{\text{weak}|Enter\} = \frac{1}{1+\theta}$

$> \frac{6}{11}$

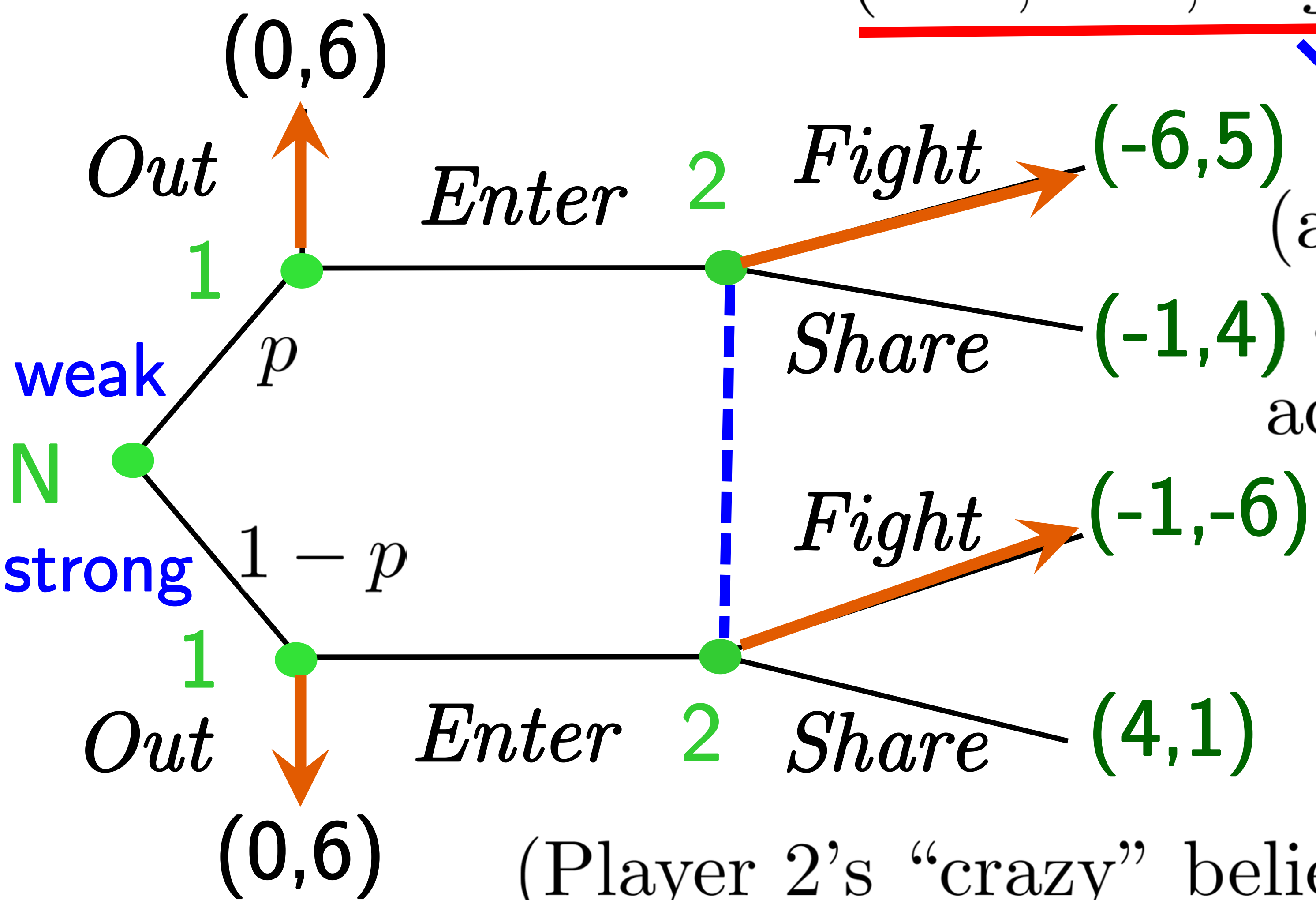


Player 2 will *Fight*

Pooling Equilibrium: (*Out*, *Out*, *Fight*)

BNE is $(Out, Out, Fight)$

(action if weak, action if strong, action if $Enter$)



(Player 2's "crazy" belief never tested)

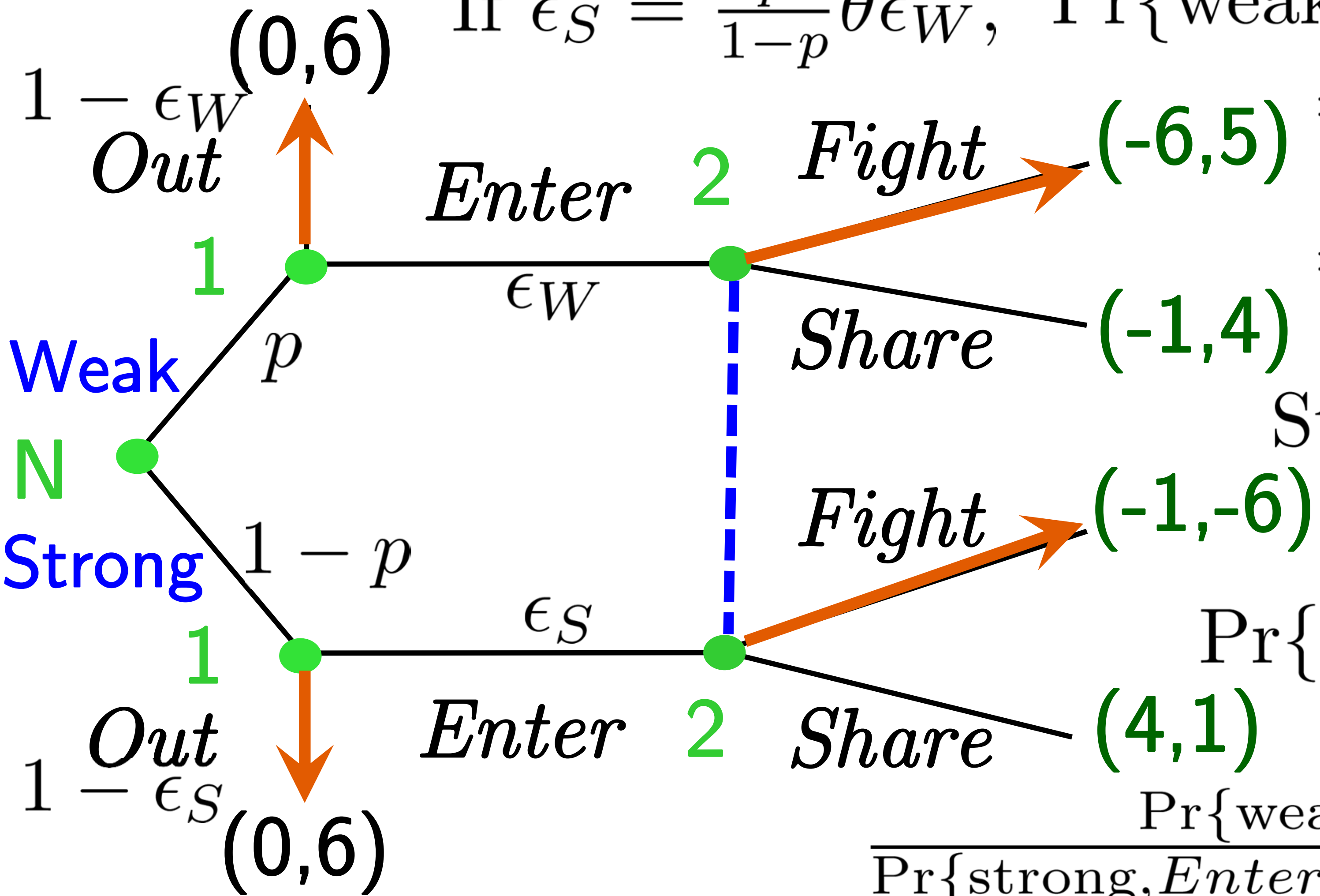
(Out, Out, $Fight$) is also a Sequential Equil.!

If $\epsilon_S = \frac{p}{1-p}\theta\epsilon_W$, $\Pr\{\text{weak}|Enter\}$

$$= \frac{p\epsilon_W}{(1-p)\epsilon_S + p\epsilon_W}$$

$$= \frac{1}{1+\theta}$$

Still Sequential!!



$$\Pr\{\text{weak}|Enter\} = \frac{\Pr\{\text{weak}, Enter\}}{\Pr\{\text{strong}, Enter\} + \Pr\{\text{weak}, Enter\}}$$

(Out, Out, $Fight$) is also a Sequential Equil.!

- (Out, Out, $Fight$) is not ruled out by THP, and hence, is also a Sequential Equilibrium…
- But why can't the strong type say,
- "If I enter, I will be credibly signaling that I am strong, since if I were weak and chose to *Enter*, my possible payoffs would be -1 or -6, smaller than 0 (equilibrium payoff if weak)."
- Seeing this, player 2's BR is *Share*
  - Profitable for strong player 1 to *Enter* & signal…

(Weak) Intuitive Criterion (Cho and Kreps)

- For first move player *1*'s action $\hat{a}$ (not in PBE)
- Let $u_1(\hat{a}, \theta, \theta')$ be player *i*'s payoff as type $\theta \in \Theta$
- if he chooses $\hat{a}$ but is believed to be type $\theta' \in \Theta$
- Let $u^N(\theta)$ be this types' PBE payoff
- The PBE fails the (Weak) Intuitive Criterion if, for some player *1* of type $\hat{\theta} \in \Theta$,

$$u_1(\hat{a}, \hat{\theta}, \hat{\theta}) > u_1^N(\hat{\theta})$$

- And, for all other types $\theta \in \Theta$, (can't signal)

$$\max_{x \in \Theta} u_1(\hat{a}, \theta, x) < u_1^N(\theta)$$

(Strong) Intuitive Criterion (for 3+ types)

- For first move player *1*'s action $\hat{a}$ (not in PBE)
- Let $u_1(\hat{a}, \theta, \theta')$be player *i*'s payoff as type $\theta \in \Theta$
- if he chooses $\hat{a}$ but is believed to be type $\theta' \in \Theta$
- Let $u_1^N(\theta)$ be this types' PBE payoff
- The PBE fails the (Strong) Intuitive Criterion if, for some player *1* of type $\hat{\theta} \in \Theta$,

$$u_1(\hat{a}, \hat{\theta}, \hat{\theta}) > u_1^N(\hat{\theta})$$

- And, for all other types $\theta \in \Theta$, (can't mimic)

$$u_1(\hat{a}, \theta, \hat{\theta}) < u_1^N(\theta)$$

Intuitive Criterion (Cho and Kreps)

- IC: I can credibly signal that I am high type
  - Cause I gain (against PBE) if you believe me, and
- Weak IC: Nobody else can make similar claims
  - Not only this claim, but any similar claim
  - Stronger requirement of failure = weaker criterion
- Strong IC: Nobody else can make this claim
  - Weaker requirement of failure = stronger criterion
- With only two types, weak and strong IC are the same...

Intuitive Criterion (Cho and Kreps)

- In the previous Example,
- (*Out*, *Out*, *Fight*) fails the Intuitive Criterion
  - "If I enter, I will be credibly signaling that I am strong, since I gain if you believe me and if I were weak and chose to *Enter*, my possible payoffs would be -1 or -6, smaller than 0 (PBE payoff if weak)."
- (*Out*, *Enter*, *Share*) meets Intuitive Criterion
  - Such argument is not credible...

Summary of 10.2

- "SPE" under incomplete information: PBE
  - Two special cases: SE and THP
- Different Types of PBE:
  - Pooling Equilibrium
  - Separating Equilibrium
  - Semi-Pooling Equilibrium (MSE)
- Intuitive Criteria
- HW 10.2: See handout