Shadow Prices

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(Lecture 1, Micro Theory I)

A Peak-Load Pricing Problem

- Consider the problem faced by Chunghwa Telecom (CHT):
- By building base stations, CHT can provide cell phone service to a certain region
  - An establish network can provide service both in the day and during the night
  - Marginal cost is low (zero?!); setup cost is huge
- Marketing research reveal unbalanced demand…
  - Day – peak; Night – off-peak (or vice versa?)

A Peak-Load Pricing Problem

- If you are the CEO of CHT, how would you price day and night usage of your service?
  - The same or different?
- Economic intuition should tell you to set off-peak prices lower than peak prices
  - But how low?
- FET's Big Broadband Service (遠傳大寬頻) faced a similar problem recently…

More on Peak-Load Pricing

- Other similar problems include:
  - How should Taipower price electricity in the summer and winter?
  - How should a theme park set its ticket prices for weekday and weekends?
- Even if demand estimations are available, you will still need to do some math to find optimal prices…
  - Either to maximize profit or social welfare

A Peak-Load Pricing Problem

- Back to CHT:
- Capacity constraints:

$$q_j \leq q_0, j = 1, ..., n$$

- CHT's Cost function:

$$C(q_0, q) = F + c_0 q_0 + c \cdot q$$

- Demand for cell phone service:

$$\text{Demand } p_j(q), \text{ Total Revenue } R(q) = p \cdot q$$

A Peak-Load Pricing Problem

- The monopolist profit maximization problem:

- How do you solve this problem?
- When does FOC guarantee a solution?
- What does the Lagrange multiplier mean?
- What should you do when FOC “fails”?

Need: Lagrange Multiplier Method

1. Write Constraints as $h_i(x) \geq 0, i = 1, \ldots, m$
   $h(x) = \left(h_1(x), \ldots, h_m(x)\right)$
2. Shadow prices $\lambda = (\lambda_1, \ldots, \lambda_m)$

- Lagrangian $\mathfrak{L}(x, \lambda) = f\left(x\right) + \lambda \cdot h\left(x\right)$
- FOC:

$$\frac{\partial \mathfrak{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \lambda \cdot \frac{\partial h}{\partial x_j} \leq 0, \text{ with equality if } \bar{x}_j > 0.$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda_i} = h_i\left(\bar{x}\right) \geq 0, \text{ with equality if } \lambda_i > 0.$$

Solving Peak-Load Pricing

- The monopolist profit maximization problem:

- The Lagrangian is

$$\mathfrak{L}(q_0, q) = R(q) - F - c_0 q_0 - \sum_{j=1}^{n} c_j q_j + \sum_{j=1}^{n} \lambda_j \left(q_0 - q_j \right)$$

$$= R(q) - \sum_{j=1}^{n} (c_j + \lambda_j) q_j + \left(\sum_{j=1}^{n} \lambda_j - c_0 \right) q_0 - F$$

Solving Peak-Load Pricing

- FOC:

$$\frac{\partial \mathfrak{L}}{\partial q_j} = MR_j - c_j - \lambda_j \leq 0, \text{ with equality if } q_j > 0.$$

$$\frac{\partial \mathfrak{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 \leq 0, \text{ with equality if } q_0 > 0.$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- For positive production, FOC becomes:

$$\frac{\partial \mathfrak{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0, \text{ since } q_j > 0.$$

$$\frac{\partial \mathfrak{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0, \text{ since } q_0 > 0.$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- Meaning of FOC:

$$\frac{\partial \mathfrak{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0, \text{ since } q_j > 0.$$

$$\frac{\partial \mathfrak{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0, \text{ since } q_0 > 0.$$

At least 1 period has shadow price > 0!

$$\frac{\partial \mathfrak{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- Meaning of FOC:

$$\frac{\partial \mathfrak{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0,$$

Hit capacity at positive shadow price!

$$\frac{\partial \mathfrak{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0,$$

Off-peak shadow price = 0

$$\frac{\partial \mathfrak{L}}{\partial \lambda_j} = q_0 - q_j \geq 0, \text{ with equality if } \lambda_j > 0.$$

Solving Peak-Load Pricing

- Meaning of FOC: Peak MR=MC+capacity cost

$$\frac{\partial \mathfrak{L}}{\partial q_j} = MR_j - c_j - \lambda_j = 0,$$

$$MR_i(\overline{q}) = c_i + \lambda_i$$

$$\frac{\partial \mathfrak{L}}{\partial q_0} = \sum_{j=1}^{n} \lambda_j - c_0 = 0,$$

Peak periods share capacity cost via shadow price

Off-peak: MR=MC!

$MR_j(\overline{q}) = c_j$ equality if $\lambda_j > 0$.

Solving Peak-Load Pricing

- Economic Insight of FOC:
- Marginal decision of the manager: MR=MC
- Off-peak: MR=operating MC
  - Since didn't hit capacity
- Peak: Need to increase capacity
  - MR of all peak periods = cost of additional capacity
    + operating MC of all peak periods
- What's the theory behind this?

Constrained Optimization: Economic Intuition

- Single Constraint Problem:

$$\underset{x}{\text{Max}}\{\, f(x) \mid x \geq 0, b - g(x) \geq 0 \}$$

- Interpretation: a profit maximizing firm
  - Produce non-negative output $x \geq 0$
  - Subject to resource constraint $g(x) \leq b$

- Example: linear constraint $a \cdot x = \sum_{j=1}^{n} a_j x_j \leq b$

Each unit of x_j requires a_j units of resource b.

Constrained Optimization: Economic Intuition

- Single Constraint Problem:

$$\underset{x}{\text{Max}}\{\, f(x) \mid x \geq 0, b - g(x) \geq 0 \}$$

- Interpretation: a utility maximizing consumer
  - Consume non-negative input $x \geq 0$
  - Subject to budget constraint $g(x) \leq b$

- Example: linear constraint $a \cdot x = \sum_{j=1}^{n} a_j x_j \leq b$

Each unit of x_j requires a_j units of currency b.

Constrained Optimization: Economic Intuition

- Suppose $\bar{x}$ solves the problem
- If increases x_j, profit changes by $\dfrac{\partial f}{\partial x_j}$
- Additional resources needed: $\dfrac{\partial g}{\partial x_j}$
- Cost of additional resources: $\lambda \dfrac{\partial g}{\partial x_j}$
  - (Market (or shadow) price is λ)

Net gain to increasing x_j is $\dfrac{\partial f}{\partial x_j}(\bar{x}) - \lambda \dfrac{\partial g}{\partial x_j}(\bar{x})$

Necessary Conditions for $\bar{x}_j$

- If $\bar{x}_j$ is strictly positive, marginal net gain $= 0$
  - i.e. $\bar{x}_j > 0 \Rightarrow \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) = 0$
- If $\bar{x}_j$ is zero, marginal net gain ≤ 0
  - i.e. $\bar{x}_j = 0 \Rightarrow \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) \leq 0$

$$\frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) \leq 0 \text{ with equality if } \bar{x}_j > 0.$$

Necessary Conditions for $\overline{x}_j$

- If resources doesn't bind, opportunity cost $\lambda = 0$
  - i.e. $b - g(\overline{x}) > 0 \Rightarrow \lambda = 0$

- Or, in other words,

$$b - g(\overline{x}) \geq 0 \text{ with equality if } \lambda > 0.$$

  - This is logically equivalent to the first statement.

Lagrange Multiplier Method

1. Write constraint as $h(x) \geq 0$
2. Lagrange multiplier = shadow price λ

- Lagrangian $\mathfrak{L}(x,\lambda) = f(x) + \lambda \cdot h(x)$
- FOC:

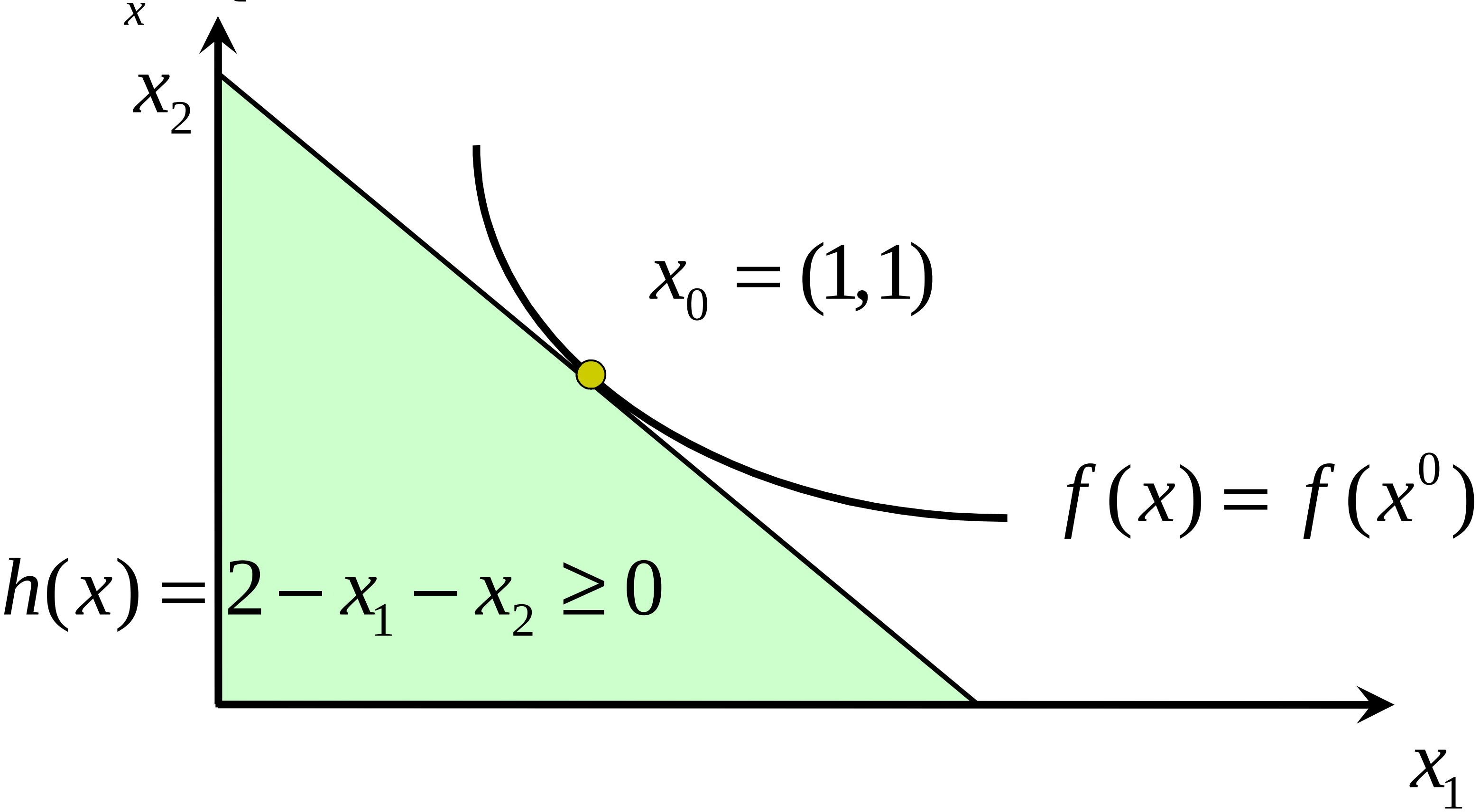
$$\frac{\partial \mathfrak{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \lambda \cdot \frac{\partial h}{\partial x_j} \leq 0, \text{ with equality if } \bar{x}_j > 0.$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda} = h(\bar{x}) \geq 0, \text{ with equality if } \lambda > 0.$$

Example 1

- A consumer problem:

$$\operatorname{Max}_{x}\{\, f(x) = \ln x_1 x_2 \mid x \geq 0, h(x) = 2 - x_1 - x_2 \geq 0 \}$$



Example 1

- Maximum at $\bar{x} = (1,1)$
- Lagrangian $\mathfrak{L}(x, \lambda) = \ln x_1 + \ln x_2 + \lambda(2 - x_1 - x_2)$
- FOC

$$\frac{\partial \mathfrak{L}}{\partial x_j} = \frac{1}{x_j} + \lambda \leq 0, \text{ with equality if } \bar{x}_j > 0.$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda} = 2 - x_1 - x_2 \geq 0, \text{ with equality if } \lambda > 0.$$

Lagrange Multiplier Method with Multiple Constraints

1. Write Constraints as $h_i(x) \geq 0, i = 1, \ldots, m$
   $h(x) = \left(h_1(x), \ldots, h_m(x)\right)$
2. Shadow prices $\lambda = (\lambda_1, \ldots, \lambda_m)$

- Lagrangian $\mathfrak{L}(x, \lambda) = f\left(x\right) + \lambda \cdot h\left(x\right)$
- FOC:

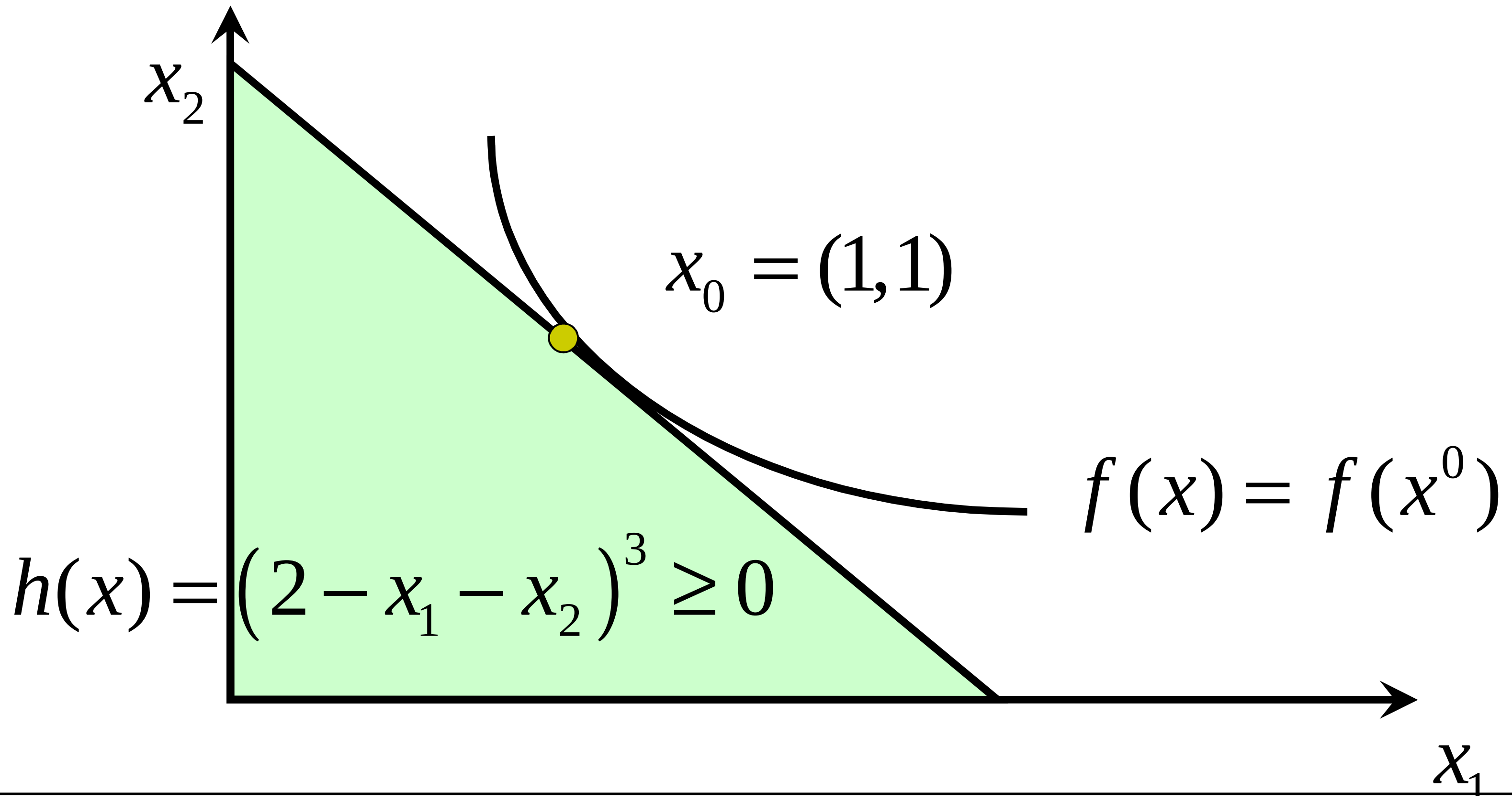
$$\frac{\partial \mathfrak{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \lambda \cdot \frac{\partial h}{\partial x_j} \leq 0, \text{ with equality if } \bar{x}_j > 0.$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda_i} = h_i\left(\bar{x}\right) \geq 0, \text{ with equality if } \lambda_i > 0.$$

When Intuition Breaks Down? Example 2

- A “new” problem:

$$\underset{x}{\text{Max}}\left\{ f(x) = \ln x_1 x_2 \mid x \geq 0, h(x) = \left(2 - x_1 - x_2\right)^3 \geq 0 \right\}$$



When Intuition Breaks Down? Example 2

- Lagrangian $\mathfrak{L}(x,\lambda) = \ln x_1 + \ln x_2 + \lambda(2 - x_1 - x_2)^3$
- FOC is violated!

$$\frac{\partial \mathfrak{L}}{\partial x_j} = \frac{1}{x_j} - 3\lambda\left(2 - x_1 - x_2\right)^2 = 1 \text{ at } \bar{x} = (1,1)$$

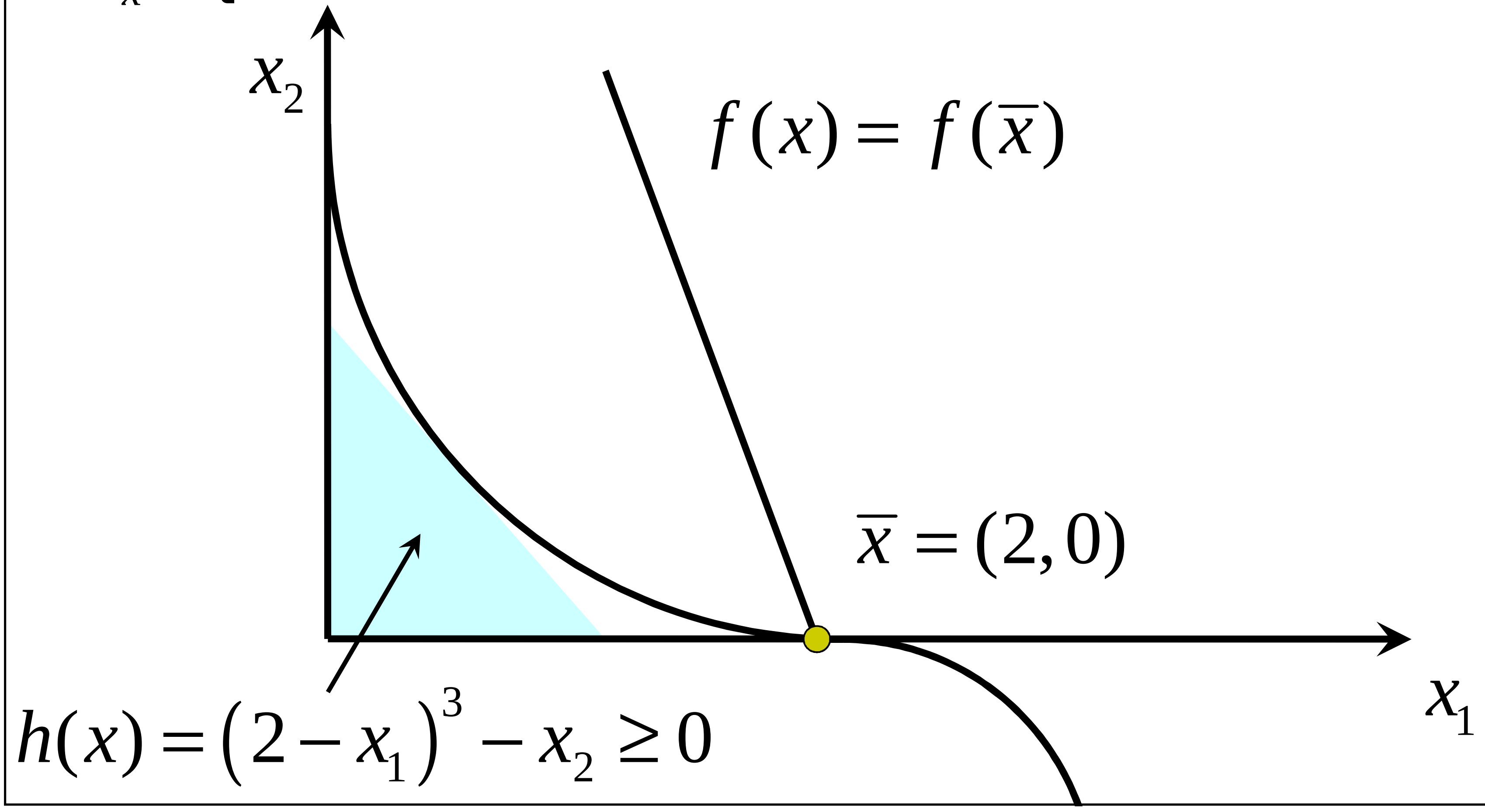
- How could this be?
- Because "linearization" fails if gradient = 0…

$$\frac{\partial h}{\partial x} = 0 \text{ at } \bar{x} = (1,1)$$

$$\bar{h}(x) = h(\bar{x}) + \frac{\partial h}{\partial x}(\bar{x}) \cdot (x - \bar{x}) = h(1,1) = 0$$

Other Breaks Down? Example 3

$$\underset{x}{\text{Max}}\left\{ f(x) = 12x_1 + x_2 \mid x \geq 0, h(x) = \left(2 - x_1\right)^3 - x_2 \geq 0 \right\}$$



Other Breaks Down? Example 3

- Lagrangian $\mathfrak{L}(x,\lambda) = 12x_1 + x_2 + \lambda\left[(2-x_1)^3 - x_2\right]$
- FOC is violated!

$$\frac{\partial \mathfrak{L}}{\partial x_1} = 12 - 3\lambda\left(2-\bar{x}_1\right)^2 = 12 \text{ at } \bar{x} = (2,0)$$

- What's the problem this time?
- Not the gradient… $\frac{\partial h}{\partial x}(\bar{x}) = (0,-1)$
- "Linearized feasible set" has no interior…

Other Breaks Down? Example 3

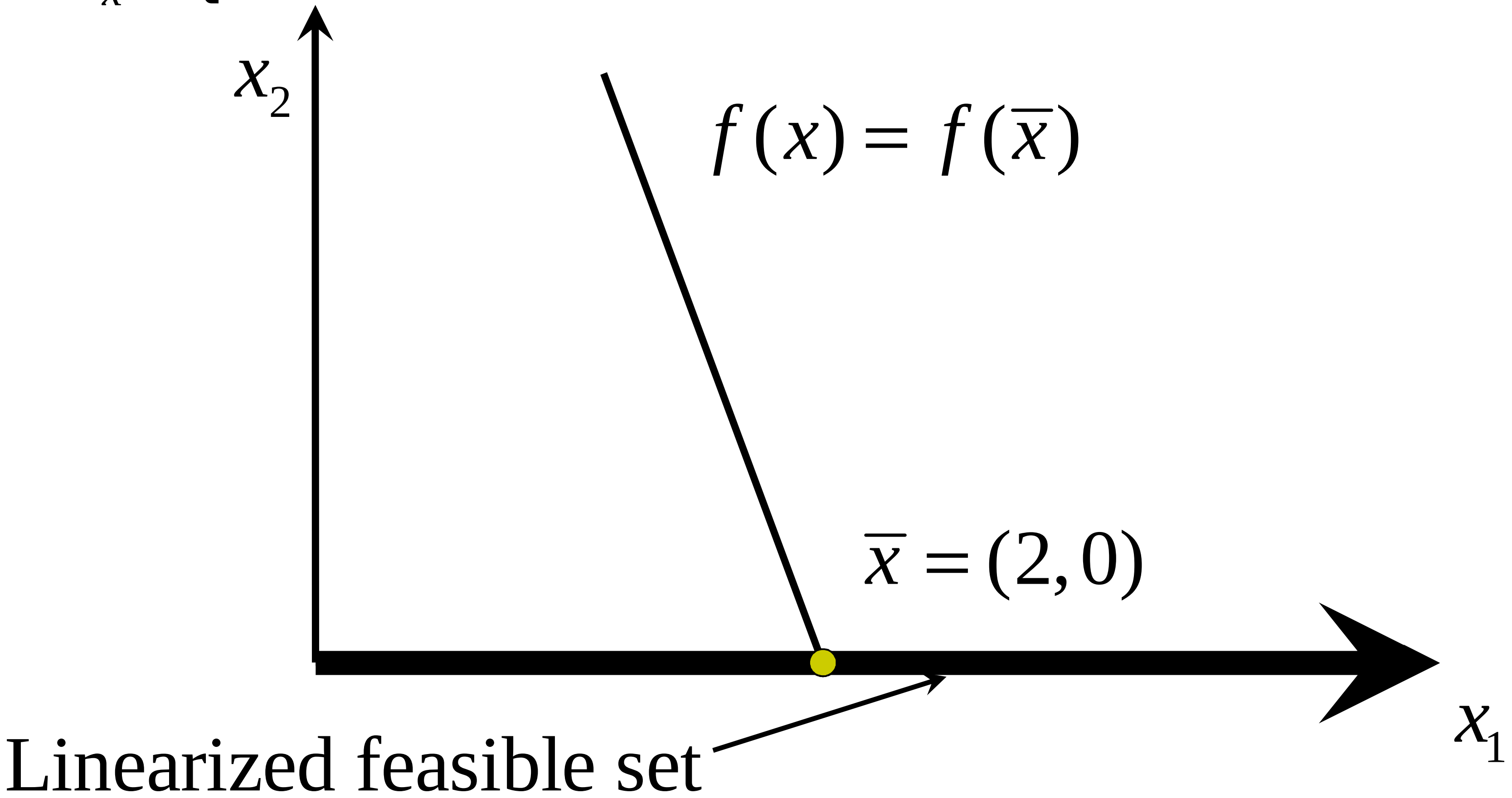
- What's the problem this time?
- Gradient is $\frac{\partial h}{\partial x}(\bar{x}) = (0,-1)$
- Hence, the linear approximation of the constraint is:

$$\frac{\partial h}{\partial x}(\bar{x})\cdot(x-\bar{x}) = \frac{\partial h}{\partial x_1}(\bar{x})\cdot(x_1-2)+\frac{\partial h}{\partial x_2}(\bar{x})\cdot x_2$$

$$= -x_2 \geq 0 \Rightarrow x_2 = 0$$

Other Breaks Down? Example 3

$$\operatorname*{Max}_{x}\left\{ f(x) = 12x_1 + x_2 \mid x \geq 0, h(x) = \left(2 - x_1\right)^3 - x_2 \geq 0 \right\}$$



Linearized Feasible Set $\bar{X}$

- Set of constraints binding at $\bar{x}$: $h_i(\bar{x}) = 0$
  - For $i \in B = \{i \mid i = 1, \ldots, m, h_i(\bar{x}) = 0\}$
- Replace binding constraints by linear approx.

$$\bar{h}_i(x) = h_i(\bar{x}) + \frac{\partial h_i}{\partial x}(\bar{x}) \cdot (x - \bar{x}) \geq 0$$

- Since these constraints also bind, we have

$$\frac{\partial h_i}{\partial x}(\bar{x}) \cdot (x - \bar{x}) \geq 0, \quad i \in B$$

  - Because $h_i(\bar{x}) = 0$

Linearized Feasible Set $\bar{X}$

- Note: These are "true" constraints if gradient

$$\frac{\partial h_i}{\partial x}(\bar{x}) \neq 0$$

- $\bar{X}$ = Linearized Feasible Set

  = Set of non-negative vectors satisfying

$$\frac{\partial h_i}{\partial x}(\bar{x}) \cdot (x - \bar{x}) \geq 0, \quad i \in B$$

Constraint Qualifications

- Set of feasible vectors:

$$X = \{x \mid x \geq 0, h_i(x) \geq 0\}$$

- The Constraint Qualifications hold at $\bar{x} \in X$ if

(i) Binding constraints have non-zero gradients

$$\frac{\partial h_i}{\partial x}(\bar{x}) \neq 0$$

(ii) The linearized feasible set $\bar{X}$ at $\bar{x}$ has a non-empty interior.

  - CQ guarantees FOC to be necessary conditions

Proposition 1.2-1
Kuhn-Tucker Conditions (FOC)

- Suppose $\bar{x}$ solves $\max_x \{ f(x) \mid x \in X \}, X = \text{feasib}$
- If the constraint qualifications hold at $\bar{x}$
- Then there exists shadow price vector $\lambda \geq 0$
- Such that (for $j = 1, \ldots, n, i = 1, \ldots m$)

$$\frac{\partial \mathfrak{L}}{\partial x_j}(\bar{x}, \lambda) \leq 0, \text{ with equality if } \bar{x}_j > 0.$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda_i}(\bar{x}, \lambda) \geq 0, \text{ with equality if } \lambda_i > 0.$$

Lemma 1.2-2 [Special Case] Quasi-Concave

- If for each binding constraint at $\bar{x}$, h_i is quasi-concave and $\frac{\partial h_i}{\partial x}(\bar{x}) \neq 0$
- Then, $X \subset \bar{X}$
  - Tangent Hyperplanes = Supporting Hyperplanes!
- Hence, if X has a non-empty interior, then so does the linearized set $\bar{X}$
  - Thus we have…

Proposition 1.2-3 [Quasi-Concave] Constraint Qualifications

- Suppose feasible set has non-empty interior

$$X = \{x \mid x \geq 0, h_i(x) \geq 0\}$$

- The Constraint Qualifications hold at $\bar{x} \in X$ if
- Binding constraints h_i is quasi-concave, and

$$\frac{\partial h_i}{\partial x}(\bar{x}) \neq 0$$

Proposition 1.2-4
Sufficient Conditions

- $\bar{x}$ solves

$$\max_{x} \{ f(x) \mid x \geq 0, h_i(x) \geq 0, i = 1, \ldots, m \}$$

- If f and $h_i, i = 1, \ldots, m$ are quasi-concave,
- The Kuhn-Tucker conditions hold at $\bar{x}$,
- Binding constraints have $\frac{\partial h_i}{\partial x}(\bar{x}) \neq 0$
- And $\frac{\partial f}{\partial x}(\bar{x}) \neq 0$.

Summary of 1.2

- Consumer = Producer
- Lagrange multiplier = Shadow prices
- FOC = "MR – MB = 0": Kuhn-Tucker
- When does this intuition fail?
  - Gradient = 0
  - Linearized feasible set has no interior

→Constraint Qualification: when it flies…

  - CQ for quasi-concave constraints

- Sufficient Conditions (Proof in Section 1.4)

Summary of 1.5

- Peak-Load Pricing requires Kuhn-Tucker
- MR="effective" MC
- Off-peak shadow price (for capacity) = 0
- All peak periods share additional capacity cost
- Can you think of situations (after you start your new job making $$$$) that requires something similar to peak-load pricing?
- Homework: J/R: A2.25, A2.28, A2.32-34
- Riley: 1.2-1, 1.2-3, 1.5-1~3