Aversion to Risk

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(Lecture 16, Micro Theory I)

Dealing with Uncertainty

- Preferences over risky choices (Section 7.1)
- One simple model: Expected Utility

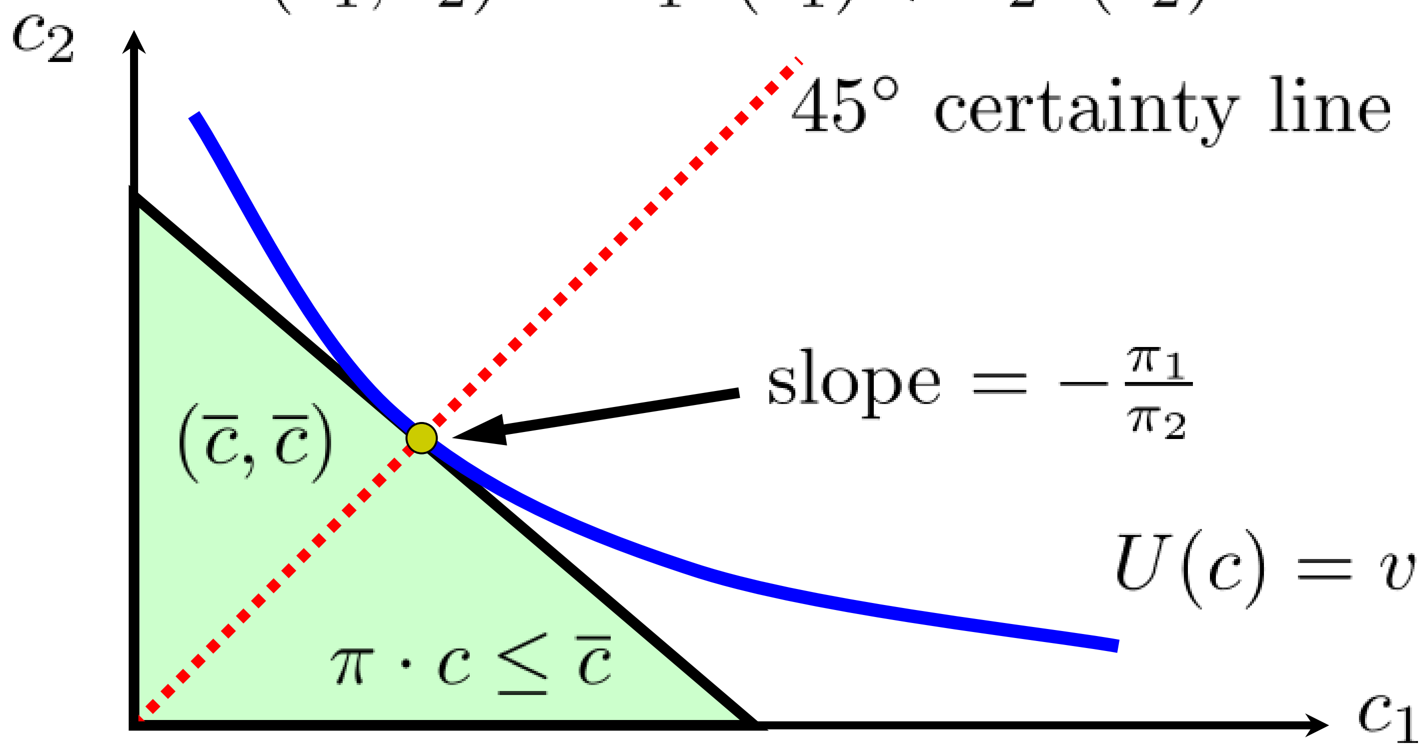
$$U(c_1, c_2) = \pi_1 v(c_1) + \pi_2 v(c_2)$$

- How can old tools be applied to analyze this?
- How is "risk aversion" measured?
- What about differences in risk aversion?
- How does a risk averse person trade state claims? (Wealth effects? Individual diff.?)

Dealing with Uncertainty

- Two states: s=1: KMT wins; s=2: DPP wins
- π_s: Prob. of state s c_s: consumption in state s

$$U(c_1, c_2) = \pi_1 v(c_1) + \pi_2 v(c_2)$$

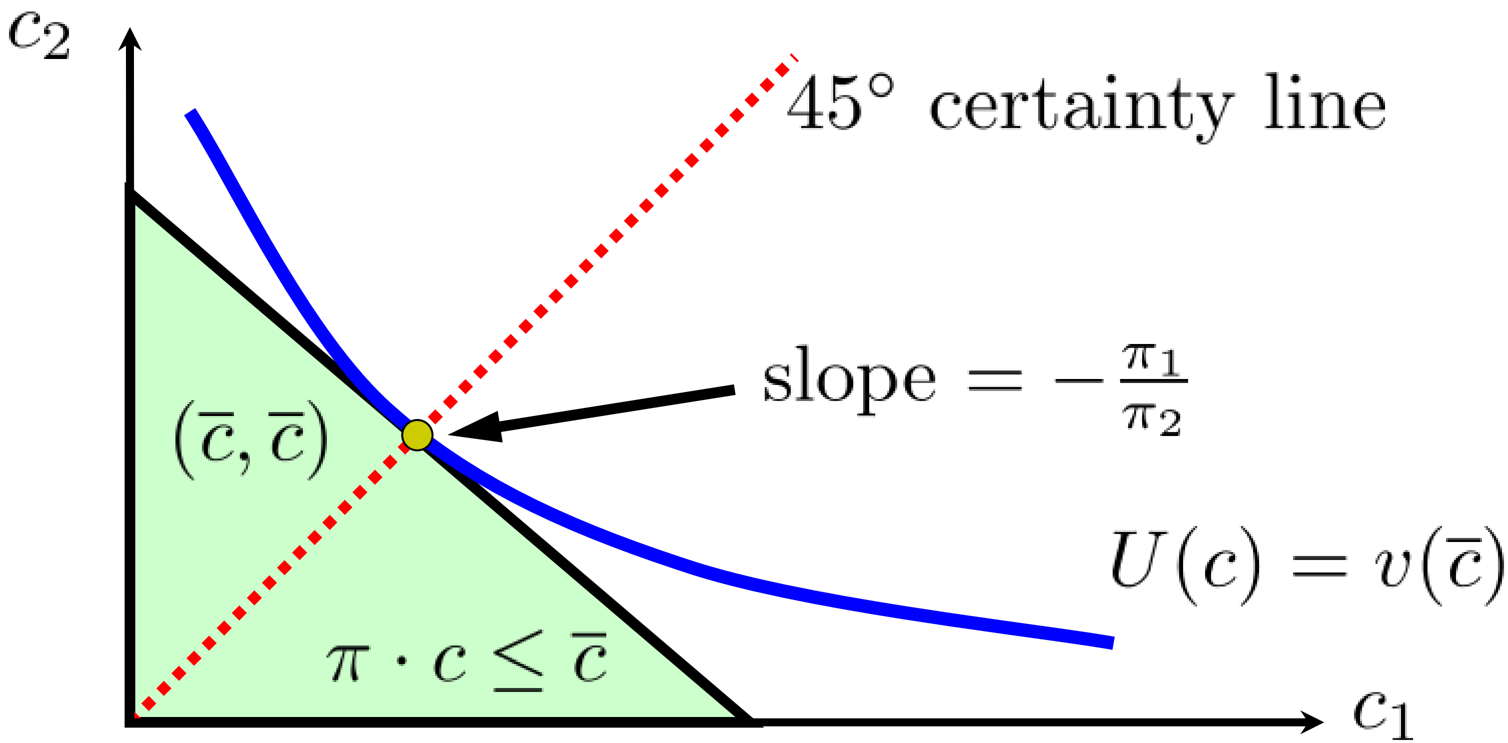


Risk Aversion: Concave *v(c)*

- Upper contour sets of *U(.)* is convex

$$U(c_1, c_2) = \pi_1 v(c_1) + (1 - \pi_1) v(c_2) \leq v(\overline{c})$$

  - Prefers certain bundle to risky ones with same EV

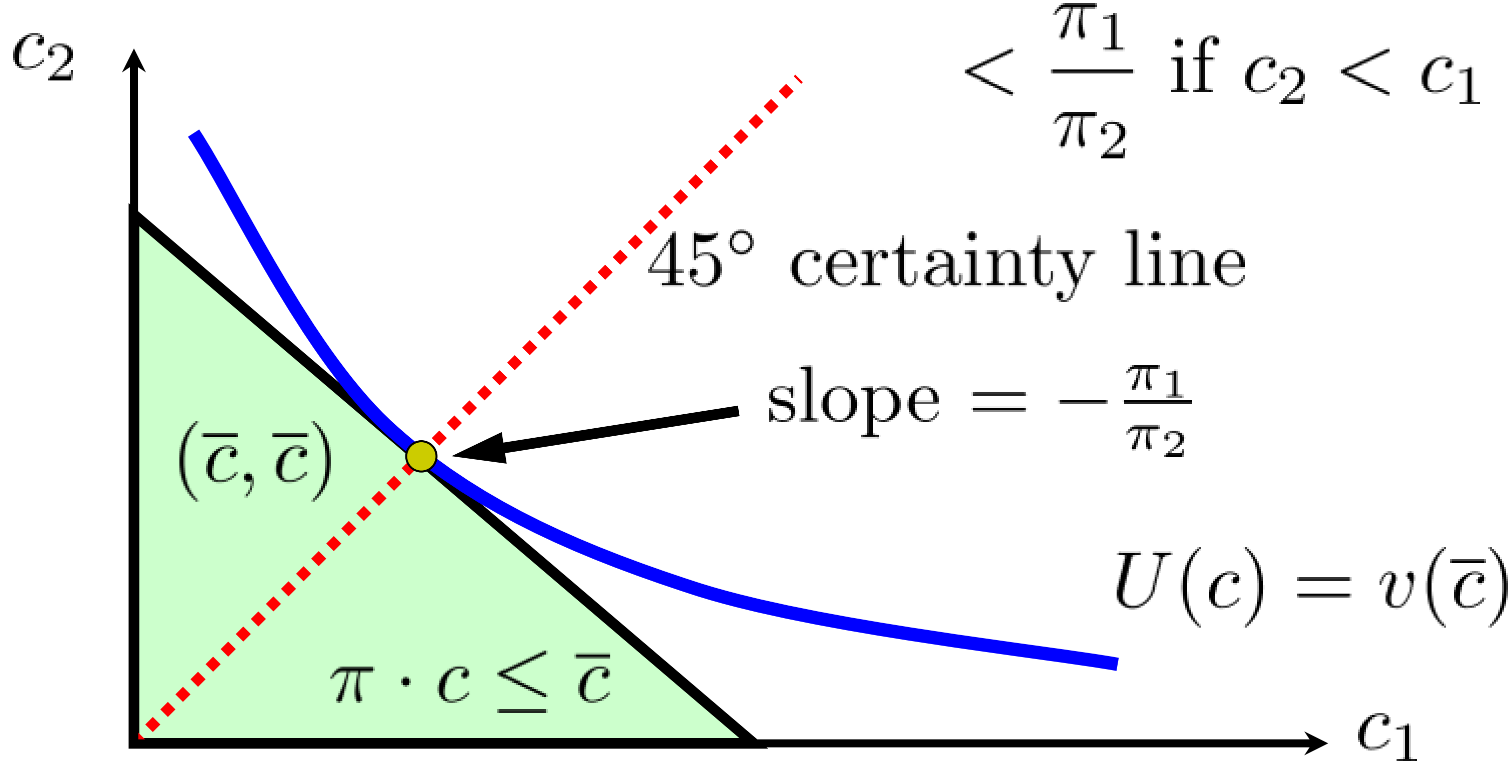


Risk Aversion: Concave *v(c)*

$$c_2 > c_1 \Rightarrow v'(c_1) > v'(c_2)$$

$$MRS(c_1, c_2) = \left.\frac{dc_2}{dc_1}\right|_{U=\overline{U}} = \frac{\frac{\partial U}{\partial c_1}}{\frac{\partial U}{\partial c_2}} = \frac{\pi_1 v'(c_1)}{\pi_2 v'(c_2)}$$

$$< \frac{\pi_1}{\pi_2} \text{ if } c_2 < c_1$$

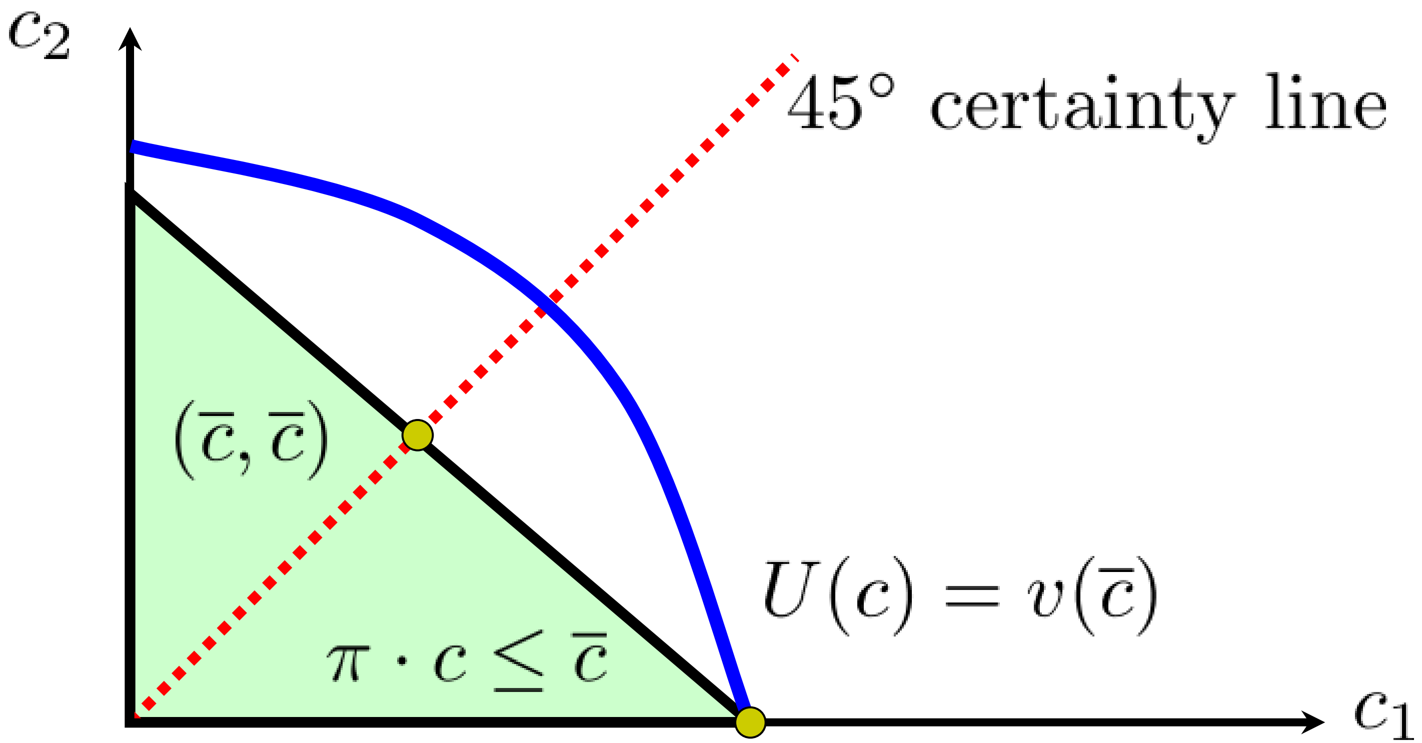


Extremely Risk Loving: Convex *v(c)*

- Upper contour sets of *U(.)* is convex

$$U(c_1, c_2) = \pi_1 v(c_1) + (1 - \pi_1) v(c_2) \geq v(\bar{c})$$

  - Prefers most risky bundles (weird!)



Jensen's Inequality

- For any probability vector π and consumption vector c, if $v(c)$ is concave, then

$$\sum_{s=1}^{S} \pi_s v(c_s) \leq v(\overline{c}) \text{ where } \overline{c} = \sum_{s=1}^{S} \pi_s c_s$$

- Proof:
- Easy if $v(c)$ is continuously differentiable, since

$$\text{Concavity implies } v(c_s) \leq v(\overline{c}) + v'(\overline{c})(c_s - \overline{c})$$

$$\sum_{s=1}^{S} \pi_s v(c_s) \leq v(\overline{c}) + v'(\overline{c}) \cdot \left(\quad 0 \quad \right)$$

QED

Measure Risk Aversion

- Let: $M = MRS(c_1, c_2) = \left.\frac{dc_2}{dc_1}\right|_{U=\overline{U}} = \frac{\pi_1 v'(c_1)}{\pi_2 v'(c_2)}$
- Then:

$$\ln M = \ln v'(c_1) - \ln v'(c_2) + \ln\left(\frac{\pi_1}{\pi_2}\right)$$

- So,

$$\frac{\partial}{\partial c_1}\ln M = \frac{v''(c_1)}{v'(c_1)}, \qquad \frac{\partial}{\partial c_2}\ln M = -\frac{v''(c_2)}{v'(c_2)}$$

- Thus,

$$\frac{1}{M}\frac{dM}{dc_1} = \frac{\partial}{\partial c_1}\ln M + \frac{\partial}{\partial c_2}\ln M \left.\frac{dc_2}{dc_1}\right|_{U=\overline{U}}$$

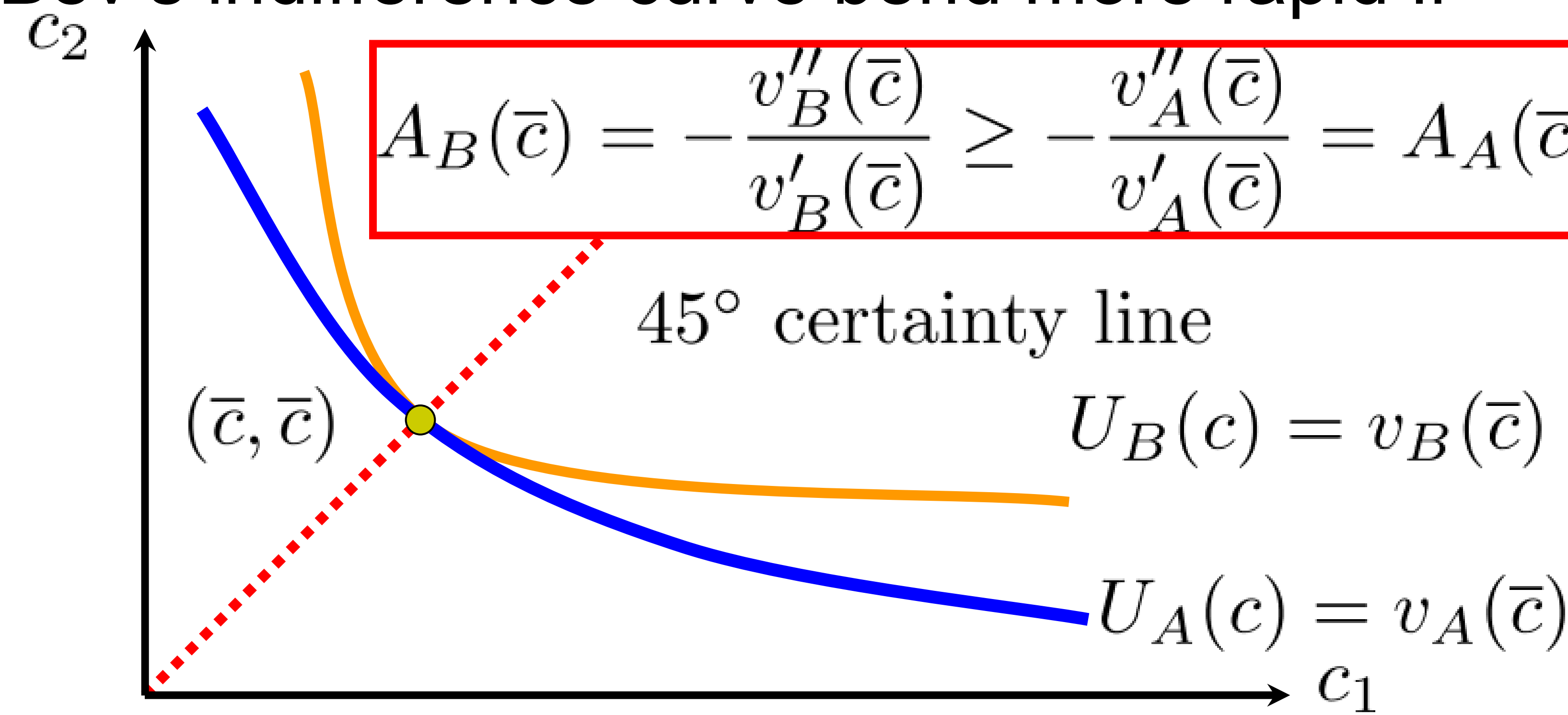
$$= \frac{v''(c_1)}{v'(c_1)} - \frac{v''(c_2)}{v'(c_2)}\cdot\left(-\frac{\pi_1}{\pi_2}\right)$$

Measuring Risk Aversion

- At $(\bar{c}, \bar{c})$, $$\frac{1}{M}\frac{dM}{dc_1} = \frac{v''(\bar{c})}{v'(\bar{c})} \cdot \left(1 + \frac{\pi_1}{\pi_2}\right)$$

- Bev's indifference curve bend more rapid if

$$A_B(\bar{c}) = -\frac{v''_B(\bar{c})}{v'_B(\bar{c})} \geq -\frac{v''_A(\bar{c})}{v'_A(\bar{c})} = A_A(\bar{c})$$



Measuring Risk Aversion

- Absolute Risk Aversion $$A(c) = -\frac{v''(\overline{c})}{v'(\overline{c})}$$
- Relative Risk Aversion $$R(c) = -\frac{cv''(\overline{c})}{v'(\overline{c})}$$
- Indifference curve bend more rapid if $A(c)$ high
- Can also obtain:
- $A(c)$ higher ➔ acceptable gambles set smaller
  - But need to first establish the relationship between two people's (risk averse) utility functions…

Proposition 7.2-1: Differences in Risk Aversion

- Two (von Neumann-Morgenstern) expected utility functions: v_A, v_B
- Then $$A_B(c) = -\frac{v''_B(c)}{v'_B(c)} \geq -\frac{v''_A(c)}{v'_A(c)} = A_A(c)$$
- iff the mapping $f(\cdot) : v_A \to v_B$ is concave.
- Proof:
- First, note that the mapping is monotonic since v_A, v_B are increasing, or $f'(c) > 0$

Proposition 7.2-1: Differences in Risk Aversion

- Proof: (Continued)
- $v_B(c) = f\left(v_A(c)\right)$ implies $v'_B(c) = f'(v_A)v'_A(c)$
- Hence,

$$\frac{\partial}{\partial c}\ln\left[v'_B(c)\right] = -\frac{v''_B(c)}{v'_B(c)} = -\frac{f''(c)}{f'(c)}v'_A(c) - \frac{v''_A(c)}{v'_A(c)}$$

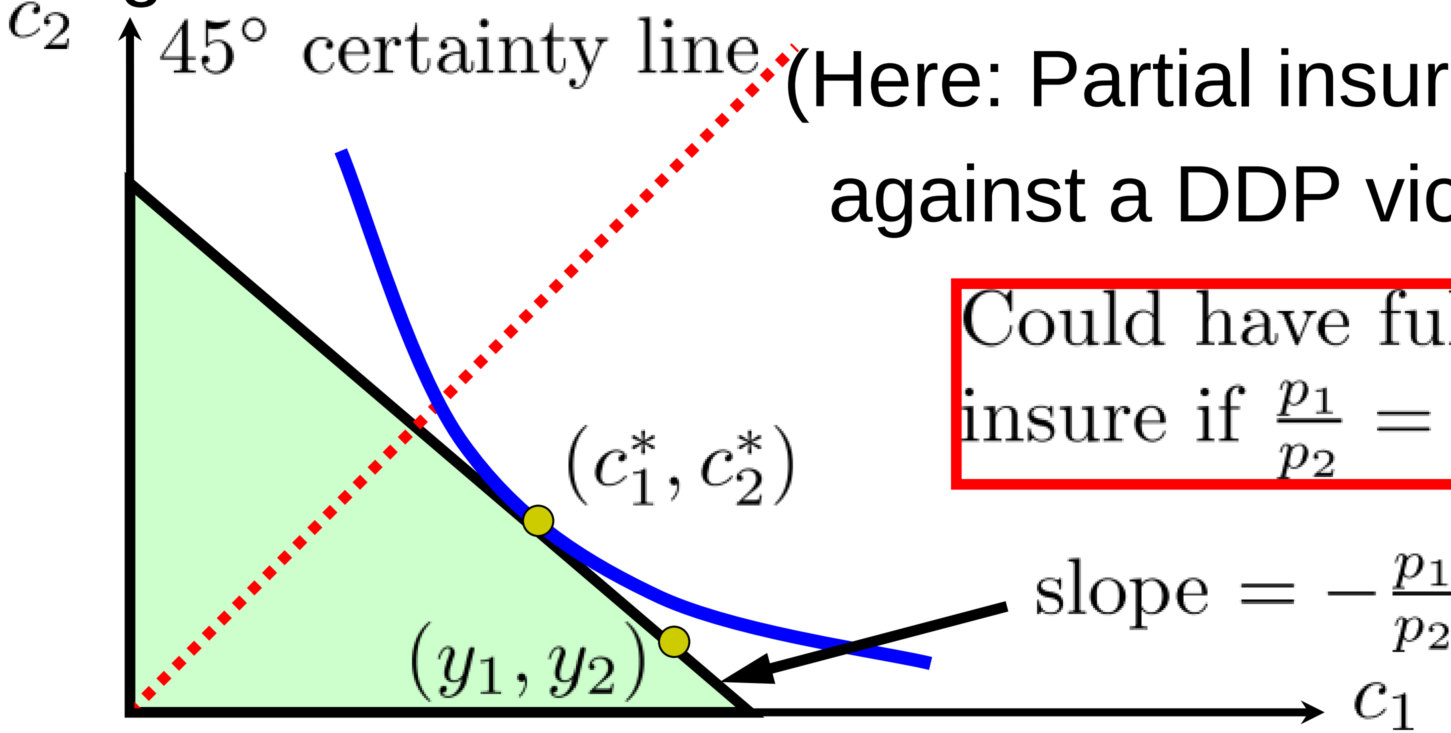
- Since $f'(c), v'_A(c) > 0$
- We have $-\frac{v''_B(c)}{v'_B(c)} \geq -\frac{v''_A(c)}{v'_A(c)} \quad \text{iff} \quad f''(v_A) \leq 0$

Proposition 7.2-2:Risk Aversion & the Set of Acceptable Gambles

- If $$A_B(c) = -\frac{v''_B(c)}{v'_B(c)} \geq -\frac{v''_A(c)}{v'_A(c)} = A_A(c)$$
- and both start with the same wealth $\bar{c}$. Then,
- The set of acceptable gambles to B is a subset of the set of gambles acceptable to A.
- Proof:
- Homework (J/R 2.33)

Trading in State Claim Markets

- y_s: Endowment in state *s*, $y_1 > y_2$
- p_s: current price of unit consumption in state *s*
- Budget Constraint: $p_1 c_1 + p_2 c_2 = p_1 y_1 + p_2 y_2$



(Here: Partial insurance against a DDP victory)

Could have fully insure if $\frac{p_1}{p_2} = \frac{\pi_1}{\pi_2}$

Wealth↑, how would riskiness of optimal choice change?

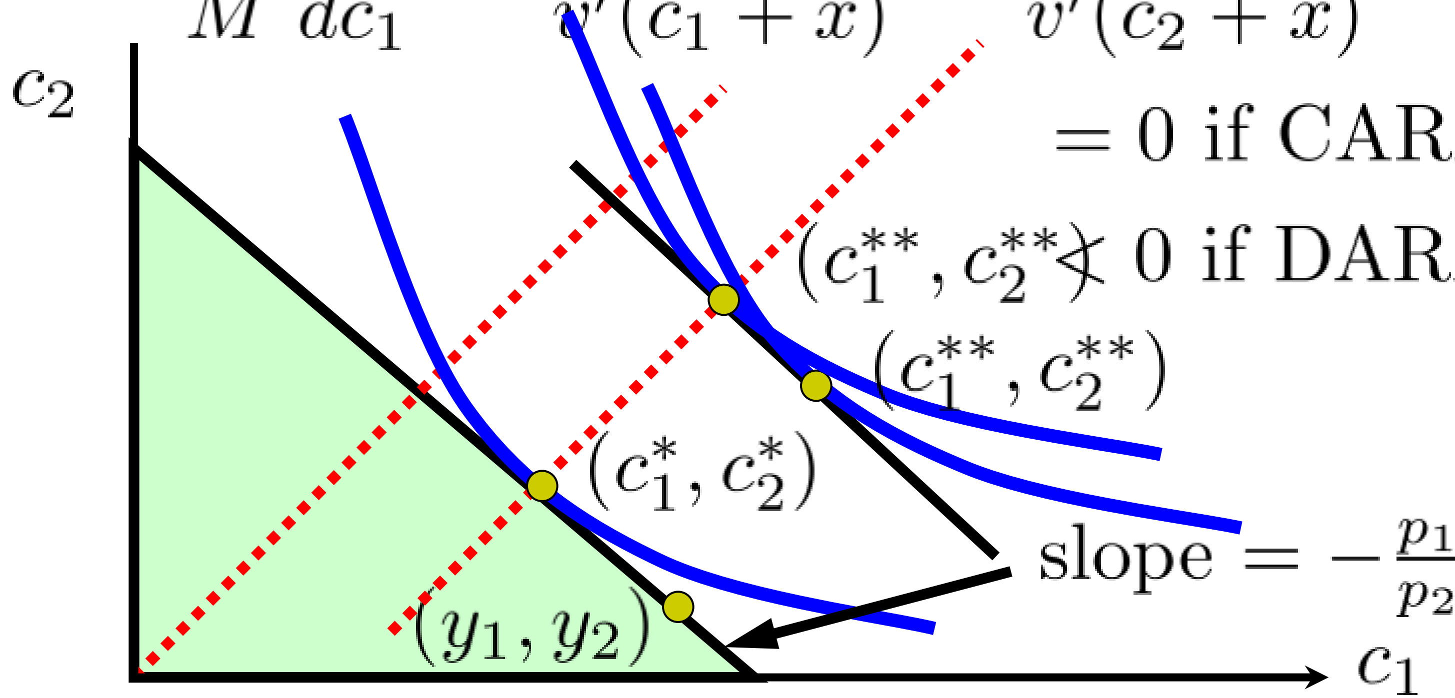
- Move from (c_1, c_2) to $(c_1 + x, c_2 + x)$

$$\ln M = \ln v'(c_1 + x) - \ln v'(c_2 + x) + \ln\left(\frac{\pi_1}{\pi_2}\right)$$

$$\frac{\partial}{\partial c}\ln M = \frac{1}{M}\frac{dM}{dc_1} = \frac{v''(c_1 + x)}{v'(c_1 + x)} - \frac{v''(c_2 + x)}{v'(c_2 + x)}$$

$$= 0 \text{ if CARA}$$

$$< 0 \text{ if DARA}$$



Wealth ↑ , how would riskiness of optimal choice change?

- In words, with CARA,
- Wealth ↑ implies parallel shift; MRS same!
  - Optimal choice is as risky as original choice

- With DARA,
- Wealth ↑ : Point lower than CARA; MRS ↑
  - Optimal choice is more risky than original choice
- Similar for IARA…

Simple Portfolio Choice: Riskless vs. Risky

- Alex can invest in either:
  - Riskless asset: $1 + r_1$
  - Risky asset: $1 + \tilde{r}_2$

- If Alex is risk averse, how high would the “risk premium” ($r_1 + \tilde{r}_2$) need to be for Alex to invest in the risky asset?

- Zero! (But risk premium affect proportions)

Simple Portfolio Choice: Riskless vs. Risky

- Using state claim formulation:
  - Risky asset yields $1+\tilde{r}_{2s}$ in state *s*
  - Probability of state *s* is $\pi_s, \;\; s = 1, \cdots, S$
- Invests *x* in risky asset, $(W - x)$ in riskless one
- Final consumption in state *s* is

$$c_s = W(1 + r_1) + x\theta_s \qquad (\theta_s = r_{2s} - r_1)$$

- Alex's utility:

$$U(x) = \sum_{s=1}^{S} \pi_s v\big(W(1 + r_1) + x\theta_s\big)$$

Simple Portfolio Choice: Riskless vs. Risky

- Marginal Gains from increasing *x*

$$U'(x) = \sum_{s=1}^{S} \pi_s v'\big(W(1+r_1) + x\theta_s\big) \cdot \theta_s$$

- So, there is a single turning point since

$$U''(x) = \sum_{s=1}^{S} \pi_s v''\big(W(1+r_1) + x\theta_s\big) \cdot \theta_s^2 < 0$$

- Should choose *x* so that $U'(0) = 0$

Simple Portfolio Choice: Riskless vs. Risky

- Since (unless infinitely risk averse)

$$U'(0) = v'\big(W(1+r_1)\big)\sum_{s=1}^{S}\pi_s\theta_s > 0 \Leftrightarrow \sum_{s=1}^{S}\pi_s\theta_s > 0$$

- Alex will always buy some risky asset!
- Intuition:
- When taking no risk, each MU weighted with the same $v'\big(W(1+r_1)\big)$, as if risk neutral!
- Not true for any $x > 0$
  - Depends on degree of risk aversion…

Would a more risk averse person invest less risky?

- Yes!
  - Choose smaller x if everywhere more risk averse
- Proof:
- Consider Bev: $v_B(c) = f\big(v_A(c)\big)$, f concave
- If Alex's optimal choice and consumption be

$$x^* \text{ and } c_s^* = W(1 + r_1) + \theta_s x^*$$

- Then, $$U_A'(x^*) = \sum_{s=1}^{S} \pi_s v'(c_s^*) \cdot \theta_s = 0$$

Would a more risk averse person invest less risky?

- Claim: $U_B'(x^*) < 0$ (And we are done!)
- Proof:
- Order states so $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_S$
- Let t be the largest state that $\theta_s = r_{2s} - r_1 > 0$
- Then, $v_A(c_s^*) \geq v_A(c_t^*) \text{ for all } s \leq t$

$$v_A(c_s^*) < v_A(c_t^*) \text{ for all } s > t$$

- And, (by concavity of *f*)

$$f'\big(v_A(c_s^*)\big) \geq f'\big(v_A(c_t^*)\big), s \leq t$$

$$f'\big(v_A(c_s^*)\big) < f'\big(v_A(c_t^*)\big), s > t$$

Would a more risk averse person invest less risky?

- Hence,

$$U_B'(x^*) = \sum_{s=1}^{S} \pi_s f'\big(v_A(c_s^*)\big) v_A'(c_s^*) \cdot \theta_s$$

$$< \sum_{s=1}^{t} \pi_s f'\big(v_A(c_t^*)\big) v_A'(c_s^*) \cdot \theta_s$$

$$- \sum_{s=t+1}^{S} \pi_s f'\big(v_A(c_t^*)\big) v_A'(c_s^*) \cdot (-\theta_s)$$

$$= f'\big(v_A(c_t^*)\big) \sum_{s=1}^{S} \pi_s v_A'(c_s^*) \cdot \theta_s = f'\big(v_A(c_t^*)\big) U_A'(c^*) = 0$$

Summary of 7.2

- Von Neumann Morganstern Utility Function
- Jensen's Inequality
- Absolute or Relative Risk Aversion
- Bev is more risk verse than Alex implies:
  - Mapping from v_A to v_B is concave
  - Bev will not accept gambles that Alex rejects
- State Claim Market
  - Wealth effect; Risk averse people invest less risky
- Homework: Riley-7.2-2, 5-8; J/R-2.25, 2.33-35